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ON FAMILIES OF 2^N-DIMENSIONAL HYPERCOMPLEX ALGEBRAS SUITABLE FOR DIGITAL SIGNAL PROCESSING

Daniel Alfsmann

Digital Signal Processing Group (DISPO), University of Bochum
Ruhr-Universität, 44780 Bochum, Germany
phone: +49 234 32 27585, fax: +49 234 32 07585, email: alfsmann@nt.rub.de, web: www.dsv.rub.de

ABSTRACT

A survey of hypercomplex algebras suitable for DSP is presented. Generally applicable properties are obtained, including a paraunitarity condition for hypercomplex lossless systems. Algebras of dimension $n = 2^N$, $N \in \mathbb{Z}$, are classified by generation methods, constituting families. Two algebra families, which hold commutative and associative properties for arbitrary N, are examined in more detail: The 2^N-dimensional hyperbolic numbers and tessarines. Since these non-division algebras possess zero divisors, orthogonal decomposition of hypercomplex numbers is investigated in general.

1. INTRODUCTION

DSP algorithms are generally based on the real number system. Nevertheless, applying complex numbers for signal and system representations often allows for compact and convenient system descriptions. Further, they double the degree of an accordingly structured real-valued system [1]. Hence, it has been investigated in how far higher dimensional hypercomplex number systems, especially 4-dimensional ones (e. g. quaternions [2, 3]), are applicable for DSP. It has turned out that the use of certain hypercomplex algebras, moreover, increases the degree of a digital system over a complex-valued system [4]. Several applications of hypercomplex numbers are related to (colour) image processing [5, 6], while only few publications cover classic (multirate) DSP applications, such as IIR filters [4] (filter banks [7]).

However, the higher the dimension, the more varieties of hypercomplex algebras exist. As a result, we try to preselect subsets of algebras that allow for efficient DSP. In doing so, the objective is to translate well-known algebra properties from mathematics to the engineering community in a systematic manner by constituting families of 2^N-dimensional algebras. However, computational efficiency on the implementation level (i. e. the number of required real multipliers and adders.) is beyond the scope of this paper, unless providing deeper insight into system analysis.

In this contribution, *i)* (general) properties of hypercomplex algebras are presented and compared with each other (Sec. 2), *ii)* two families favourable for DSP are proposed (Sec. 3) and *iii)* some conclusions are drawn (Sec. 4). The following notation is employed: real number (a), complex number ($\underline{a}$), any hypercomplex number ($\boldsymbol{a}$), matrix with any type of elements ($\mathbf{A}$). Both the set of numbers of an algebra and the algebra itself are denoted by $\mathbb{A}$.

2. HYPERCOMPLEX ALGEBRAS

Define an n-dimensional *hypercomplex algebra* $\mathbb{A}$, expanded on the real vector space

$$\boldsymbol{a} = a_1\boldsymbol{i}_1 + a_2\boldsymbol{i}_2 + \ldots + a_n\boldsymbol{i}_n = \sum_{\nu=1}^{n} a_\nu \boldsymbol{i}_\nu \in \mathbb{A}, \quad a_1,\ldots,a_n \in \mathbb{R}, \tag{1}$$

based on *imaginary vector units* $\boldsymbol{i}_1,\ldots,\boldsymbol{i}_n$, where $\boldsymbol{i}_1 = \mathbf{1}$ represents the vector *identity element*, **and** a *multiplication table* which defines the products of any imaginary unit with each other or with itself. In the following, the notation of the identity vector is abbreviated: $\boldsymbol{i}_1\boldsymbol{i}_\nu = \boldsymbol{i}_\nu\boldsymbol{i}_1 = \boldsymbol{i}_\nu\mathbf{1} = \boldsymbol{i}_\nu$, $\nu = 1,\ldots,n$. Moreover, we restrict every multiplication table entry to $\boldsymbol{i}_\lambda\boldsymbol{i}_\mu = \pm\boldsymbol{i}_\nu$, $\lambda,\mu,\nu = 1,\ldots,n$.

Alternatively, any hypercomplex algebra $\mathbb{A}$ can be represented by a matrix algebra, where the entries of the matrix belong to a *subalgebra* $\mathbb{B} \subset \mathbb{A}$. As an example,

$$\mathscr{F}_{\mathbb{C},\mathbb{R}} : \mathbb{C} \mapsto \mathbb{M}(\mathbb{R},2), \quad \underline{a} = a_1 + a_2\mathrm{i} \mapsto \mathbf{A}_{\mathbb{C},\mathbb{R}} = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} \tag{2}$$

maps a complex number $\underline{a}$ to a real matrix $\mathbf{A}_{\mathbb{C},\mathbb{R}}$. Many properties of an algebra can be observed by analysing the elements of (and the operations with) *isomorphism matrices* $\mathbf{A}$ of a number $\boldsymbol{a}$. Obviously, the matrix representation is redundant.

Two hypercomplex algebras are called *isomorphic* to each other iff their multiplication tables can be made identical by interchanging or linear combination of their base units (imaginaries).

2.1 Properties and operations

Beside addition, which is always performed componentwise, multiplication is an operation commonly requiring n^2 real multiplications according to the algebra's multiplication table. Multiplication is generally *distributive* with respect to addition, but it has to be distinguished between left and right multiplication $\boldsymbol{u}(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{uv}+\boldsymbol{uw}$, $(\boldsymbol{v}+\boldsymbol{w})\boldsymbol{u} = \boldsymbol{vu}+\boldsymbol{wu}$, $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \mathbb{A}$. The most characteristic property of a hypercomplex algebra is whether or not its multiplication is *commutative* $\boldsymbol{uv} = \boldsymbol{vu}$ and/or *associative* $(\boldsymbol{uv})\boldsymbol{w} = \boldsymbol{u}(\boldsymbol{vw})$.

Frequently, not only a general *hypercomplex conjugation* [2]

$$\bar{\boldsymbol{a}} = a_1 - \sum_{\nu=2}^{n} a_\nu \boldsymbol{i}_\nu = a_1 - a_2\boldsymbol{i}_1 - \ldots - a_n\boldsymbol{i}_n \tag{3}$$

is defined, but also modifications of (3), as required for DSP (e. g. in [4]). In order to reduce the variety of different types of conjugations we introduce the *poly-conjugation*

$$\boldsymbol{a}^{\ddagger} = \sum_{\nu=1}^{n} a_\nu \boldsymbol{i}_\nu^3 = a_1 + \sum_{\nu=2}^{n} a_\nu \boldsymbol{i}_\nu^3. \tag{4}$$

Note that common complex conjugation $\{\cdot\}^*$ of a complex number $\underline{z} \in \mathbb{C}$ is consistent with both conjugations (3) and (4), and will be treated as a special case thereof: $\underline{z}^* = \bar{\underline{z}} = \underline{z}^{\ddagger}$. In contrast to the *hypercomplex norm*

$$\|\boldsymbol{a}\| = \sqrt[n]{\det \mathbf{A}_{\mathbb{A},\mathbb{R}}} \in \mathbb{R}, \quad \text{where } \|\boldsymbol{ab}\| = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|, \tag{5}$$

the quadratic identity is *not* generally valid for the modulus $|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2} \in \mathbb{R}$ of a hypercomplex number: $|\boldsymbol{ab}| \neq |\boldsymbol{a}| \cdot |\boldsymbol{b}|$.

If (5) is valid for $\|\boldsymbol{a}\| := |\boldsymbol{a}|$ (as for real and complex numbers), then $\mathbb{A}$ is a division algebra, for which an inverse element $\boldsymbol{a}^{-1} \in \mathbb{A}$ exists for all elements $\boldsymbol{a} \in \mathbb{A}$ except $\boldsymbol{a} = 0$. However, in some hypercomplex algebras the product $\boldsymbol{u} = \boldsymbol{vx}$, $\boldsymbol{u},\boldsymbol{v},\boldsymbol{x} \in \mathbb{A}$, may be zero even if $\boldsymbol{v} \neq 0$ and $\boldsymbol{x} \neq 0$. In these cases, $\boldsymbol{v}$ and $\boldsymbol{x}$ are called *zero divisors* (in matrix representation, $\det \mathbf{V}_{\mathbb{A},\mathbb{R}} = \det \mathbf{X}_{\mathbb{A},\mathbb{R}} = 0$ holds). Since the implementation of a division is hardware-consuming, it is generally circumvented in DSP. Nevertheless, for non-division algebras the interpretation of signal energy becomes involved, since their norm is non-Euclidean: $\|\boldsymbol{a}\| \neq |\boldsymbol{a}|$.

Type	$\boldsymbol{i}_c^2$	(Anti-)Commutativity	Isomorphism
A	-1	$\boldsymbol{i}_{n+\nu} = \boldsymbol{i}_\nu \boldsymbol{i}_c = -\boldsymbol{i}_c \boldsymbol{i}_\nu$	$\begin{pmatrix} \mathbf{A}_1 & -\mathbf{A}_2 \\ \mathbf{A}_2 & \mathbf{A}_1 \end{pmatrix}$
B	1	$\boldsymbol{i}_{n+\nu} = \boldsymbol{i}_\nu \boldsymbol{i}_c = -\boldsymbol{i}_c \boldsymbol{i}_\nu$	$\begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2 & \mathbf{A}_1 \end{pmatrix}$
C	-1	$\boldsymbol{i}_{n+\nu} = \boldsymbol{i}_\nu \boldsymbol{i}_c = \boldsymbol{i}_c \boldsymbol{i}_\nu$	$\begin{pmatrix} \mathbf{A}_1 & -\mathbf{A}_2 \\ \mathbf{A}_2 & \mathbf{A}_1 \end{pmatrix}$
D	1	$\boldsymbol{i}_{n+\nu} = \boldsymbol{i}_\nu \boldsymbol{i}_c = \boldsymbol{i}_c \boldsymbol{i}_\nu$	$\begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2 & \mathbf{A}_1 \end{pmatrix}$

Table 1: Four types of doubling procedures, $\nu = 2, \dots, n$

2.2 Generation of 2^N-dimensional hypercomplex algebras

We apply the *doubling procedure* introduced by CAYLEY and DICKSON [2, 3]:

$$\boldsymbol{u} = \boldsymbol{w}_1 + \boldsymbol{w}_2 \boldsymbol{i}_c = \boldsymbol{w}_1 + \boldsymbol{w}_2 \boldsymbol{i}_{n+1} \in \mathbb{B}, \quad \boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathbb{A}, \tag{6}$$

where $\boldsymbol{i}_c = \boldsymbol{i}_c \boldsymbol{i}_1 = \boldsymbol{i}_{n+1}$ is the additional imaginary unit for doubling the dimension of $\mathbb{A} \subset \mathbb{B}$ from n to $2n$. For instance, in case of $\mathbb{A} = \mathbb{R}$, we obtain $\mathbb{B} = \mathbb{C}$ with $\boldsymbol{i}_c^2 = \boldsymbol{i}_2^2 = \mathrm{i}^2 = -1$.

It is generally assumed for *common* CAYLEY-DICKSON doubling procedure [2] that $\boldsymbol{i}_c$ anticommutes with the existing imaginaries except of the identity element $\boldsymbol{i}_1 = 1$ **and** its square is negative:

$$\boldsymbol{i}_{n+\nu} = \boldsymbol{i}_\nu \boldsymbol{i}_c = -\boldsymbol{i}_c \boldsymbol{i}_\nu, \quad \boldsymbol{i}_c^2 = -1, \quad \nu = 2 \dots n. \tag{7}$$

In this contribution, four modification cases of the basic properties (7) are distinguished by four types A, B, C, D and presented in Tab. 1. For the complex extensions A/C (hyperbolic extensions B/D) we have $\boldsymbol{i}_c^2 = -1$ ($\boldsymbol{i}_c^2 = 1$), whereas the additional imaginary unit $\boldsymbol{i}_c$ either commutes for types C/D or anticommutes for types A/B with the imaginary units $\boldsymbol{i}_\nu$, $\nu = 2, \dots, n$ of $\mathbb{A}$. Also the matrix representations of $\mathbb{B}$ are given in Tab. 1, which show the consistency of isomorphisms with doubling procedure. Type A conforms with the original method by CAYLEY and DICKSON.

2.3 Transfer functions and matrices, paraunitarity condition

In order to generally define hypercomplex transfer functions, the z-Transform is employed componentwise to a signal $\boldsymbol{x}(k)$

$$\boldsymbol{X}(z) = \mathscr{Z}\{\boldsymbol{x}(k)\} = \mathscr{Z}\left\{\sum_{\nu=1}^{n} x_\nu(k) \boldsymbol{i}_\nu\right\} = \sum_{\nu=1}^{n} \sum_{\kappa=-\infty}^{\infty} x_\nu(\kappa) z^{-\kappa} \boldsymbol{i}_\nu,$$

(utilising the linearity property of $\mathscr{Z}\{\cdot\}$). This is not always the best choice, yet sufficient for the requirements of this section. The common form of the convolution theorem

$$\boldsymbol{Y}(z) = \boldsymbol{H}(z)\boldsymbol{X}(z) \leftrightarrow \boldsymbol{y}(k) = \boldsymbol{h}(k) * \boldsymbol{x}(k) \tag{8}$$

is *only valid if* the hypercomplex algebra is commutative [5, 6]. Note that a hypercomplex convolution consists of several real convolutions interconnected according to the multiplication table. Often it is useful to display a hypercomplex transfer function $\boldsymbol{H}(z)$ by its real matrix (MIMO) representation $\mathbf{H}(z)$. If the first column of an algebra's isomorphism matrix $\mathbf{A}_{\mathbb{A},\mathbb{R}}$ has positive entries only (which is always true for doubling procedure type C/D), this matrix directly corresponds to the transfer matrix $\mathbf{H}(z)$. For instance, the transfer matrix for systems consisting of complex numbers reads according to (2): $\mathbf{H}(z) = \begin{pmatrix} H_1(z) & -H_2(z) \\ H_2(z) & H_1(z) \end{pmatrix}$.

The essential DSP *losslessness property* of complex systems $\underline{H}(z)\underline{H}_*(z^{-1}) = 1$ (with *paraconjugation* $\{\cdot\}_*$) can also be stated for the corresponding real MIMO system matrix $\mathbf{H}(z)\mathbf{H}_*^{\mathrm{T}}(z^{-1}) = \mathbf{H}(z)\mathbf{H}^{\dagger}((z^*)^{-1}) = \mathbf{I}$ (with $\{\cdot\}^{\dagger} = \{\cdot\}^{*\mathrm{T}}$): *paraunitarity condition*. Using (4), paraunitarity can likewise be expressed for arbitrary hypercomplex algebras by

$$\mathbf{H}(z)\mathbf{H}_{\ddagger}^{\mathrm{T}}(z^{-1}) = \mathbf{H}(z)\mathbf{H}^{\ddagger\mathrm{T}}((z^*)^{-1}) = \mathbf{I}, \quad \text{or} \quad \boldsymbol{H}(z)\boldsymbol{H}_{\ddagger}(z^{-1}) = 1, \tag{9}$$

respectively (with *poly-paraconjugation* $\{\cdot\}_{\ddagger}$).

	Direct computation		Orthogonal comp.	
$\mathbb{A}$ operation	$\mathbb{R}$ add.	$\mathbb{R}$ mult.	$\mathbb{R}$ add.	$\mathbb{R}$ mult.
addition	n	0	n	0
multiplication	$n(n-1)$	n^2	$n\left(\frac{n}{d}-1\right)$	$\frac{n^2}{d}$

Table 2: Real-valued operations required for basic hypercomplex operations utilising an n-dim. algebra (decomposed to level d)

2.4 Orthogonal decomposition of non-division algebras

Any number of an n-dimensional non-division algebra $\mathbb{A}$ can be decomposed into d *orthogonal components* $\tilde{\boldsymbol{a}}_1, \dots, \tilde{\boldsymbol{a}}_d \in \mathbb{E} \subset \mathbb{A}$

$$\boldsymbol{a} = \sum_{\nu=1}^{n} a_\nu \boldsymbol{i}_\nu = \sum_{\delta=1}^{d} \tilde{\boldsymbol{a}}_\delta \boldsymbol{e}_\delta := [\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2, \dots, \tilde{\boldsymbol{a}}_d], \tag{10}$$

scaling the base vectors $\boldsymbol{e}_1, \dots, \boldsymbol{e}_d$. In general, there are options for both the *decomposition level d* (the number of orthogonal components) and the choice of the orthogonal base, resulting in an associated subalgebra $\mathbb{E}$ of dimension $h = \frac{n}{d}$. However, iff $d = n \Rightarrow h = 1$, the only feasible decomposition is $\mathbb{E} = \mathbb{R} \Rightarrow \tilde{\boldsymbol{a}}_\delta = \tilde{a}_\delta$. The reduction of the computational load depends on d (Tab. 2). Therefore its maximum decomposition level $d_{\max} \le n$ is a significant characteristic of an algebra.

2.4.1 Orthogonal base generation and (de-)orthogonalisation

How to establish a favourable base? In order to gain desirable properties of the decomposition, we call for an *idempotent orthogonal system*. In such a system the base vectors $\boldsymbol{e}_1, \dots, \boldsymbol{e}_d$ meet the following conditions:

$$\begin{aligned} \boldsymbol{e}_\delta \cdot \boldsymbol{e}_\varepsilon &= 0, \quad \delta \neq \varepsilon, \quad \delta, \varepsilon = 1, \dots, d && (11) \\ (\boldsymbol{e}_\delta)^k &= \boldsymbol{e}_\delta, \quad \delta = 1, \dots, d, \quad k \in \mathbb{N} && (12) \\ \sum_{\delta=1}^{d} \boldsymbol{e}_\delta &= \sum_{\delta=1}^{d} \sum_{\nu=1}^{n} e_{\nu\delta} \boldsymbol{i}_\nu = 1 \in \mathbb{R}. && (13) \end{aligned}$$

For any non-division algebra infinitely many, linearly dependent pairs of zero divisors $\alpha \boldsymbol{e}_\delta \cdot \beta \boldsymbol{e}_\varepsilon = \alpha\beta \cdot \boldsymbol{e}_\delta \cdot \boldsymbol{e}_\varepsilon = 0$, $\alpha, \beta \in \mathbb{E}$ fulfil (11). However, as it has been observed, (12) is only met by

$$e_{\nu\delta} = \begin{cases} 0, & \boldsymbol{i}_\nu^2 = -1 \\ \pm\frac{1}{d}, & \boldsymbol{i}_\nu^2 = 1. \end{cases} \tag{14}$$

Thus, in connection with (11), matched pairs of base vectors can be chosen.

Although $\boldsymbol{e}_1, \dots, \boldsymbol{e}_d$ in (10)-(13) are hypercomplex numbers (particularly zero divisors), they can alternatively be expressed by their real vector representation $\mathbf{e}_\delta = \begin{pmatrix} e_{1\delta} & e_{2\delta} & \cdots & e_{n\delta} \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^n$, and combined to a (not necessarily quadratic) *base matrix*

$$\mathbf{E} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_d \end{pmatrix} = \left(\underbrace{\begin{matrix} e_{11} & e_{12} & \cdots & e_{1d} \\ e_{21} & e_{22} & \cdots & e_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \cdots & e_{nd} \end{matrix}}_{\text{base vectors}} \right) \in \mathbb{R}^{n \times d}. \tag{15}$$

In order to change between representations $\boldsymbol{a}$ and $[\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2, \dots, \tilde{\boldsymbol{a}}_d]$ of a hypercomplex number, *orthogonalisation* (determination of orthogonal components) $\tilde{\mathbf{a}} = \mathbf{F}\mathbf{a}$, $\tilde{\mathbf{a}} = \begin{pmatrix} \tilde{\boldsymbol{a}}_1 & \tilde{\boldsymbol{a}}_2 & \cdots & \tilde{\boldsymbol{a}}_d \end{pmatrix}^{\mathrm{T}} \in \mathbb{E}^d$ and *de-orthogonalisation* (restoring common components) $\mathbf{a} = \mathbf{E}\tilde{\mathbf{a}}$, $\mathbf{a} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^n$ procedures are required. To make the matrices $\mathbf{E}$ and $\mathbf{F}$ quadratic, a suitable extension of the orthogonal components to the vector $\tilde{\mathbf{a}} = \begin{pmatrix} \tilde{a}_1 & \tilde{a}_2 & \cdots & \tilde{a}_n \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^n$ is suggested. Hence, the *orthogonalisation matrix* $\mathbf{F}$ is the inverse of the *base matrix*: $\mathbf{F} = \mathbf{E}^{-1}$.

2.4.2 *Calculation in the orthogonal representation*

The orthogonal representation (10) *both i)* simplifies system analysis *and ii)* increases computational efficiency, since each hypercomplex operation $\boldsymbol{f}(\boldsymbol{a})$ can be performed componentwise [8, 9]

$$\boldsymbol{f}(\boldsymbol{a}) = \boldsymbol{f}([\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2, \ldots, \tilde{\boldsymbol{a}}_d]) \equiv [\boldsymbol{f}(\tilde{\boldsymbol{a}}_1), \boldsymbol{f}(\tilde{\boldsymbol{a}}_2), \ldots, \boldsymbol{f}(\tilde{\boldsymbol{a}}_d)]. \quad (16)$$

This applies e. g. for multiplication, yielding a reduction of computational load (*ii*), as given in Tab. 2. Regarding division, (16) states that $\boldsymbol{a}$ is a zero divisor if at least one orthogonal component $\tilde{\boldsymbol{a}}_\delta \equiv 0$ (*i*). Considering a complete hypercomplex LTI system, it requires an orthogonaliser at the input, and a de-orthogonaliser at the output of a processing chain: $\mathbf{H}(z) = \mathbf{E} \cdot \tilde{\mathbf{H}}(z) \cdot \mathbf{F}$.

2.5 Classification of hypercomplex algebra families

An overview interrelating the algebra families to be examined in the following is given in Fig. 1. Obviously, many algebra families coincide at lower n and then branch into different arms.

The most common algebra family are the CAYLEY-DICKSON algebras constructed by N-fold doubling procedure type A, notably the quaternions and octonions [2, 3]. These are the only hypercomplex algebras providing normed division systems up to $n = 8$ [2, 3]. However, they also imply disadvantageous properties: Commutativity vanishes for $n \geq 4$ (quaternions $\mathbb{H}$), associativity for $n \geq 8$ (octonions $\mathbb{O}$), while for $n \geq 16$ (sedenions $\mathbb{S}$) zero divisors appear. From the DSP point of view, division algebras are neither required nor desirable, since they imply $d_{\max} = 1$ (no computational economy) and have no useful extension to the $n = 2^N$ case.

In contrast, CLIFFORD algebras are associative in any dimension [3, 6]. These are generated by multiplication table extensions with imaginary units $\boldsymbol{i}_c^2 = \pm 1$, always not commutative to existing imaginaries (similar to doubling procedure types A/B). Lack of commutativity is a major drawback for DSP applications, since rules and processing become involved, e. g. invalidity of (8).

Therefore we decide to look for algebras which are *associative and commutative* (constructed with doubling type C/D), regardless of zero divisors. In order to maximise decomposition level and following (14), algebras could be generated with doubling type D only, resulting in *hyperbolic numbers* $\mathbb{D}_N$: Sec. 3.1. Furthermore, extending ordinary complex numbers by complex extensions (type C), or mixing complex and hyperbolic numbers (types C/D) is outlined in Sec. 2.5.1, and as suitable specimen, the *tessarines* $\mathbb{CD}_N$ in Sec. 3.2.

2.5.1 *Multicomplex and mixed complex and hyperbolic algebras*

Fig. 1 presents, amongst others, 3 isomorphic algebra families:

1. SCHÜTTE et al. were the first to apply the 4-dim. *bicomplex numbers*, or *Reduced Biquaternions* $\mathbb{C}_2 = \mathbb{C} \otimes \mathbb{C}$ to DSP (doubling pattern CC) [4]. Their extensions to $n = 2^N$ can be called *Multicomplex numbers* $\mathbb{C}_N$ (doubling pattern CC…C) [9].
2. FELSBERG extended *Hypercomplex Commutative Algebras* (HCA) from DAVENPORT to $n = 2^N$, which are generated by a procedure similar to CLIFFORD algebras, yet commutative [6].
3. COCKLE proposed the tessarines $\mathbb{CD} = \mathbb{D} \otimes \mathbb{C}$, a 4-dim. algebra (doubling pattern CD) [10]. In [5] an identical algebra is used for colour image processing. We propose an extension to $n = 2^N$ by applying doubling pattern CDD…D since, this way, the straightforward hyperbolic numbers are available as a backbone.

Next, we compare the squares of the imaginary units for 8-dimensional instances of these 3 algebra families: Tab. 3. All algebras are mutually isomorphic, since they only differ in the imaginary units' ordering. It can be observed that for any algebra generated with *at least one* doubling pattern C and possibly any number of D, there are always $\frac{n}{2}$ imaginary units with $\boldsymbol{i}_\nu^2 = -1$, and $\frac{n}{2}$ imaginary units with $\boldsymbol{i}_\nu^2 = 1$. The latter ones correspond to hyperbolic extensions of type D. For instance, the bicomplex numbers $\mathbb{C}_2 = \mathbb{C} \otimes \mathbb{C}$ are decomposable to the subalgebras $\mathbb{C} = \mathbb{C}(\boldsymbol{i}) \subset \mathbb{C}_2$, $\mathbb{C}(\boldsymbol{j}) \subset \mathbb{C}_2$, $\mathbb{D}(\boldsymbol{k}) \subset \mathbb{C}_2$, although no hyperbolic extensions participated! Due to this fact and in conjunction with (14), for these algebras applies $d_{\max} = 2^{N-1} = \frac{n}{2}$.

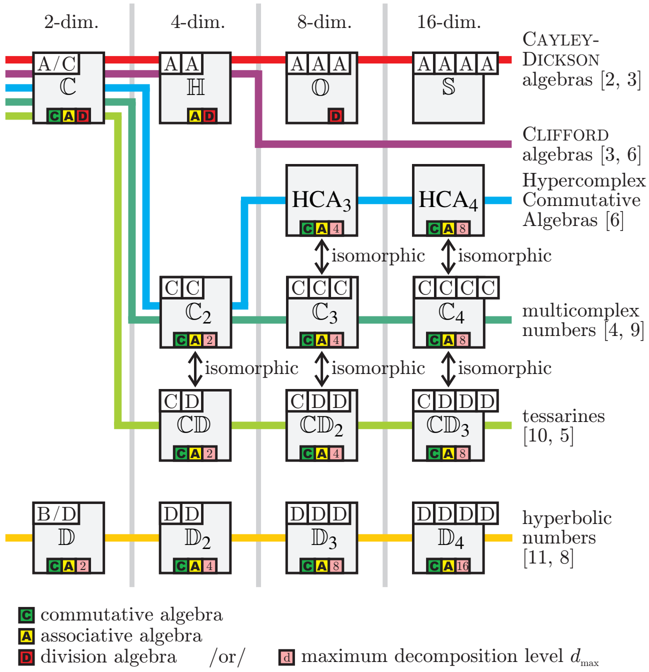

Figure 1: Algebra families, properties and doubling procedure types, if existing (for dimensions $n = 2, 4, 8, 16$)

Algebra	$\boldsymbol{i}_1^2$	$\boldsymbol{i}_2^2$	$\boldsymbol{i}_3^2$	$\boldsymbol{i}_4^2$	$\boldsymbol{i}_5^2$	$\boldsymbol{i}_6^2$	$\boldsymbol{i}_7^2$	$\boldsymbol{i}_8^2$
HCA_3	$+1$	-1	-1	-1	$+1$	$+1$	$+1$	-1
$\mathbb{C}_3$	$+1$	-1	-1	$+1$	-1	$+1$	$+1$	-1
$\mathbb{CD}_2$	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1

Table 3: Squares of imaginary units of the mutually isomorphic algebras HCA_3, tricomplex numbers and 8-dimensional tessarines

3. SUITABLE HYPERCOMPLEX ALGEBRAS

3.1 Hyberbolic numbers

3.1.1 *2-dimensional hyperbolic numbers* $\mathbb{D}$

The algebra of hyperbolic numbers (also called double or split-complex numbers) [11, 8] is derived from real numbers and doubling construction type D (Tab. 1). It features the isomorphism

$$\mathscr{F}_{\mathbb{D},\mathbb{R}} : \mathbb{D} \mapsto \mathbb{M}(\mathbb{R}, 2), \quad \boldsymbol{a} = a_1 + a_2\mathrm{j} \mapsto \mathbf{A}_{\mathbb{D},\mathbb{R}} = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}, \quad (17)$$

where $\boldsymbol{i}_2^2 = \mathrm{j}^2 = 1$ is the *unipotent* imaginary unit [11, 10]. Since it does not possess a complex number structure, complex conjugation $\boldsymbol{a}^*$ is not defined for a hyperbolic number, and poly-conjugation has no effect: $\boldsymbol{a}^\ddagger = \boldsymbol{a}$. The hypercomplex norm of a hyperbolic number differs from its Euclidean modulus:

$$\|\boldsymbol{a}\| = \sqrt{\det \mathbf{A}_{\mathbb{D},\mathbb{R}}} = \sqrt{a_1^2 - a_2^2} \neq |\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2}.$$

The orthogonal representation (10) of a hyperbolic number

$$\boldsymbol{a} = (a_1 + a_2)\frac{1+\mathrm{j}}{2} + (a_1 - a_2)\frac{1-\mathrm{j}}{2} = \tilde{a}_1 \boldsymbol{e}_1 + \tilde{a}_2 \boldsymbol{e}_2 =: [\tilde{a}_1, \tilde{a}_2], \quad (18)$$

$\tilde{a}_1, \tilde{a}_2 \in \mathbb{R}$, meets conditions (11)-(13), for instance:

$$\boldsymbol{e}_1^2 = \frac{1^2 + 2\mathrm{j} + \mathrm{j}^2}{4} = \frac{1+\mathrm{j}}{2} = \boldsymbol{e}_1, \quad \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = \frac{1 - \mathrm{j}^2}{4} = 0, \quad \boldsymbol{e}_1 + \boldsymbol{e}_2 = 1.$$

3.1.2 *2^N-dimensional hyperbolic numbers* $\mathbb{D}_N$

The extension of hyperbolic numbers $\mathbb{D}$ to $n = 2^N$ is straightforward. Applying doubling generation type D (Tab. 1) to the real numbers N times, the commutative and associative 2^N-dimensional hyperbolic algebras [8] emerge:

$$\mathbb{D}_N = \left\{ \boldsymbol{a}_1 + \boldsymbol{a}_2 \boldsymbol{i}_{2^{N-1}+1} \mid \boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{D}_{N-1} \right\} = \underbrace{\mathbb{D} \otimes \mathbb{D} \otimes \ldots \otimes \mathbb{D}}_{N \text{ times}}. \quad (19)$$

As an example, the 4-dimensional hyperbolic algebra $\mathbb{D}_2$ comprises the following isomorphism to the real matrix algebra $\mathbb{M}(\mathbb{R}, 4)$:

$$\boldsymbol{a} \mapsto \mathbf{A}_{\mathbb{D}_2,\mathbb{R}} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_4 & a_3 \\ a_3 & a_4 & a_1 & a_2 \\ a_4 & a_3 & a_2 & a_1 \end{pmatrix} = \mathbf{A}^{\mathrm{T}}_{\mathbb{D}_2,\mathbb{R}}.$$

These non-division algebras are highly decomposable: $d_{\max} = n$. In order to retrieve a feasible base for the orthogonal representation, following construction is applicable:

$$\mathbf{E} = \frac{1}{2^d} \bigotimes_{\delta=1}^{d} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \mathbf{E}^{\mathrm{T}} = \frac{1}{2^d}\mathbf{E}^{-1} = \frac{1}{2^d}\mathbf{F} = \frac{1}{2^d}\mathbf{K}_d. \quad (20)$$

For instance, the base matrix of $\mathbb{D}_2$ is given by

$$\mathbf{E} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = \frac{1}{4}\mathbf{F}. \quad (21)$$

It follows that the orthogonalisation matrix $\mathbf{F} = \mathbf{E}^{-1}$ of decomposition level d coincides with a WALSH-KRONECKER matrix $\mathbf{K}_d$ of order d. Hence the orthogonalisation (de-orthogonalisation) procedure results in a Hadamard Transform (its inverse), which can be carried out by an efficient algorithm: Fast Hadamard Transform (FHT). With such a decomposition, the subalgebra $\mathbb{E}$ of the orthogonal components $\tilde{a}_1, \ldots, \tilde{a}_d$ is an h-dimensional hyperbolic algebra. Furthermore, in the case of maximum decomposition $d = n$, the eigenvectors of $\mathbf{A}_{\mathbb{D}_N,\mathbb{R}}$ are proportional to the orthogonal base vectors $\boldsymbol{e}_\nu = \frac{1}{n}\mathbf{Eig}_\nu \left\{ \mathbf{A}_{\mathbb{D}_N,\mathbb{R}} \right\}$, $\nu = 1, \ldots, n$, while the eigenvalues of $\mathbf{A}_{\mathbb{D}_N,\mathbb{R}}$ are the components used in the orthogonal representation. Hence, such a decomposition resembles matrix diagonalisation. Multiplication complexity is greatly reduced: $n^2 \to n$ real multiplications per hypercomplex multiplication (see Tab. 2).

3.2 Tessarines

3.2.1 4-dimensional tessarines $\mathbb{CD}$

In the following, hyperbolic numbers with complex coefficients (generation type CD, Tab. 1) will be called tessarines according to the 4-dimensional algebra introduced by J. COCKLE in 1848 [10]:

$$\boldsymbol{a} = \underline{a}_1 + \underline{a}_2 \boldsymbol{j} \in \mathbb{CD}, \quad \boldsymbol{j} \neq \pm 1, \quad \boldsymbol{j}^2 = 1, \quad \underline{a}_1, \underline{a}_2 \in \mathbb{C}.$$

Combining (2), (17) and following instructions from Tab. 1 (types C and D), its isomorphism to the real matrix algebra $\mathbb{M}(\mathbb{R}, 4)$ yields

$$\boldsymbol{a} = a_1\boldsymbol{i}_1 + a_2\boldsymbol{i}_2 + a_3\boldsymbol{i}_3 + a_4\boldsymbol{i}_4 \mapsto \mathbf{A}_{\mathbb{CD},\mathbb{R}} = \begin{pmatrix} a_1 & -a_2 & a_3 & -a_4 \\ a_2 & a_1 & a_4 & a_3 \\ a_3 & -a_4 & a_1 & -a_2 \\ a_4 & a_3 & a_2 & a_1 \end{pmatrix}.$$

Thus, the tessarine multiplication table is

$$\boldsymbol{i}^2 = \boldsymbol{i}_2^2 = -1, \quad \boldsymbol{j}^2 = \boldsymbol{i}_3^2 = 1, \quad \boldsymbol{k}^2 = \boldsymbol{i}_4^2 = -1,$$
$$\boldsymbol{ij} = \boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{ik} = \boldsymbol{ki} = -\boldsymbol{j}, \quad \boldsymbol{jk} = \boldsymbol{kj} = \boldsymbol{i}.$$

If complex conjugation of both the complex coefficients $\underline{a}_1, \underline{a}_2$ is merged to $\boldsymbol{a}^*$, it coincides with the poly-conjugation (4): $\boldsymbol{a}^{\ddagger} = \boldsymbol{a}^*$. The orthogonal representation of a 4-dimensional tessarine is based on the 2-dimensional hyperbolic number's decomposition (18):

$$\boldsymbol{a} = (\underline{a}_1 + \underline{a}_2)\frac{1+\boldsymbol{j}}{2} + (\underline{a}_1 - \underline{a}_2)\frac{1-\boldsymbol{j}}{2} = \underline{\tilde{a}}_1\boldsymbol{e}_1 + \underline{\tilde{a}}_2\boldsymbol{e}_2 =: [\underline{\tilde{a}}_1, \underline{\tilde{a}}_2].$$

3.2.2 2^N-dimensional tessarines $\mathbb{CD}_N$

We propose an extension of the tessarines to dimension $n = 2^N$, resulting in hyperbolic numbers (19) of dimension $\frac{n}{2} = 2^{N-1}$ with complex coefficients: doubling pattern CDD...D. The squares of imaginaries follow the simple rule $\boldsymbol{i}_\nu^2 = (-1)^{\nu-1}$, $\nu = 1, \ldots, n$. Due to this fact, poly-conjugation reduces to: $\boldsymbol{a}^{\ddagger} = \sum_{\nu=1}^{n} (-1)^{\nu-1} a_\nu \boldsymbol{i}_\nu = \boldsymbol{a}^*$. Hence, the hypercomplex paraunitarity condition (9) also reduces to the common formulae $\mathbf{H}(z)\mathbf{H}_*^{\mathrm{T}}(z^{-1}) = \mathbf{I}$ and $\boldsymbol{H}(z)\boldsymbol{H}_*(z^{-1}) = 1$, respectively.

The base matrices for orthogonal representation resemble (20), considering that for a tessarine $d_{\max} = \frac{n}{2}$. For instance, (21) can be employed as a complex base matrix for the 8-dimensional tessarine algebra $\mathbb{CD}_2$. In these cases, the eigenvalues of the real isomorphism matrix $\mathbf{A}_{\mathbb{CD}_{N-1},\mathbb{R}}$ turn up in complex conjugated pairs.

4. CONCLUSION

In contrast to the widely used quaternions and CLIFFORD algebras, we consider algebra families, accomplishing both the commutativity and associativity property for any dimension 2^N, most favourable for DSP application. However, these algebras generally exhibit zero divisors for $n \geq 4$ (which is also the case for any noncommutative, but associative algebra with $n \geq 8$). This feature can be a great advantage, since it allows for orthogonal decomposition, resulting in both improved computational efficiency and system analysis. As suitable specimen, 2^N-dimensional hyperbolic and tessarine algebras are presented. The major drawback of these non-division algebras regarding DSP is their non-Euclidean norm complicating the definition of signal energy, which will be an issue of further investigation. Nevertheless, a paraunitarity condition for hypercomplex digital systems ensuring losslessness is provided.

The lacking examples related to DSP applications will be provided in an extended forthcoming publication. Finally, I greatly acknowledge the careful review by Prof. H. G. Göckler (DISPO).

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