14th European Signal Processing Conference (EUSIPCO 2006), Florence, Italy, September 4-8, 2006, copyright by EURASIP

DETECTION OF UNKNOWN SIGNALS BASED ON SPECTRAL CORRELATION MEASUREMENTS

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ABSTRACT

The problem of detecting an unknown signal embedded in white Gaussian noise is addressed. A CFAR detector based on combining the information of the whole frequency-cyclefrequency plane is proposed. An analytical characterization of the detector is provided, and its detection capability evaluated. The FFT-Accumulation Method (FAM) is used to measure the SCF. Approximate analytic expressions of the probability of false alarm are provided.

1. INTRODUCTION

The radiometer is commonly considered as the most appropriate solution to the unknown-signal detection problem, when the signal is stationary and the noise is white [1]. However, most of man-made signals obey an (almost) cyclostationary model rather than a stationary one [2]. This means that the statistics of the signal are (almost) periodic functions of time or, in other words, that all man-made signals exhibit periodicities such as the carrier frequency, the bit rate, the pulse repetition interval, etc. [3]. The objective of this work is to exploit the (second order) cyclostationary properties of man-made signals in order to develop detection schemes outperforming the radiometric approach. Moreover, an approximate analytical expression for the probability of false alarm is provided, and its accuracy discussed.

The cyclostationary approach to the signal detection problem has been formally addressed in [4]. The spectral correlation function (SCF), also known as cyclic spectral density function [2], represents the correlation between two spectral signal components at frequencies $f+\alpha/2$ and $f-\alpha/2$ [3]. f, which is called the frequency, is the mean frequency of the two components, and α, the cyclefrequency, is their frequency separation. In [4], the SCF of the signal is known except for the signal phase. The knowledge of the signal SCF allows the optimum coherent integration of the SCF along the frequency dimension, for each signal cyclic spectral component. Moreover, the values of cyclefrequency where the signal is present are also known and therefore, the SCF is computed only for these values simplifying the estimation process.

However, the work presented herein deals with the signal detection problem when no assumptions are made about the signal. This produces some differences with the work mentioned previously: First, all the frequency-cyclefrequency plane must be computed, since the cyclefrequencies exhibiting signal components are unknown. This suggests the use of efficient algorithms for measuring the SCF. Second, it is impossible to integrate coherently the signal SCF along the frequency dimension without some previous knowledge about the SCF. Thus, the detection statistics must process the information incoherently.

In the following, we develop two detection statistics based on the spectral correlation measurement (Section 2). These statistics result from decoupling the information acquired through the SCF: That concerning to the conventional power spectral density (PSD), i.e. the SCF for null cyclefrequency, from that concerning to the rest of the SCF. We shall see that this decoupling is adequate because of the different statistical properties of the SCF within these two regions. In Section 3 we study the statistical properties of these statistics and provide an approximate analytical expression for their probability of false alarm (P_{FA}). The accuracy and range of application are discussed and supported with simulation results. Finally, a new detector based on combining these two statistics for improving radiometric detection is presented in Section 5. Also, the performance results provided shall show the advantages of the cyclostationary approach.

2. DESCRIPTION OF THE DETECTION STATISTICS

Let $\mathbf{x}$ be the sequence of collected data, the detection problem can be formulated through the hypothesis test:

$$\begin{aligned} H_0 &: \mathbf{x} = \mathbf{n} \\ H_1 &: \mathbf{x} = \mathbf{s} + \mathbf{n} \end{aligned} \tag{1}$$

where $\mathbf{s}$ is the unknown signal to be detected and $\mathbf{n}$ is an additive white noise. Hereinafter, the noise will be Gaussian, with zero mean and unknown variance $\sigma_{\mathbf{n}}^2$.

For solving the previous test, we will use an approach based on measuring the SCF of the collected data. Then, two statistics (γ_1 and γ_2) are extracted from the measured data and used to detect the signal. The SCF represents the correlation between two spectral signal components at frequencies $f_1 = f+\alpha/2$ and $f_2 = f-\alpha/2$ [3] where f, the *frequency*, is the mean of f_1 and f_2 and α, the *cyclefrequency*, is their frequency separation $\alpha = f_1 - f_2$.

Since we have made no assumptions about the signal, we cannot know a priori either the spectral or the cyclic spectral components of the signal. Therefore, the SCF must be measured throughout the frequency-cyclefrequency plane. Then, after appropriate normalizations of the SCF for the two different regions in which it is divided, we calculate and analyze two disjoint statistics which will be combined after for a global detector.

This work has been supported by the National Grant Program FPU and the project TEC2005-07010-C02-01/TCM.

The SCF information on the whole frequency-cyclefrequency plane has been decoupled in order to analyze the problem. The key reason for this decoupling is the different statistical properties, under hypothesis H_0, of the SCF within two regions: $\alpha = 0$ and $\alpha \neq 0$. In the first case, the SCF is always real and positive since it represents the autocorrelation of the signal spectral components, i.e. for $\alpha = 0$ (the frequency axis) the expected SCF matches the conventional power spectral density (PSD) of the signal. On the contrary, for $\alpha \neq 0$, the SCF is not zero (as ideal), but complex valued with zero mean. The reason is that, in practice, the SCF cannot be known, only estimated (what is a noise-dependent operation) [5].

On the other hand, measuring the SCF is computationally very expensive and then, an efficient algorithm should be used. We have chosen the FFT Accumulation Method (FAM) [6]. This algorithm is more efficient than those based on frequency smoothing [6]. Besides, it provides SCF estimates distributed over a rectangular grid, an important feature for detection purposes since many signals exhibit their SCF maxima along straight lines of constant frequency [7]. Nevertheless, the use of FAM introduces non-constant power of the measurement noise, which decreases as the reliability product (defined in the next section) increases [8]. Thus, a normalization procedure is required due to the fluctuation of the measurement noise power.

Then, let γ_1 be the detection statistic concerning the PSD region ($\alpha = 0$):

$$\gamma_1 = \max_f \left[\widetilde{S}^0_{\mathbf{x}}(f)\right] \tag{2}$$

where $\widetilde{S}^\alpha_{\mathbf{x}}(f)$ is the SCF measurement conveniently (mean and variance) normalized with the aim to obtain a set of identically distributed RV. For $\alpha = 0$, we obtain:

$$\widetilde{S}^0_{\mathbf{x}}(f) = \frac{\widehat{S}^0_{\mathbf{x}}(f) - \widehat{\sigma}^2_{\mathbf{x}}}{K(f)\widehat{\sigma}^2_{\mathbf{x}}} \tag{3}$$

where $\widehat{S}^\alpha_{\mathbf{x}}(f)$ is the SCF measured through FAM. $\widehat{\sigma}^2_{\mathbf{x}}$ is the mean of $\widehat{S}^0_{\mathbf{x}}(f)$ and the noise power MLE under the H_0 hypothesis:

$$\widehat{\sigma}^2_{\mathbf{x}} = \frac{\|\mathbf{x}\|^2}{N} \tag{4}$$

where N is the length of the collected data. In (3), $K(f)$ is a function which compensates the variance fluctuation due to FAM [8]:

$$\begin{aligned} K(f) &= \sqrt{1 + \frac{\sum_n |h(n;f)|^2}{\sum_n [h(n;0)]^2}} \\ h(n;f) &= \sum_m a(m)a(m+nL)e^{-j4\pi fm} \end{aligned} \tag{5}$$

where, in the FAM implementation, $a(n)$ is the window of the first FFT and L is the decimation factor [6].

The inclusion of $\widehat{\sigma}^2_{\mathbf{x}}$ in the denominator of (3) makes γ_1 invariant to scales and therefore, γ_1 exhibits CFAR properties with respect to the input noise power.

Now, let γ_2 be the detection statistic involving the rest of the frequency-cyclefrequency plane, i.e. $\alpha \neq 0$:

$$\gamma_2 = \max_{f,\alpha\neq 0} \left|\widetilde{S}^\alpha_{\mathbf{x}}(f)\right|^2 \tag{6}$$

In this region, the mean of the SCF is zero under H_0 and thus, only the variance fluctuations are compensated in the normalization:

$$\widetilde{S}^\alpha_{\mathbf{x}}(f) = \frac{\widehat{S}^\alpha_{\mathbf{x}}(f)}{K(f,\alpha)\widehat{\sigma}^2_{\mathbf{x}}} \qquad ,\alpha \neq 0 \tag{7}$$

where $K(f,\alpha)$ is the function compensating the variance fluctuations due to FAM, which also depends on α [8]:

$$\begin{aligned} K(f,\alpha) &= \frac{1}{2}\sqrt{\frac{1}{P}\mathbb{R}e\left[H_*\left(j\frac{2\pi q(\alpha)}{P};f\right)\right]} \\ h_*(n;f) &= [h(n;0)]^2 + |h(n;f)|^2 \\ q(\alpha) &= \mathrm{mod}\left(\alpha\Delta t, [\Delta f]^{-1}\right) \\ P &= \frac{\Delta t}{L} \end{aligned} \tag{8}$$

where $H_*(j\omega;f)$ is the Fourier Transform of $h_*(n;f)$, Δf is the frequency resolution of FAM, Δt is the inverse of the cyclefrequency resolution (approximately the input length N), P is the length of the second FFT in FAM, and q represents the cyclefrequency offset of the estimate within each channel-pair region. For a better understanding of these FAM parameters, the reader is encouraged to visit [6]. Similarly to γ_1, the detection statistic γ_2 is invariant to scales and then presents CFAR properties too.

3. ANALYTIC P_{FA} AND VALIDITY OF THE APPROXIMATION

In this section we will focus on finding an approximate P_{FA} analytical expression for detection statistics γ_1 and γ_2. The discussion on the best way of combining γ_1 and γ_2 will be deferred until Section 4.

Let Γ_1 be the set of identically distributed RV maximized by γ_1, i.e. the set of all SCF measures for $\alpha = 0$, $\widetilde{S}^0_{\mathbf{x}}(f)$. By using the central limit theorem (CLT) for m-dependent variables [9], each of the RV in Γ_1 can be approximated as Gaussian with zero mean and variance $\sigma^2_{\Gamma_1}$. The accuracy of the Gaussian approximation mainly depends on the so-called reliability product $\Delta f \Delta t$, which represents a quality measure of the SCF estimate [5]. In fact, the higher this product is, the more accurate the Gaussian approximation is.

Hereinafter, let us assume the RV in Γ_1 to be Gaussian. The distribution of γ_1 is then the distribution of the maximum of a set of *correlated* Gaussian RV. Indeed, in order to avoid estimation gaps in the frequency-cyclefrequency plane (when FAM is used), they cannot be independent [6]. Nevertheless, assuming the first FFT window exhibits low sidelobes, the approximate P_{FA} curve can be obtained by continuing as if they were independent:

$$\begin{aligned} P_{FA\,\gamma_1}(Th) &= 1 - \left[Q\left(\frac{Th}{\sigma_{\Gamma_1}}\right)\right]^{K_1} \\ \sigma^2_{\Gamma_1} &= \frac{1}{P}\sum_n [h(n;0)]^2 \\ K_1 &= \frac{1}{2\Delta f} + 1 \end{aligned} \tag{9}$$

where Th is the detection threshold, $Q(z)$ is the cumulative distribution function of a normal RV $N(0,1)$, and K_1 is the cardinal of the set Γ_1, i.e the number of frequency channels in the range $f = 0$ to 0.5.

Now, let us focus on the detection statistic γ_2. In this case, and making use of the CLT for m-dependent variables again, it can be seen that each of the RV maximized in (6) exhibit an exponential distribution [8]. Analogously to results for statistic γ_1, the exactitude of the exponential assumption is intimately related to $\Delta f \Delta t$. Then, the asymptotic distribution of the extreme value of a set of exponential RV has been studied in [10]. Using the results therein, the $\mathrm{P_{FA}}$ of the detection statistic γ_2 can be approximated by the analytic expression:

$$\mathrm{P_{FA}}_{\gamma_2}(Th) = 1 - \exp(-\exp(-(Th - \log(K_2)))) \\ K_2 = \left[\left(\frac{2}{\Delta f} + 1 \right)^2 - 2 \right] \frac{\Delta t}{\Delta f} - 1 \tag{10}$$

where K_2 is the cardinal of the set maximized by γ_2.

In Figure 1, the simulated (100,000 trials, thicker trace) and analytical approximate $\mathrm{P_{FA}}$ (thinner trace) for both detection statistics γ_1 (on the left) and γ_2 (on the right) are plotted. For each statistic, three cases are represented: constant reliability product, constant frequency resolution and constant data length.

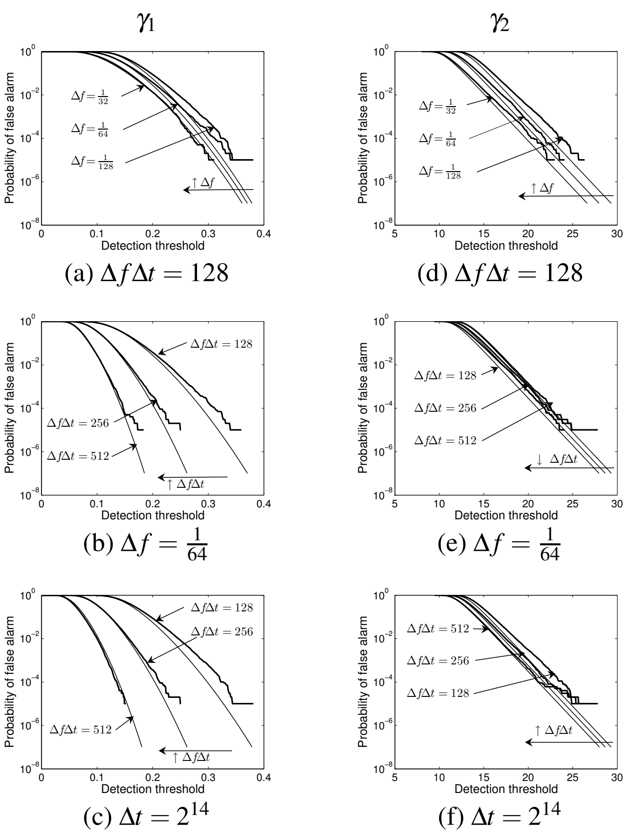


Figure 1: $\mathrm{P_{FA}}$ for detection statistics γ_1 and γ_2. The analytical approximation is more accurate as $\Delta f \Delta t$ increases.

A general result is that increasing the reliability product $\Delta f \Delta t$ turns into more accuracy of analytical $\mathrm{P_{FA}}$ (Note that a high reliability product is advisable since increasing the reliability product also reduces the measurement noise.) However, the approximations made for both statistics are no longer accurate for small values of $\Delta f \Delta t$. On the other hand, for a constant $\Delta f \Delta t$, the higher Δf is, the more precise analytical $\mathrm{P_{FA}}$. As a result, in order to preserve the accuracy, a decrease in Δf (better frequency resolution) requires a proportionally greater increase in Δt (the collected data length).

4. THE DETECTOR

Then, both statistics are combined in order to obtain a single detector. The best solution is to find out the optimum detector in the Neyman-Pearson sense. The optimum detector results from applying the GLRT for statistics γ_1 and γ_2:

$$\frac{f_{H_1}(\gamma_1, \gamma_2)}{f_{H_0}(\gamma_1, \gamma_2)} \underset{H_0}{\overset{H_1}{\gtrless}} Th \tag{11}$$

where $f_{H_1}(\gamma_1, \gamma_2)$ and $f_{H_0}(\gamma_1, \gamma_2)$ are the joint pdf of γ_1 and γ_2 under hypothesis H_1 and H_0, respectively. Th is the detection threshold, which is set to attain the desired $\mathrm{P_{FA}}$. Unfortunately, this detector is hard to implement in practice since $f_{H_1}(\gamma_1, \gamma_2)$ is highly variable with the SNR and many signal features such as its modulation, bandwidth, etc.

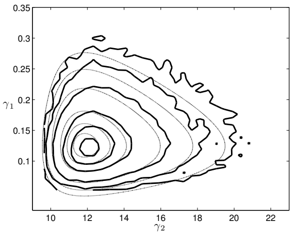
For this reason, we use a simple detector consisting of the logical OR of the two single detectors which use, separately, statistics γ_1 and γ_2, that is, a detection is declared if any of the single detectors declare a detection. The motivation of this detector makes sense after checking the results shown in the next section, where it can be appreciated that in some cases the best detection probability is achieved with the single detector employing γ_1, and in other cases the single detector using γ_2 is better. Thus, the OR detector shall approach the best of them. Moreover, the probability of false alarm of the OR detector can be expressed analytically by using (9) and (10) in the following equation:

$$\mathrm{P_{FA}}_{OR}(Th_1, Th_2) = \mathrm{P_{FA}}_{\gamma_1}(Th_1) + \mathrm{P_{FA}}_{\gamma_2}(Th_2) \\ - \mathrm{P_{FA}}_{\gamma_1}(Th_1)\mathrm{P_{FA}}_{\gamma_2}(Th_2) \tag{12}$$

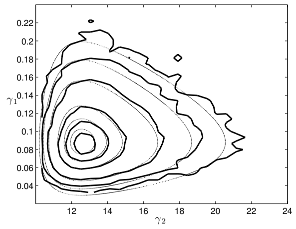
where Th_1 and Th_2 are the detection thresholds of the single detectors. Furthermore, we set $\mathrm{P_{FA}}_{\gamma_1}(Th_1) = \mathrm{P_{FA}}_{\gamma_2}(Th_2)$ and, assuming they are low, the last term in (12) can be obviated. Then, Th_1 and Th_2 are set to achieve in each single detector, respectively, half the probability of false alarm of the OR detector. On the other hand, more complex (and ad hoc) detection schemes than the OR detector have been used too, with no significant improve in the detection.

Finally, (12) is valid only if statistics γ_1 and γ_2 are independent from each other, which we discuss in the following. The results obtained by simulation are in concordance with the independence assertion, and although it does not represent a formal justification, it is useful to provide an idea of the validity of the independence supposition. We can see in Figure 2 the contour map of the simulated joint pdf (thicker trace) and the analytical joint pdf (thinner trace) obtained by multiplying the marginal pdf, which can be easily obtained from (9) and (10). It can be seen that as the reliability product increases, the simulated curves fit to the analytical curves better.

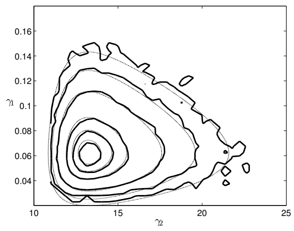
Additionally, an independence chi-square test [11] has been applied to each of the cases represented. As a figure of merit, it can be used the maximum significance level for which the test declares that statistics γ_1 and γ_2 are independent. The significance level represents the probability that



(a) $\Delta f \Delta t = 128$



(b) $\Delta f \Delta t = 256$



(c) $\Delta f \Delta t = 512$

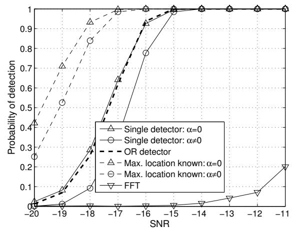
Figure 2: Joint pdf of detection statistics γ_1 and γ_2. As $\Delta f \Delta t$ increases, the joint pdf (thick trace) approaches the product of marginal pdf's (thin trace).

γ_1 and γ_2 are declared dependent, given that they are independent. Thus, the higher the significance level is, the likelier the independence assumption is [11]. The results show that statistics γ_1 and γ_2 are independent with a significance level 0.09 for $\Delta f \Delta t = 128$, 0.77 for $\Delta f \Delta t = 256$ and 0.73 for $\Delta f \Delta t = 512$. Thus, the independence assumption of (12) is less clear in the first case, although a commonly used value of the significance level is 0.05 [11].

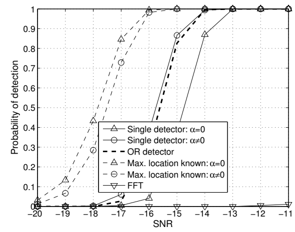
5. RESULTS

In Figure 3, it is plotted the probability of detection (P_D) resulting for the three detectors described: That using only statistic γ_1 (PSD region: $\alpha = 0$); that using only statistic γ_2 ($\alpha \neq 0$); and the third one, proposed herein, which combines both statistics by the OR scheme. The examples include three different signal modulations, BPSK, QPSK and MSK, all with a symbol duration $Tb = 32$ samples, and the detection threshold has been fixed to attain, in each case, $P_{FA} = 10^{-3}$ (on the left) and $P_{FA} = 10^{-6}$ (on the right). The results shown have been computed for a reliability product $\Delta f \Delta t = 256$, frequency resolution $\Delta f = 1/64$ (and therefore, $\Delta t = 2^{14}$), and decimation factor $L = 8$ (and thus, $P = \Delta t / L = 2^{11}$). It is noteworthy that the single detector based on statistic γ_1 is equivalent to a channelized energy detector and can be used as a reference too. This detector is the result of applying a bank of filters with bandwidth Δf, and then, in each branch, an amplitude-squaring device followed by a non-coherent integrator. Then, the maximum of all these outputs normalized according to (3), i.e. γ_1, is compared with a detection threshold. In addition, just for comparison purposes, the plots also show the probability of detection of three more detectors. The first one is an FFT-based detector, which compares the squared magnitude of the outputs of the FFT with a threshold and assumes that the noise power is known. The other two detectors are detectors which employ statistics γ_1 and γ_2, respectively, but supposing that the locations of the SCF maximum amplitude in both regions are known, and therefore, the detection is always made by using these single points of the SCF. These last two detectors represent an upper P_D bound for the single detectors in the unknown maximum location case.

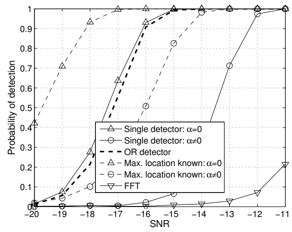
$P_{FA} = 10^{-3}$ $\qquad$ $P_{FA} = 10^{-6}$



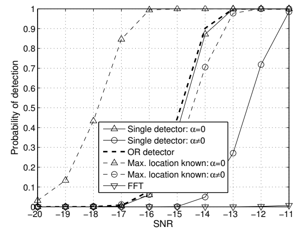
(a) BPSK signal



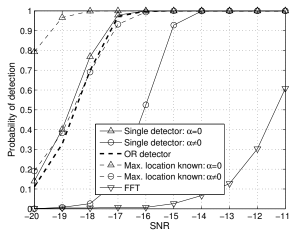
(d) BPSK signal



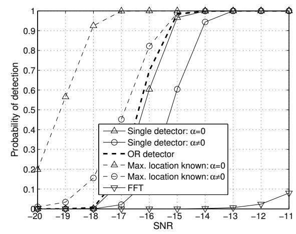
(b) QPSK signal



(e) QPSK signal



(c) MSK signal



(f) MSK signal

Figure 3: P_D for different signal modulations and P_{FA}. The OR detector is as good as the best of both single detectors. *Maximum-location-known* detectors need some signal knowledge (just used as reference).

In the figures, it can be appreciated that the OR detector practically equals the best of both single detectors is all cases. Furthermore, the developed detector presents a sensitivity (input SNR required to attain objective P_D and P_{FA}) improvement of near 1 dB compared to the channelized energy detector (that based only on γ_1), in the best case. The FFT-based detector fails to detect these signals due to their wide bandwidth. Moreover, the improvement when the maximum location is known is, for the single detector based on γ_1, 2 dB at $P_{FA} = 10^{-3}$ and 3 dB at $P_{FA} = 10^{-6}$. For the single detector based on γ_2, the improvement when the location of the maximum is known is about 2 dB for both values of P_{FA}. For clarity, the sensitivities at $P_D = 90\%$ are shown in Table 1.

Other important result is that, when decreasing the P_{FA}, the detector based on γ_2 requires a smaller increase in SNR than the detector based on γ_1 in order to preserve their P_D. The reason is their different statistics which result in an intersection of the ROC (receiver operating characteristic) curves of both detectors. Thus, for a high P_{FA} the detector based on γ_1 is better than that based on γ_2, but as the P_{FA} decreases,

$P_{FA} = 10^{-3}$			
Signal Modulation	BPSK	QPSK	MSK
OR	-16.1	-16.0	-17.3
CED[a] ($\alpha = 0, \gamma_1$)	-16.1	-16.1	-17.4
$\alpha \neq 0, \gamma_2$	-15.5	-12.3	-15.1
Known max. location, $\alpha = 0$	-18.1	-18.1	-19.4
Known max. location, $\alpha \neq 0$	-17.6	-14.4	-17.1
Matched filter	-28.0	-28.0	-28.0
$P_{FA} = 10^{-6}$			
Signal Modulation	BPSK	QPSK	MSK
OR	-14.5	-14.0	-15.3
CED[a] ($\alpha = 0, \gamma_1$)	-13.7	-13.8	-15.2
$\alpha \neq 0, \gamma_2$	-14.7	-11.3	-14.1
Known max. location, $\alpha = 0$	-16.6	-16.6	-18.1
Known max. location, $\alpha \neq 0$	-16.3	-13.3	-15.6
Matched filter	-26.0	-26.0	-26.0

[a] Channelized Energy Detector, $\Delta f = 1/64$

Table 1: Sensitivities (dB) at $P_D = 90\%$ for the detectors and probabilities of false alarm shown in Figure 3.

the detector based on γ_2 gets better. A similar result was also obtained in [4] for the known signal case.

6. SUMMARY

In this paper, we have used a cyclostationary approach to the problem of detecting an unknown signal embedded in white Gaussian noise. The proposed detection scheme first decouples, and after recombines, the information of the spectral correlation function, one concerning $\alpha = 0$ (the PSD) and the other concerning $\alpha \neq 0$. This scheme results in the OR detector described above. Since the SCF should be computed throughout the frequency-cyclefrequency plane, an efficient algorithm (FAM) is used for measuring the SCF. Besides, the OR detector exhibits CFAR properties and equals the best of the single detectors, improving the channelized energy detector sensitivity up to 1 dB. This improvement is better when the P_{FA} decreases, due to an intersection of the ROC curves, and when the signal exhibits a high maximum of the SCF amplitude for $\alpha \neq 0$. Of course the cyclostationary approach only has sense when the signals we want to detect present a SCF with a moderate maximum amplitude for $\alpha \neq 0$. Otherwise, a conventional radiometer will provide similar detection performance. An interesting future work could be the extension of the results herein to higher order cyclostationarity, in order to improve the sensitivity for signals with a low level of second-order cyclostationarity. On the other hand, the statistical analysis of the detector leads to an approximate analytic expression for the P_{FA}. The accuracy of the formula depends on the reliability product $\Delta f \Delta t$, which also represents a quality measure of the SCF estimator.

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