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Reconstruction of *M*-Periodic Nonuniformly Sampled Signals Using Multivariate Polynomial Impulse Response Time-Varying FIR Filters

Håkan Johansson, Per Löwenborg, and Kameswaran Vengattaramane

Division of Electronics Systems, Department of Electrical Engineering, SE-581 83 Linköping University, Sweden
Email: hakanj@isy.liu.se, perl@isy.liu.se, kamve460@student.liu.se, Phone: +4613281676, Fax: +4613139282

ABSTRACT

This paper introduces multivariate polynomial impulse response time-varying FIR filters for reconstruction of M-periodic nonuniformly sampled signals. The main advantages of these reconstruction filters are that 1) they do not require on-line filter design, and 2) most of their multipliers are fixed and can thus be implemented using low-cost dedicated multiplier elements. This is in contrast to existing filters that require on-line design as well as many general multipliers in the implementation. By using the proposed filters, the overall implementation cost may therefore be reduced in applications where the sampling pattern changes now and then. Design examples are included demonstrating the usefulness of the proposed filters.

1. INTRODUCTION

Nonuniform sampling takes place in many applications either intentionally or unintentionally [1]. One application belonging to the latter category is constituted by *M*-channel time-interleaved analog-to-digital converters (ADCs) [2] where static time-skew channel mismatch errors give rise to an *M*-periodic nonuniform sampling pattern instead of the desired uniform pattern, as exemplified in Fig. 1 for $M = 3$. In such an application, it is necessary to reconstruct the "uniform sequence" from the "nonuniform sequence" In this paper, a time-varying FIR filter is employed for the reconstruction. If the sampling pattern is known and fixed, this filter can be designed in an optimum way using, e.g., least-squares or minimax design [3], [4]. However, problems arise when the sampling pattern is changed now and then during normal operation, like in time-interleaved ADCs, as the filters then need to be redesigned. This results in a large overhead cost in the implementation as general on-line design is cumbersome.

To circumvent this problem, this paper introduces an FIR filter class for which the time-varying impulse response can be expressed as a multivariate polynomial with the variables being the distances between the undesired nonuniform sampling instances and the desired uniform sampling instances. The main advantage of using these filters is that the need for on-line design thereby is eliminated. Instead, it is enough to perform only one design before the filter is implemented. In the implementation, it suffices to adjust only *M* variable parameters. Although the number of multipliers (filter coefficients) is larger for the proposed filters than for the ones in [3], [4], most of these multipliers are fixed which means that they can be implemented using dedicated multipliers which can be implemented to a substantially lower cost than general multipliers [5]. This is in contrast to the filters in [3], [4], which do need general multipliers. Therefore, using the proposed filters, the implementation cost of the filtering can be kept low and the overall cost reduced due to the removal of the on-line design.

It should be noted, however, that depending on the maximum distance between the undesired nonuniform sampling instances and the desired uniform sampling instances, and the

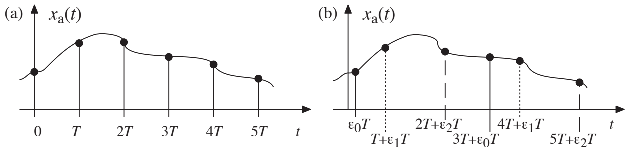

Figure 1. (a) Uniform sampling. (b) *M*-periodic nonuniform sampling for $M = 3$.

approximation error allowed, the number of terms in the multivariate polynomial may become rather high which can result in an intolerably high complexity. The proposed filter class may therefore be less attractive in such cases. On the other hand, it turns out that many of the terms in the general multivariate polynomial a priori can be eliminated and, in addition, it is possible to impose symmetries among the polynomial coefficients in order to further reduce the complexity. Therefore, the proposed filter class it is indeed useful for many practical cases, in particular applications where the sampling instance deviations from the uniform grid are small. This is the case in, e.g. time-interleaved ADCs where the deviations (time-skew mismatch errors) are typically only a few per cent.

The proposed filters are generalizations of the filters in [6] that target the special case of two-periodic nonuniform sampling with one of the sampling instance deviations being zero, which corresponds to a reference channel in two-channel time interleaved ADCs. It should also be noted that similar approaches have been reported earlier [7], [8], but the technique in [7] is more restricted in the sense that it requires an additional amount of oversampling, whereas the technique in [8] is not well understood; both the design and potentials are unclear. The advantage of the proposed filters comes from the fact that high-order terms in the polynomial are included. In [7], [8], only first-order terms are present which imposes restrictions.

2. PREREQUISITES

Throughout the paper, it is assumed that the nonuniform sampling of the continuous-time (CT) signal $x_a(t)$ generates the sequence $v(n)$ given by

$$v(n) = x_a(t_n) \tag{1}$$

where

$$t_n = nT + \varepsilon_n T \tag{2}$$

with $\varepsilon_n T$ representing the distance between the nonuniform sampling instance t_n and the uniform sampling instance nT, and the average sampling frequency being $1/T$. Furthermore, we assume that 1) $\varepsilon_n T$ is *M*-periodic, which means that $\varepsilon_n T = \varepsilon_{n+M} T$, and 2) $-\varepsilon_{\max} \le \varepsilon_n \le \varepsilon_{\max}$.

The second condition means that we know beforehand that ε_n are fixed and lie in the interval $-\varepsilon_{\max} \le \varepsilon_n \le \varepsilon_{\max}$, but we do not know beforehand the exact values ε_n will take on in a specific realization. This is the situation in, e.g., M-channel time-interleaved ADCs where one typically knows that ε_n (the time-skew mismatch errors) are bounded in magnitude by a few percent of the sampling period, but one does not know the exact values. These values are obtained through estimations, and they may change now and then during operation [1]. Regardless the values of $\varepsilon_n T$, satisfying $-\varepsilon_{\max} \le \varepsilon_n \le \varepsilon_{\max}$, the reconstruction filter must be able to reconstruct the nonuniformly sampled signal within a certain approximation error. The simplest way to do this is to redesign the reconstruction filter once a new set of M fixed values $\varepsilon_n T = \varepsilon_{n+M} T$, n = 0, 1, ..., M–1, has been obtained. This is costly though in an implementation. To avoid this problem, we introduce the multivariate polynomial impulse response time-varying FIR filters in Section 3 for which redesign is not needed.

Furthermore, it is assumed that 3) $x_a(t)$ is bandlimited according to

$$X_a(j\omega) = 0, \quad 0 < \omega_0 < |\omega|, \quad \omega_0 < \pi/T. \tag{3}$$

and 4) $x_a(t)$ is slightly oversampled which means that there is a certain "don't-care region" between ω_0 and π/T where the signal contains no frequency components. The practical advantages of assuming a slight oversampling were thoroughly discussed and demonstrated in [3], [4].

3. PROPOSED MULTIVARIATE POLYNOMIAL IMPULSE RESPONSE TIME-VARYING FIR FILTERS

Using a noncausal (for convenience in the design) time-varying FIR filter of order 2N with the impulse responses $h_n(k)$, the output $y(n)$ is formed according to

$$y(n) = \sum_{k=-N}^{N} v(n-k)h_n(k). \tag{4}$$

It is desired to select $h_n(k)$ so that $y(n)$ approximates $x(n) = x_a(nT)$ as close as possible in some sense as perfect reconstruction is not feasible in general. This is done by minimizing $e(n) = y(n) - x(n)$ (in some sense) where [3], [4]

$$e(n) = \frac{1}{2\pi}\int_{-\omega_0 T}^{\omega_0 T} (A_n(j\omega T) - 1)X(e^{j\omega T})e^{j\omega T n}d(\omega T) \tag{5}$$

with

$$A_n(j\omega T) = \sum_{k=-N}^{N} h_n(k)e^{-j\omega T(k-\varepsilon_{n-k})}. \tag{6}$$

Apparently, to goal is to make $A_n(j\omega T)$ as close to one as possible in some sense.

The formulas above hold for general nonuniform sampling [3], [4] in which case a new impulse response has to be determined for each n. In the M-periodic case, it suffices to determine only M impulse responses because, then, $h_n(k) = h_{n+M}(k)$. To this end, we let $h_n(k)$ be expressible as a multivariate polynomial according to

$$h_n(k) = \delta(k) + \sum_{\boldsymbol{r}\in \boldsymbol{S}} \alpha_{r_0 r_1 \ldots r_{M-1}}(k)\varepsilon_n^{r_0}\varepsilon_{n+1}^{r_1}\ldots\varepsilon_{n+M-1}^{r_{M-1}} \tag{7}$$

1. The deviations $\varepsilon_n T$ cannot change from sample to sample, but between batches of samples in which case reconstruction of M-periodic nonuniform sampling still applies, but only "batchwise".

where $\delta(k)$ is the unit impulse sequence,

$$\boldsymbol{r} = \begin{bmatrix} r_0 \; r_1 \; \ldots \; r_{M-1} \end{bmatrix}^T, \tag{8}$$

and $\boldsymbol{S}$ is a set of vectors for which

$$r_0 \in [1, 2, \ldots, L], \tag{9}$$

$$r_m \in [0, 1, \ldots, L-1], \quad m = 1, 2, \ldots, M-1, \tag{10}$$

and

$$\sum_{m=0}^{M-1} r_m \in [1, 2, \ldots, L]. \tag{11}$$

This means that only those combinations of r_m for which the sum in (7) contains at least one of ε_n, ε_n^2, ..., ε_n^L, are utilized. The rationale behind this is the observation that, when $\varepsilon_n = 0$, we already have available the "uniform sample" $v(n) = x_a(nT)$. This means that the filter output should simply produce $y(n) = v(n)$ which corresponds to $h_n(k) = \delta(k)$, i.e., the first term in (7). This is independent of the values of the remaining deviations ε_{n+1}, ε_{n+2}, ..., ε_{n+M-1}, which implies that it is not meaningful to include terms in the sum in (7) that does not contain ε_n.

For L = 1 and L = 2, (7) can be written in a simpler form. For L = 1, (7) reduces to

$$h_n(k) = \delta(k) + \alpha_{10\ldots0}(k)\varepsilon_n \tag{12}$$

and for L = 2 one obtains

$$h_n(k) = \delta(k) + \alpha_{10\ldots0}(k)\varepsilon_n + \alpha_{20\ldots0}(k)\varepsilon_n^2 + \sum_{m=1}^{M-1}\alpha_{10\ldots m\ldots0}(k)\varepsilon_n\varepsilon_{n+m}. \tag{13}$$

For L = 1 and L = 2, it is seen from (12) and (13) that the number of terms in $h_n(k)$ is small. These cases can however only be used when $\varepsilon_{\max}$ is small. As $\varepsilon_{\max}$ increases, one has to increase L to obtain an acceptably small approximation error. This comes with an increase of the number of terms that grows faster and faster with increasing L. For example, for M = 4, we need 1, 5, 15, and 35 terms for L = 1, 2, 3, 4, respectively. The corresponding realizations require 1, 5, 15, and 35 fixed subfilters and equally many general multipliers (see Section 4). This should be compared to four filters (M filters in general) with general multipliers plus the additional block that implements the on-line design when using the traditional approach [3], [4]. From these figures, it is clear that the proposed filter class is most attractive for cases where $\varepsilon_{\max}$ is small like in time-interleaved ADCs. In such cases, the number of fixed subfilters and general multipliers is small which results in a substantially lower overall implementation complexity than that of the traditional approach [3], [4]. More detailed examples are given in Section 5.

Both the design and implementation complexity of the proposed filters can be reduced by utilizing symmetries among the filter coefficients. When $-\varepsilon_{\max} \le \varepsilon_n \le \varepsilon_{\max}$, it can be shown that the following symmetries can be imposed.

First, we have

$$\alpha_{r_0 0\ldots0}(k) = \begin{cases} \alpha_{r_0 0\ldots0}(-k), & \text{for even } r_0 \\ -\alpha_{r_0 0\ldots0}(-k), & \text{for odd } r_0 \end{cases}. \tag{14}$$

That is, the impulse responses $\alpha_{r_0 0 \ldots 0}(k)$ for the terms $\varepsilon_n^{r_0}$ are symmetric (anti-symmetric) [10] for even (odd) r_0. It is noted that the anti-symmetry in (14) implies

$$\alpha_{r_0 0 \ldots 0}(0) = 0, \quad \text{for odd } r_0 . \tag{15}$$

Second, we have

$$\alpha_{r_0 \ldots r_m \ldots r_{M-m} \ldots r_{M-1}}(k) =$$

$$= \begin{cases} \alpha_{r_0 \ldots r_{M-m} \ldots r_m \ldots r_{M-1}}(-k), & \text{for even } R \\ -\alpha_{r_0 \ldots r_{M-m} \ldots r_m \ldots r_{M-1}}(-k), & \text{for odd } R \end{cases} \tag{16}$$

for $m = 1, 2, ..., M–1$, where

$$R = \sum_{m=0}^{M-1} r_m . \tag{17}$$

That is, the impulse responses $\alpha_{r_0 \ldots r_m \ldots r_{M-m} \ldots r_{M-1}}(k)$ for terms containing $\varepsilon_{n+m}^{r_m} \varepsilon_{n+M-m}^{r_{M-m}}$ equal the reversed (and sign-inverted for odd R) versions of the impulse responses $\alpha_{r_0 \ldots r_{M-m} \ldots r_m \ldots r_{M-1}}(-k)$ for terms containing $\varepsilon_{n+m}^{r_{M-m}} \varepsilon_{n+M-m}^{r_m}$.

When $r_m = r_{M-m}$, the two α impulse responses in (16) coincide which means that this impulse response is symmetric or anti-symmetric, like (14). When $r_m \neq r_{M-m}$, the two α impulse responses do not coincide but exhibit pair-wise symmetry or anti-symmetry. Further, for M even, we have $r_m = r_{M-m}$ for $m = M/2$ which means that all terms containing $\varepsilon_{n+M/2}^{r_{M/2}}$, $r_{M/2} > 0$, are symmetric or anti-symmetric, like (14).

4. REALIZATION

The transfer functions corresponding to (7) are

$$H_n(z) = \sum_{k=-N}^{N} h_n(k) z^{-k} . \tag{18}$$

Inserting (7) into (18), $H_n(z)$ can be rewritten as

$$H_n(z) = 1 + \sum_{r=0}^{L} \varepsilon_n^{r_0} \varepsilon_{n+1}^{r_1} \ldots \varepsilon_{n+M-1}^{r_{M-1}} Q_{r_0 r_1 \ldots r_{M-1}}(z) \tag{19}$$

where

$$Q_{r_0 r_1 \ldots r_{M-1}}(z) = \sum_{k=-N}^{N} \alpha_{r_0 r_1 \ldots r_{M-1}}(k) z^{-k} . \tag{20}$$

The reconstruction filter can consequently be implemented with the aid of the *fixed* 2*N*th-order FIR filters $Q_{r_0 r_1 \ldots r_{M-1}}(z)$ and the set of variable multipliers $\varepsilon_n^{r_0} \varepsilon_{n+1}^{r_1} \ldots \varepsilon_{n+M-1}^{r_{M-1}}$ as illustrated in Fig. 2. It is noted that this structure reduces to the well-known Farrow structure [9] in the one-dimensional case, i.e., when only $\varepsilon_n^{r_0}$ are present.

5. DESIGN EXAMPLES

In general, the reconstruction system is designed by minimizing the “size” of $A_n(j\omega T) - 1$ in some sense. Two ways to do it is to minimize $A_n(j\omega T) - 1$ in the minimax (Chebyshev) sense and least-squares sense. A third option is to formulate the problem as a filter bank design problem and then design a distortion and M–1 aliasing functions [4]. Due to the limited space, we cannot go into design details in this paper. Instead, we provide two examples that demonstrate the usefulness of the proposed filters. Design details will be published elsewhere.

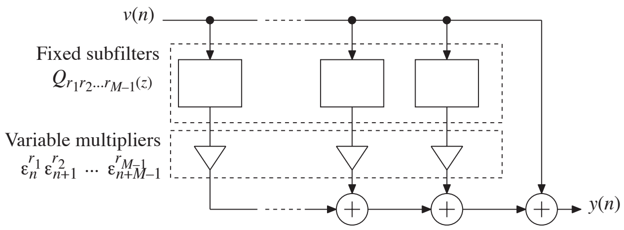

Figure 2. Realization of the proposed time-varying FIR filters using fixed subfilters and variable multipliers.

Example 1 – The case with $M = 4$, $L = 2$, $\varepsilon_{\max} = 0.02$, and $\omega_0 T = 0.8\pi$ is considered. The reconstruction filter is designed by minimizing the maximum of $A_n(j\omega T)$ over $[-\omega_0 T, \omega_0 T]$ for all combinations of $\varepsilon_n, \varepsilon_{n+1}, \varepsilon_{n+2}$, and ε_{n+3} within $-\varepsilon_{\max} \le \varepsilon_n, \varepsilon_{n+1}, \varepsilon_{n+2}, \varepsilon_{n+3} \le \varepsilon_{\max}$ [2]. This is a convex problem for which a unique optimum exists. We have used minimax.m in MATLAB to solve the corresponding discretized problem. Using filters of order $2N = 20$, the maximum error becomes 0.0000825 (–81.67 dB). Figure 3 plots the magnitude of $A_n(j\omega T) - 1$ for three different combinations of ε_{n+p}, $p = 0, 1, 2, 3$.

The filter realization consists of five subfilters $Q_{1000}(z)$, $Q_{2000}(z)$, $Q_{1100}(z)$, $Q_{1010}(z)$, and $Q_{1001}(z)$, and the corresponding five variable multipliers ε_n, ε_n^2, $\varepsilon_n \varepsilon_{n+1}$, $\varepsilon_n \varepsilon_{n+2}$, and $\varepsilon_n \varepsilon_{n+3}$, respectively. The filter $Q_{1000}(z)$ is anti-symmetric whereas $Q_{2000}(z)$ and $Q_{1010}(z)$ are symmetric. The filters $Q_{1100}(z)$ and $Q_{1010}(z)$ do not possess symmetries, but they contain the same coefficients but in reversed order according to (16). The number of distinct multiplier values in the design is therefore only 53 plus the four variable ones. Using transposed direct-form FIR structures (for $Q_{1100}(z)$ and $Q_{1010}(z)$), the number of fixed multipliers in the implementation is also 53, but this comes with an increased number of delay elements. Using instead direct-form FIR structures for all five subfilters (and thereby the minimum number of delay elements) the implementation requires 74 fixed multipliers.

Using the filters in [3], [4], the same specification (–80 dB) is met with orders of 18. The corresponding implementation requires four different filter impulse responses, and thus 76 distinct multiplier values, and at least 19 general multipliers (if a single filter with time-varying coefficients is implemented). This should be compared to 53 (or 76) fixed multipliers, which are substantially less complex to implement, and four general multipliers. For the fixed multipliers, one can take full advantage of so called multiple-constant techniques [11], [12], which is not possible for general multipliers. In addition, the filters in [3], [4] require on-line filter design which increases their overall implementation cost dramatically. This illustrates that, for small $\varepsilon_{\max}$, the use of the proposed filters can reduce the over-

2. Since the optimization is done over all combinations of ε_{n+p}, $p = 0, 1, 2, 3$, in their interval $[-\varepsilon_{\max}, \varepsilon_{\max}]$, the problem formulation and solution are independent on the value of n. Hence, one can in the filter design fix n to an arbitrary number, and perform only one optimization.

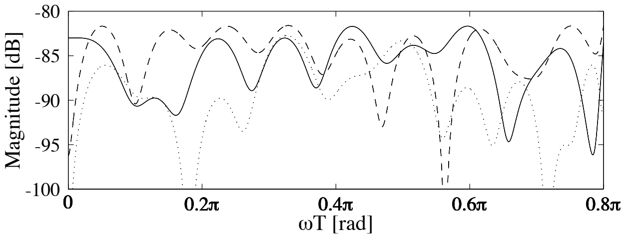

Figure 3. Magnitude of $A_n(j\omega T) - 1$ in Example 1 for three different combinations of ε_{n+p}, $p = 0, 1, 2, 3$.

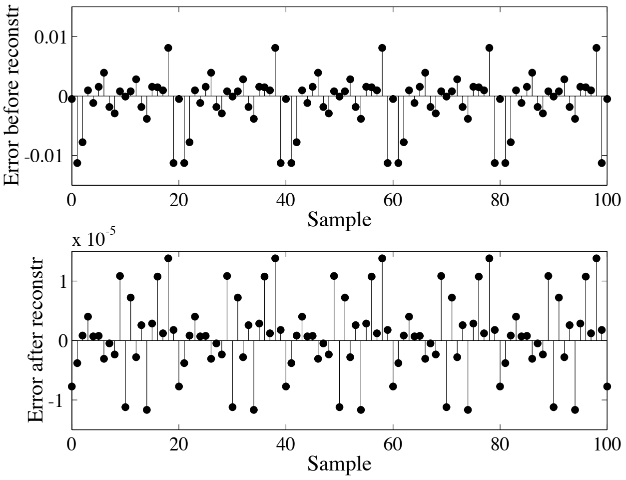

Figure 4. Time-domain error before (top) and after (bottom) reconstruction in Example 1. A nearly 60 dB improvement is seen.

all complexity substantially.

To verify the function of the proposed filter, we have applied a multi-sine input with frequencies at $k\pi/8$ for $k = 1, 2, ..., 8$, for the case where the sampling instance errors are 0.01, –0.02, –0.01, and 0.02, respectively. Figure 4 plots the time-domain error before reconstruction [i.e., $v(n)$–$x(n)$] and the error after reconstruction [i.e., $y(n)$–$x(n)$]. We see that the error after reconstruction is far below 0.0000825 (–81.67 dB) which is the upper bound here due to the minimax design [4]. It is also seen that the maximum error has been suppressed by some 60 dB.

Example 2 – The case with $M = 2$, $L = 3$, $\varepsilon_{\max} = 0.05$, and $\omega_0 T = 0.8\pi$ is considered. As opposed to Example 1, the reconstruction filter is here designed in the least-squares sense instead of minimax sense. This is done by extending the method in [6] where one of ε_n and ε_{n+1} is fixed. This amounts to solving a triple integral over the region $\omega T \in [-\omega_0 T, \omega_0 T]$, $\varepsilon_n, \varepsilon_{n+1} \in [-0.05, 0.05]$, with the integrand being $|A_n(j\omega T) - 1|^2$. This is a convex problem to which a unique solution exists. Figure 3 plots the magnitude of $A_n(j\omega T) - 1$ for three different combinations of ε_n and ε_{n+1} and the order being $2N = 22$. As expected, the solution is more "least-squares like" compared to the minimax optimized filters in Example 1 (Fig. 3).

Using the filters in [3], [4] with the same order, roughly the same error is obtained. The corresponding implementation requires two different filters, and thus 46 distinct multiplier values for each combination of ε_n and ε_{n+1}, at least 23 general multipliers (if a single time-varying filter is implemented), and on-line design. This should be compared to 68 fixed multipliers,

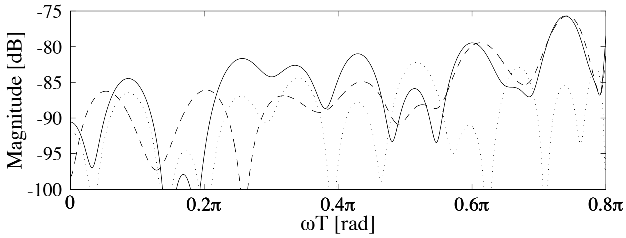

Figure 5. Magnitude of $A_n(j\omega T) - 1$ in Example 2 for three different combinations of ε_n, and ε_{n+1}.

two general multipliers, but no on-line design. This demonstrates that the proposed filters can be efficient also for larger values of L, at least as long as M is small.

6. CONCLUSION

This paper introduced multivariate polynomial impulse response time-varying FIR filters for reconstruction of M-periodic nonuniformly sampled signals. The main advantages of these reconstruction filters are that they do not require on-line filter design and, in cases of small uniform sampling instance deviations like in time-interleaved ADCs, most of their multipliers are fixed and can thus be implemented using low-cost dedicated multiplier elements. The proposed filters are therefore attractive for applications where the sampling pattern may change now and then. The features and benefits of the proposed filters were demonstrated through design examples. Design details were not included but will be considered elsewhere.

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