OPTIMAL ESTIMATION OF A RANDOM SIGNAL FROM PARTIALLY MISSED DATA

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ABSTRACT

We provide a new technique for random signal estimation under the constraints that the data is corrupted by random noise and moreover, some data may be missed. We utilize nonlinear filters defined by multi-linear operators of degree r, the choice of which allows a trade-off between the accuracy of the optimal filter and the complexity of the corresponding calculations. A rigorous error analysis is presented.

1. INTRODUCTION

The tasks considered in this paper are motivated by restrictions arising in real applications. They mainly concern with a practical way for gathering data. In many practical cases, to estimate a component \( s_k \) of the reference signal \( x = (x_1, \ldots, x_m)^T \), a filter \( \mathcal{A} \) uses (or "remembers") no more than the past \( v_k \) most recent components \( y_{x_1}, \ldots, y_{x_k} \) from the measurement data \( y = (y_1, \ldots, y_m)^T \), where \( s_k \) and \( v_k \) are respectively defined by

\[
s_k = v_k - p_k + 1 \quad \text{and} \quad v_k = 1, \ldots, k.
\]

We say that such a filter \( \mathcal{A} \) has arbitrarily variable incomplete memory \( p = \{p_1, \ldots, p_m\} \).

Such a restriction makes the problem of finding the best \( \mathcal{A} \) quite specific. This is, perhaps, a reason that despite a long history of the subject [3], even for a simplest structure of the filter \( \mathcal{A} \) when \( \mathcal{A} \) is defined by a matrix, the problem of determining the best \( \mathcal{A} \) has only been solved under the hard assumption of positive definiteness of an associated covariance matrix (see [3, 4, 5]) and for the case of complete memory only. We avoid such bottlenecks and solve the problem in the general case of the polynomial filter with the arbitrarily variable incomplete memory \( p \). The proposed technique is substantially different from those considered, for the case of linear filters, in [3, 4, 5] and for the case of non-linear regression, in [6].

A distinguishing feature of our solution is that the filter should be non-linear and causal with incomplete memory. The simplest particular case of our desired filter is an optimal linear filter defined by a lower \( p \)-band matrix. There is no published solution even for this simplest case but Fomin and Ruzhansky [4, 5] have recently proposed an optimal causal linear filter where no restriction is imposed on the memory. In effect their filter has unlimited memory.

The novelty of our approach derives from our formulation of a new class of filters, the new reduction technique to determine an optimal representative and a rigorous error analysis. The main results are Proposition 1 and Theorems 1 and 2.

1.1 Computational impact

Many known results assume the relevant covariance matrices are invertible. Such an assumption is quite restrictive from a computational point of view and is often associated with numerical instability. Our formula for the optimal filters can always be computed. In addition the reduction procedure (Section 4) means that our optimal filters are defined by matrices of reduced size. Consequently the computational load should compare favorably with known methods. On the other hand we use non-linear filters to provide improved accuracy and it is natural to expect additional computation in problems where increased accuracy is desired.

2. NONLINEAR CAUSAL FILTERS WITH INCOMPLETE MEMORY

The vector \( y \) is interpreted as observable data containing information about the reference signal \( x \), but may be contaminated by noise. No specific relationship between the reference signal and noise is assumed. Using an heuristic idea of causality, we expect that the present value of the estimate is not affected by future values of the data. Since the filters under consideration must have incomplete memory \( p = \{p_1, \ldots, p_m\} \) the estimate of \( x_k \) must be obtained from the data components \( y_{s_k}, \ldots, y_{s_k} \). For each \( k = 1, 2, \ldots, m \) and random \( m \)-vector \( y \), we define \( \tau_k \) by

\[
\tau_k(y) = (y_{s_k}, y_{s_k+1}, \ldots, y_{s_k}).
\]

Definition 1 An \( m \)-vector operator \( \mathcal{A} \) acting on \( y \) is called a causal filter with incomplete memory \( p = \{p_1, \ldots, p_m\} \) if \( \mathcal{A}(y) \) takes the form

\[
\mathcal{A}(y) = \begin{bmatrix} \mathcal{A}_1(y) \\ \vdots \\ \mathcal{A}_m(y) \end{bmatrix}.
\]

For a brevity, we often omit the term “causal” for such a filter.

For an appropriate choices of \( \mathcal{A} \), the vector \( \mathcal{A}(y) \) provides an estimate of \( x \).

A simple example in which \( \mathcal{A}(y) \) is a first-order polynomial is

\[
\mathcal{A}(y) = a + B_1 y
\]

with a an \( m \)-vector and \( B_1 \) an \( m \times m \) matrix. By the definition, the filter \( \mathcal{A} \) is causal with incomplete memory \( p = \{p_1, \ldots, p_m\} \) if the matrix \( B = \{b_{kj}\} \) is such that

\[
b_{kj} = 0 \quad \text{for } j = \{1, \ldots, s_k - 1\} \cup \{v_k + 1, \ldots, m\}.
\]
We call $B_1$ a lower $p$-band matrix. The set of lower $p$-band matrices in $\mathbb{R}^{m \times m}$ is denoted by $B_{p1,\ldots,p_m}$.

For instance, if

\[
\begin{align*}
m &= 4, \\
p_1 &= 1, \\
p_2 &= 2, \\
p_3 &= 3, \\
p_4 &= 3, \\
v_1 &= 1, \\
v_2 &= 2, \\
v_3 &= 3, \\
v_4 &= 3,
\end{align*}
\]

then $B_1 \in B_{1223}$ is given by

\[
B_1 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

where $\bullet$ denotes an entry which is not necessarily zero.

More generally, we employ as in [7] an operator $\mathcal{P}_r$ of degree $r \in \mathbb{N}$ given by

\[
\mathcal{P}_r(y) = a + \sum_{q=1}^{r} B_q(y^q).
\]

Here $y^q = (y_1, \ldots, y_q)$ ($q$ terms), $a$ is an $m$–vector and $B_q$ is $q$–linear, that is, $B_q(z_1, \ldots, z_q)$ is linear in each of $z_1, \ldots, z_q$. We wish to employ $\mathcal{P}_r Y$ as a filter for $x$. Motivation for such usage follows from known results [7, 9] where strong estimating properties of $\mathcal{P}_r$ have been demonstrated.

For $j = 1, \ldots, m$, denote by $e_j$ the $m$–vector with unity in place $j$ and zeros elsewhere. Then by $q$–linearity

\[
B_q[y^q] = B_q\left[ \sum_{j=1}^{m} y_{j_1} e_{j_1} \ldots \sum_{j_q=1}^{m} y_{j_q} e_{j_q} \right] = \sum_{j_1=1}^{m} \ldots \sum_{j_q=1}^{m} y_{j_1} \ldots y_{j_q} B_q(e_{j_1}, \ldots, e_{j_q}) = \sum_{j_1=1}^{m} \ldots \sum_{j_q=1}^{m} y_{j_1} \ldots y_{j_q} B_{q,j_1,\ldots,j_q},
\]

say, for $q \geq 1$. Here each $B_{q,j_1,\ldots,j_q}$ is an $m$–vector. Thus

\[
\mathcal{P}_r(y) = a + \sum_{q=1}^{r} \sum_{j_1=1}^{m} \ldots \sum_{j_q=1}^{m} y_{j_1} \ldots y_{j_q} B_{q,j_1,\ldots,j_q}. \tag{3}
\]

We denote by $\mathcal{P}_{rk}(y)$ the $k$-th element of $\mathcal{P}_r(y)$ ($k = 1, 2, \ldots, m$), with corresponding definitions for $B_{q,j_1,\ldots,j_q}(k)$ and $a_k$. Then

\[
\mathcal{P}_{rk}(y) = a_k + \sum_{q=1}^{r} \sum_{j_1=1}^{m} \ldots \sum_{j_q=1}^{m} y_{j_1} \ldots y_{j_q} B_{q,j_1,\ldots,j_q}(k). \tag{4}
\]

The products $y_{j_1} \ldots y_{j_q}$ in (3) and (4) are not all distinct. Suppose we collect together terms with the common factors $y_{i_1} \ldots y_{i_m}$ into a class $\mathcal{I}_i$ for each nonzero vector $l = (i_1, \ldots, i_m)$ of nonnegative integers. We may then express (4) as

\[
\mathcal{P}_{rk}(y) = a_k + \sum_{l \in \mathcal{I}_m} y_{i_1} \ldots y_{i_m} B_{l}(k), \tag{5}
\]

where

\[
\mathcal{I}_{m,r} = \bigcup_{i_1 + \ldots + i_m \leq r} \mathcal{I}(i)
\]

and

\[
B_1 = \sum_{j_1,\ldots,j_q} B_{q,j_1,\ldots,j_q}.
\]

Here the summation $\sum_{i}$ is over all $(j_1, \ldots, j_q)$ for which

\[
i_n = \sum_{\ell=1}^{q} \delta(n, j_{\ell}) \quad (1 \leq n \leq m),
\]

where $\delta$ is the Kronecker delta.

In fact with a filter of incomplete memory $p = \{p_1, \ldots, p_m\}$, the number of terms is further reduced. Let us temporary denote $s = s_k$ and $v = v_k$. We set

\[
\mathcal{I}_r y = \begin{bmatrix}
\mathcal{I}_{r1}(y_1) \\
\mathcal{I}_{r2}(y_1, y_2) \\
\vdots \\
\mathcal{I}_{rm}(y_1, \ldots, y_m)
\end{bmatrix} = \begin{bmatrix}
\mathcal{I}_{r1}(y_{i_1}) \\
\mathcal{I}_{r2}(y_{i_1}, y_{i_2}) \\
\vdots \\
\mathcal{I}_{rm}(y_{i_1}, \ldots, y_{i_m})
\end{bmatrix},
\]

where for each $k = 1, \ldots, m$ we have

\[
\mathcal{I}_{rk}(y_{i_k}, \ldots, y_{i_r}) = a_k + \sum_{\mu \notin \mathcal{I}_{m,r}} y_{i_1} \ldots y_{i\mu} \beta_{\mu}(\tau_k(y)), \tag{7}
\]

where $i^k = (0, i_1, \ldots, i_k, 0, \ldots, 0)$ and $\beta_{\mu}$ is an appropriate restriction of $\mathcal{I}_{1,k}$. Thus $\mathcal{I}_{rk}$ is constructed from $\mathcal{I}_{rk}$ when the general terms $y_{i_1} \ldots y_{i_m}$ $\mathcal{I}_{1,k}(y)$ in (5) are restricted to terms of the form $y_{i_1} \ldots y_{i_k} \beta_{\mu}(\tau_k(y))$. Note that it is possible to have different degree operators for each component. In such cases we simply replace $r$ by $r_k$ in (7).

Although $\mathcal{I}_{rk}(y_{i_1}, \ldots, y_{i_k})$ is multilinear in the original variables $y_{11}, \ldots, y_{1m}$, the dependence on the product terms $y_{i_1} \ldots y_{i_k}$ is linear (ones again, we write $s = s_k$ and $v = v_k$ here). We denote by $N_k = N_k(r)$ the number of such terms, the terms by $h_k = h_k(l)$ and the corresponding coefficients by $\eta_{l,k} = \eta_{l,k}(r)$ ($j = 1, 2, \ldots, N_k$) in any convenient ordering. Thus we write

\[
\mathcal{I}_{rk}(y_{i_k}, \ldots, y_{i_r}) = a_k + \sum_{j=1}^{N_k} \eta_{l,k} h_k = a_k + \eta_{l,k}^T h_k, \tag{8}
\]

where $\eta_{l,k}^T = (\eta_{l_1,k}, \ldots, \eta_{N_k,k})$ and $h_k^T = (h_{l_1,k}, \ldots, h_{N_k,k})$ for $k = 1, 2, \ldots, m$.

### 3. FORMULATION OF THE PROBLEM

Let

\[
J(\mathcal{P}_r) = E ||x - \mathcal{P}_r(y)||^2,
\]

where $\mathcal{P}_r$ is defined by (6) and (8). The problem is to find $\mathcal{P}_r^0$ such that

\[
J(\mathcal{P}_r^0) = \min_{\mathcal{P}_r} J(\mathcal{P}_r). \tag{9}
\]

An optimal filter $\mathcal{P}_r^0$ in the class of causal $r$-th degree filters with incomplete memory $p$ takes the general form.
\[ P^0_r(y) = \begin{bmatrix} P^0_{r,1}(y_1, \ldots, y_k) \\ \vdots \\ P^0_{r,m}(y_1, \ldots, y_k) \end{bmatrix}, \] (10)

where the component \( P^0_{r,k}(y_1, \ldots, y_k) \) is given by

\[ P^0_{r,k}(y_1, \ldots, y_k) = \frac{1}{\lambda_k} + \frac{1}{h_k^T} \] (11)

for each \( k = 1, \ldots, m \). Finding an optimal representative \( P^0_r \) is therefore a matter of finding optimal values \( \frac{1}{\lambda_k} \) and \( \frac{1}{h_k^T} \) for the constants \( \lambda_k \) and vectors \( h_k \).

4. REDUCTION OF ORIGINAL PROBLEM TO SEVERAL INDEPENDENT PROBLEMS

The special structure of the operators makes direct solution of (9) a difficult problem. Suffice it to say that a solution is known only for the special case where \( T_r \) is linear and has complete memory [4, 5]. Moreover the solution in [4, 5] has been obtained with a quite restrictive assumption that the co-variance matrix \( E[y_1 y_1^T] \) is nonsingular. Indeed we observe that direct determination of \( P^0_r \) from (9) is not straightforward because of difficulties imposed by the embedded lower \( p \)-band structure of the matrices. To avoid these difficulties we show that the problem (9) can be reduced to \( m \) independent problems. Define

\[ J_k(T_r) = E[|x_k - T_r(y_1, \ldots, y_k)|^2] \] (12)

for \( k = 1, \ldots, m \), where \( T_r \) is defined by (8). We have the following result.

**Proposition 1** We have

\[ \min_{T_r} J_r(T_r) = \min_{T_r} J_k(T_r). \] (13)

Expression (13) allows us to reformulate (9) in an equivalent form: for each \( k = 1, \ldots, m \), find \( P^0_{r,k} \) such that

\[ J_k(P^0_{r,k}) = \min_{T_r} J_k(T_r). \] (14)

Any optimal operator \( P^0_{r,k} \) is the \( k \)-th component of an optimal estimator \( P^0_r \). Hence an optimal estimator \( P^0_r \) can be constructed from any solutions \( P^0_{r,1}, \ldots, P^0_{r,m} \) to the \( m \) independent problems (14).

5. DETERMINATION OF THE OPTIMAL CAUSAL FILTER WITH INCOMPLETE MEMORY

To describe the solution to (9) we introduce some additional notation. If \( A \) is a nonnegative square matrix then we write \( A^{1/2} \) for a nonnegative square matrix satisfying \( A^{1/2}A^{1/2} = A \). We also denote by \( A^\dagger \) the generalised inverse matrix for \( A \). Both matrices are normally defined via the singular–value decomposition. For any two random vectors \( w, q \) we write \( \hat{w} = w - E[w] \) and \( \hat{q} = q - E[q] \) and define

\[ E_{wq} = E[\hat{w} \hat{q}^T] = E[wq^T] - E[w]E[q^T]. \]

Then we have

\[ E_{wq}E_{qw}^T E_{wq} = E_{wq} \] (15)

(see [7]).

It is convenient to use a special notation in two particular cases. We introduce \( H_k := H_k(r) \) and \( Q_k := Q_k(r) \) by the formulæ

\[ H_k = E[h_k h_k^T] - E[h_k^T]E[h_k^T] \]

and

\[ Q_k = E[x_k] - E[x_k h_k^T]. \]

We have from (15) that

\[ Q_k H_k^\dagger H_k = Q_k. \] (17)

Theorem 1 below solves problem (14) for \( k = 1, 2, \ldots, m \).

**Theorem 1** For each \( k = 1, 2, \ldots, m \), an optimal causal \( r \)-th degree filter \( P^0_{r,k} \) with incomplete memory \( p \) is defined by

\[ \eta_k^T = Q_k H_k^\dagger + M_k - E(h_k^T), \] (18)

where the \( N_k \)-vector \( M_k \) is arbitrary, \( I_k \) is the \( N_k \times N_k \) identity matrix and

\[ a_k^0 = E[x_k] - \eta_k^T E[h_k]. \] (19)

**Remark 1** The solution structure given by Theorem 1 may formally be recognized in terms of the known statistical least–squares prediction problem (see, for example, [6]). By the known technique, we seek a least–squares prediction of a random variable \( X \) in the form

\[ \hat{X} = a_0 + \sum_{i=1}^n a_i W_i. \]

Let

\[ \gamma = \begin{bmatrix} \text{cov}(X, W_1) \\ \vdots \\ \text{cov}(X, W_n) \end{bmatrix}, \]

\[ \mu_w = \begin{bmatrix} E(W_1) \\ \vdots \\ E(W_n) \end{bmatrix}, \]

\( \Gamma \) be the \( n \times n \) matrix with \( (i, j) \)-th entry \( \text{cov}(W_i, W_j) \), and suppose that these quantities, \( E(X) \) and \( \text{var}(X) \) are all known. If \( a = (a_1, \ldots, a_n)^T \), the solution of this problem is known to be

\[ a_0 = E(X) - a^T \mu_w, \]

where \( a \) is any solution to

\[ \Gamma a = \gamma. \]

Further, the associated mean–square error is

\[ \text{var}(X) - a^T \gamma. \]

The differences of our approach (and as a result, of Theorem 1) from the known methods are implied by the specific structure of the filter used. In turn, such a specific structure is implied by the incomplete memory condition (Definition 1) and by the reduction to \( m \) independent problems (Proposition 1). Moreover, nonlinear terms involving the data values \( y_i \) are used for the estimation. The latter implies the significant improvement in the accuracy of estimation. In this regard, see also references [2, 7].

**Remark 2** The covariances used in (18) and (19) and in similar relationships below, are assumed to be known. This assumption is a well-known and significant limitation in problems dealing with estimation of random signals. The covariances can be estimated in various ways and particular techniques are given in many papers (e.g., see [10]).
Figure 1: The performance of the proposed method.
Theorem 2 The mean-square error $E[\|x - \mathcal{F}_r^0(y)\|^2]$, for any optimal filter $\mathcal{F}_r$ defined by (11), (18) and (19) is

$$E[\|x - \mathcal{F}_r^0(y)\|^2] = \sum_{k=1}^{m} J_k(\mathcal{F}_r^0) = \sum_{k=1}^{m} \mathbb{E}_{q_k q_k} - Q_k H_k^T Q_r^T.$$  

(20)

6. SIMULATIONS

To illustrate the performance of the method, we apply the proposed filters to the problem of extracting information about images of the surface of the earth obtained by air observations. The $256 \times 256$ reference matrix $\mathcal{F}$ to be estimated is a numerical representation of the image of a chemical plant. The data can be found in http://sipi.usc.edu/services/database/Database.html.

We consider two different cases. In the first the data is disturbed by additive noise and in the second by multiplicative noise. In each case the raw data set is represented by $\mathcal{F}$ with 256 separate blocks. Since the procedure is essentially complete memory proposed above to filters that are not restricted to exhibit superior performance but there are many realistic applications where such filters cannot be used.

The proposed method has also been tested with other simulations including EEG data. Those tests were also consistent with the theoretical results obtained above.

It is inappropriate to compare causal filters with incomplete memory proposed above to filters that are not restricted in this way. One would naturally expect unrestricted filters to exhibit superior performance but there are many realistic applications where such filters cannot be used.

<table>
<thead>
<tr>
<th>Case</th>
<th>$\Delta_{1,ij}$</th>
<th>$\Delta_{2,ij}$</th>
<th>$\Delta_{1,ij}$</th>
<th>$\Delta_{2,ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.16 \times 10^5</td>
<td>0.02 \times 10^5</td>
<td>5.32 \times 10^5</td>
<td>0.71 \times 10^5</td>
</tr>
<tr>
<td>2</td>
<td>2.85 \times 10^5</td>
<td>0.54 \times 10^5</td>
<td>1.05 \times 10^5</td>
<td>0.29 \times 10^5</td>
</tr>
</tbody>
</table>

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REFERENCES