ON SIGNAL REPRESENTATION FROM GENERALIZED SAMPLES: MINMAX APPROXIMATION WITH CONSTRAINTS

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ABSTRACT

Signal representation plays a major role in many DSP applications. In this paper we consider the task of calculating representation coefficients of an analog signal, where the only available data is its samples based on practical non-ideal acquisition devices. We adopt a minmax approach and further incorporate regularity constrains on the original continuous-time signal. These constraints stem from the nature of applications where smooth signals serve as input data. The ensued solution is shown to consist of an orthogonal projection within a Sobolev space. Illustrating examples are given, utilizing this constrained minmax approach. Our conclusion is that this new approach to signal representation could improve presently available systems, especially in non-ideal situations.

1. INTRODUCTION

Signal processing applications are concerned mainly with digital information, although the origin of many sources of information is analog; e.g. speech and audio, optics, radar, sonar and biomedical applications, to name a few. Within the context of signal representation, calculating L_2 inner products relies on discrete data. This is the case in Gabor analysis and in the initialization of the discrete wavelet transform [1, 2, 3, 4, 5]. A common approach to this task is to approximate each representation coefficient by a Riemann-type sum [6]

$$d[k] = \langle x(t), w_k(t)\rangle \cong T \cdot \sum_n x(nT) \cdot \overline{w_k(nT)}. \qquad (1)$$

Here, $x(t) \in L_2$ is the input signal, $\{w_k(t)\}_k$ is a set of analytically known analysis functions, $d \in \ell_2$ denotes the representation coefficients, and T is the sampling interval. An alternative minmax approximation scheme for $d[k]$ was suggested in [7], by interpreting the ideal sampling scheme as a linear bounded operator in a Sobolev space. This minmax approximation exploits the analytically known function $w_k(t)$ rather than utilizing only partial information given by the samples $\{w_k(nT)\}$ applied to (1).

Nevertheless, it remains to consider practical aspects of the acquisition process. That is, practical sampling schemes involve generalized rather than ideal sampling schemes. Consider a signal $x(t)$, its generalized samples are described by means of consecutive L_2 inner products with a set of sampling functions $\{s_n(t)\}_n$ associated with the acquisition device [8, 9, 10, 11, 12]

$$c[n] = \langle x(t), s_n(t)\rangle . \qquad (2)$$

This sampling model is general enough to describe a large set of practical acquisition devices. As an example, consider an analog to digital converter which performs pre-filtering prior to sampling. In such a setting the sampling functions are $\{s_n(t) = s(t - nT)\}$, where $s(t)$ is a mirrored version of the corresponding impulse response [8].

A minmax approach to the approximation of d was recently introduced in [13] for a generalized sampling scheme. Given the samples sequence $c \in \ell_2$, the maximum possible error was minimized over all admissible signals, yielding an analytic minmax solution.

However, there are applications in which not all L_2 signals serve as a possible input. For instance, such prior information is available in biomedical applications where the underlying mechanism that generates the measured signal is physiologically understood (e.g. ECG, EEG, EMG). Motivated by this observation, this work further extends the results of [13] by incorporating additional information, available a priori. In particular, we consider cases for which the input signal is known to comply with a certain regularity criterion and adopt the Sobolev space framework for solving a minmax approximation problem.

2. MATHEMATICAL PRELIMINARIES

A Sobolev space H_2 of order $p = 1$ is a Hilbert space consisting of all finite energy functions on the real line having a derivative of finite energy as well [14]. The corresponding inner product is defined by

$$\langle x(t), y(t)\rangle_{H_2} = \langle x(t), y(t)\rangle_{L_2} + \langle x'(t), y'(t)\rangle_{L_2}, \qquad (3)$$

where $x'(t)$ denotes the derivative of $x(t)$. In this paper all inner products are within L_2, unless otherwise stated.

The operator P_{A} represents an orthogonal projection onto a closed subspace A. $\mathsf{A}^{\perp}$ is the orthogonal complement of A. The Moore-Penrose pseudo inverse and the adjoint of a bounded transformation are denoted by the † and $*$ superscripts, respectively. The sampling space, S, and the analysis space, W, are defined by

$$\mathsf{S} = \overline{Span\{s_n(t)\}} \qquad (4)$$
$$\mathsf{W} = \overline{Span\{w_n(t)\}}. \qquad (5)$$

It is assumed that both $\{s_n(t)\}$ and $\{w_k(t)\}$ constitute frames for $\mathsf{W} \subseteq L_2$ and for $\mathsf{S} \subseteq L_2$, respectively, giving rise to the following set transformation:

$$S : \ell_2 \to L_2 \qquad (6)$$
$$Sc = \sum_n c[n] \cdot s_n(t),$$

with its adjoint being

$$S^* : L_2 \to \ell_2 \qquad (7)$$

$$S^*x = \sum_n \langle x(t), s_n(t) \rangle \cdot e_n.$$

The set $\{e_n\}$ denotes the standard basis of ℓ_2. A similar definition applies to W. Adopting this notation, the generalized samples of $x(t)$ are given by $c = S^*x$, and the representation coefficients of $x(t)$ satisfy $d = W^*x$. A shift-invariant space is defined by its generator function,

$$\mathsf{S} = \overline{Span\{s(t-nT)\}_n}, \qquad (8)$$

where T is the translation parameter. It is assumed that the generator function and its translated versions constitute a frame for S. The corresponding sampled autocorrelation function and its Fourier transform are given by

$$r_{s,s}[m] = \langle s(t), s(t-mT) \rangle, \qquad (9)$$

$$R_{s,s}(\omega) = \frac{1}{T}\sum_m \left| S(\omega + \frac{2\pi}{T}m) \right|^2, \qquad (10)$$

where $S(\omega)$ is the Fourier transform of $s(t)$. The support of $R_{s,s}(\omega)$ is given by

$$\Omega_{s,s} = \{\omega \in [0, 2\pi) \,|\, R_{s,s}(\omega) \neq 0\}. \qquad (11)$$

Similarly, one can define

$$r_{w,s}[m] = \langle w(t), s(t-mT) \rangle, \qquad (12)$$

$$R_{w,s}(\omega) = \frac{1}{T}\sum_m W\left(\omega + \frac{2\pi}{T}m\right)\overline{S\left(\omega + \frac{2\pi}{T}m\right)}, \qquad (13)$$

having the support $\Omega_{w,s}$ within the frequency domain.

3. THE PROBLEM

We consider the problem of determining the representation coefficients $d = W^*x$ while having $c = S^*x$, the generalized samples of the signal, as the only available data. Adopting a minmax approach, this approximation problem can be formulated as follows:

$$\arg\min_{\hat{d}\in\ell_2} \left\{ \max_{c=S^*x,\ \|x\|\leq L} \left\| W^*x - \hat{d} \right\|_{\ell_2}^2 \right\}. \qquad (14)$$

The constant L is arbitrarily chosen. It serves as an upper bound on the norm of the signal x, ensuring a bounded approximation error. It has been shown in [13] that the unique solution of (14) is given by

$$\hat{d} = W^*P_{\mathsf{S}}x = W^*S(S^*S)^{\dagger}c. \qquad (15)$$

That is, applying W^* to the orthogonal projection of x onto S rather than to x. Also, the constant L has no effect on the ensued solution. It does determine, however, the worst-case scenario yielding the maximum possible approximation error. For example, when considering a single analysis function, w, the minmax solution gives rise to a vector interpretation shown in Figure 1. In such a case, approximating $W^*x = \langle x, w \rangle$ is equivalent to calculating $\langle P_{\mathsf{S}}x, P_{\mathsf{S}}w \rangle$ and the approximation error is upper-bounded by

$$\begin{aligned} \left| d - \hat{d} \right| &= |\langle P_{\mathsf{S}^\perp}x, w \rangle| \\ &\leq \sqrt{L^2 - \|P_{\mathsf{S}}x\|^2} \cdot \|P_{\mathsf{S}^\perp}w\|. \end{aligned} \qquad (16)$$

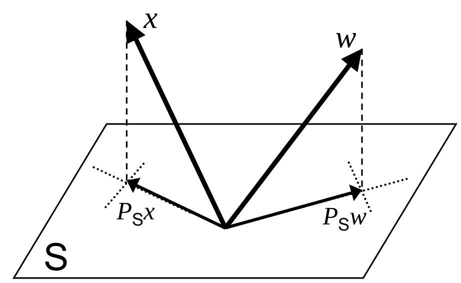

Figure 1: Vector interpretation for the minmax approximation problem of a single representation coefficient. Here, $P_{\mathsf{S}}x$, w and $P_{\mathsf{S}}w$ are analytically known.

A possible input achieving this upper-bound is

$$x = P_{\mathsf{S}}x + \frac{\sqrt{L^2 - \|P_{\mathsf{S}}x\|^2}}{\|P_{\mathsf{S}^\perp}w\|} \cdot P_{\mathsf{S}^\perp}w. \qquad (17)$$

The problem we address involves a minmax approximation scheme for d, while imposing a regularity condition of the form of $x \in H_2$:

$$\arg\min_{\hat{d}\in\ell_2} \left\{ \max_{c=S^*x,\ \|x\|_{H_2}\leq L} \left\| W^*x - \hat{d} \right\|_{\ell_2}^2 \right\}. \qquad (18)$$

4. MINMAX APPROXIMATION

The objective (18) involves two forms of inner products, one being the Sobolev norm evident in the $\|x\|_{H_2} \leq L$ prior and the other being the L_2 inner product evident in the set transformations W^* and S^*. In order to solve this minmax objective in the manner applied to (14) in [13], one may recast (18) to include inner products of the same type.

Let $U : H_2 \to L_2$ be the bounded operator satisfying

$$\begin{aligned} &U^* : L_2 \to H_2 \\ &U^*w = w(t) * u(t) = w(t) * \frac{1}{2}e^{-|t|}. \end{aligned} \qquad (19)$$

This description of the adjoint stems from expressing an H_2 inner product in the frequency domain:

$$\begin{aligned} \langle x, \widetilde{w} \rangle_{H_2} &= \frac{1}{2\pi}\int X(\omega) \cdot \overline{\widetilde{W}(\omega)} \cdot (1+\omega^2)\, d\omega \\ &= \frac{1}{2\pi}\int X(\omega) \cdot \overline{W(\omega)}\, d\omega \\ &= \langle x, w \rangle_{L_2}, \end{aligned} \qquad (20)$$

resulting in $\widetilde{W}(\omega) = W(\omega) \cdot U(\omega)$, where $U(\omega)$ is the Fourier transform of $u(t)$, *i.e.*,

$$U(\omega) = \frac{1}{1+\omega^2}. \qquad (21)$$

This in turn, implies that the set transform W^*x involving L_2 inner products can be alternatively described by means of a set transform of H_2 inner products, denoted by $\widetilde{W}^*$, where

$\tilde{w}_n(t) = w_n(t) * u(t)$. The same holds for the set transform S^*. Eq. (18) can now be written as

$$\arg\min_{\hat{d}\in\ell_2} \left\{ \max_{c=\widetilde{S}^* x,\ \|x\|_{H_2}\le L} \left\| \widetilde{W}^* x - \hat{d} \right\|_{\ell_2}^2 \right\}, \tag{22}$$

where $\tilde{S}^*, \tilde{W}^*$ are operators with a domain in H_2. Those operators are bounded. For example, let $0 < A \le B < \infty$ be the frame bounds of S. That is, for any $x \in \mathsf{S}$

$$A \cdot \|x\|_{L_2}^2 \le \sum_k \left| \langle x, s_k \rangle_{L_2} \right|^2 \le B \cdot \|x\|_{L_2}^2 . \tag{23}$$

A proper choice for the upper frame bound of $\tilde{S}$ would be $\tilde{B} = B$. However, it is not guaranteed that the corresponding lower frame bound, $\tilde{A}$, is strictly positive. This property of $\tilde{A} > 0$ is important. If $\tilde{A} = 0$ then the operator $\widetilde{S}^* \widetilde{S}$ has no closed range, thus no bounded pseudo-inverse operator can be defined for it [15]. This in turn, precludes the possibility of reconstructing $P_{\tilde{\mathsf{S}}} x$ from the samples c.

Nevertheless, a sufficient condition can be derived for the shift-invariant case. Let $R_{s,\tilde{s}}(K;\omega)$ be the partial sum

$$R_{s,\tilde{s}}(K;\omega) = \frac{1}{T} \sum_{m=-K}^{K} \left| S\left(\omega + \frac{2\pi}{T} m\right) \right|^2 U\left(\omega + \frac{2\pi}{T} m\right). \tag{24}$$

It can then be shown that if $R_{s,\tilde{s}}(K;\omega)$ converges uniformly to $R_{s,\tilde{s}}(\omega)$ on $\Omega s, s$ then $\tilde{A} > 0$. In such a case, the minmax solution of (22) is given by

$$\hat{d} = \widetilde{W}^* P_{\widetilde{\mathsf{S}}} x = \widetilde{W}^* \widetilde{S} (\widetilde{S}^* \widetilde{S})^{\dagger} c, \tag{25}$$

where $\widetilde{\mathsf{S}}$ is the space spanned by $\{s_n(t) * u(t)\}$ and all operators are within the Sobolev space.

The approximation error for considering a single analysis function is then given by

$$\begin{aligned} \left| d - \hat{d} \right| &= \left| \widetilde{W}^* P_{\widetilde{\mathsf{S}}^\perp} x \right| \\ &= \left| \langle P_{\widetilde{\mathsf{S}}^\perp} x, \tilde{w} \rangle_{H_2} \right| \\ &\le \left\| P_{\widetilde{\mathsf{S}}^\perp} x \right\|_{H_2} \cdot \left\| P_{\widetilde{\mathsf{S}}^\perp} \tilde{w} \right\|_{H_2} \\ &\le \sqrt{L^2 - \left\| P_{\widetilde{\mathsf{S}}} x \right\|_{H_2}^2} \cdot \left\| P_{\widetilde{\mathsf{S}}^\perp} \tilde{w} \right\|_{H_2}, \end{aligned} \tag{26}$$

and a possible input signal achieving this upper-bound is

$$x = P_{\widetilde{\mathsf{S}}} x + \frac{\sqrt{L^2 - \left\| P_{\widetilde{\mathsf{S}}} x \right\|_{H_2}^2}}{\left\| P_{\widetilde{\mathsf{S}}^\perp} \tilde{w} \right\|_{H_2}} \cdot P_{\widetilde{\mathsf{S}}^\perp} \tilde{w}. \tag{27}$$

It is possible to extend these derivations to Sobolev spaces of arbitrary orders, corresponding to the degree of regularization (smoothness). All the results still apply with a minor change: the function $u(t)$ given in (19) would correspond now to the inverse Fourier transform of

$$U(\omega) = \frac{1}{1 + \omega^2 + \ldots + \omega^{2p}}, \tag{28}$$

where p is the order of the Sobolev space.

For the shift-invariant case of

$$\begin{aligned} \mathsf{S} &= \overline{Span\{s(t - nT)\}}, & (29)\\ \mathsf{W} &= \overline{Span\{w(t - nT)\}}, & (30) \end{aligned}$$

it can be shown that the minmax solution of (15) can be obtained by filtering the generalized samples c with a digital filter [16] having a Fourier transform of

$$G(\omega) = \begin{cases} \frac{R_{w,s}(\omega)}{R_{s,s}(\omega)}, & \omega \in \Omega_{s,s} \\ 0, & \omega \notin \Omega_{s,s} \end{cases}. \tag{31}$$

This form of solution is applicable to the Sobolev case, too:

$$G(\omega) = \begin{cases} \frac{R_{w,\tilde{s}}(\omega)}{R_{s,\tilde{s}}(\omega)}, & \omega \in \Omega_{s,s} \\ 0, & \omega \notin \Omega_{s,s} \end{cases}. \tag{32}$$

5. EXAMPLES

5.1 Example 1

We consider the case of approximating a single representation coefficient obtained by calculating an L_2 inner product with a modulated version of a normalized Gaussian

$$w(t) = \frac{1}{\sqrt[4]{\pi}} e^{-t^2/2} \cos(2\pi t). \tag{33}$$

Denoting $\beta^1(t)$ to be the B-spline of order one, the input signal is

$$x(t) = \beta^1(t) \cdot \cos(2\pi t). \tag{34}$$

The generalized samples correspond to the ZOH scheme

$$s_n(t) = \begin{cases} 1/\Delta, & t \in [nT - \Delta, nT] \\ 0, & otherwise \end{cases}, \tag{35}$$

where T is the sampling interval. Figure 2 depicts $x(t)$ and the analysis function $w(t)$. The uniform convergence criterion of (24) holds surely for $T > \Delta$, guaranteeing $\tilde{A} > 0$. Figure 3 then depicts $P_{\mathsf{S}} x$ and $P_{\widetilde{\mathsf{S}}} x$ for the case of $T = 0.15, \Delta = 0.05$. Having known the generalized samples of x only rather than the signal itself, one may consider a worst possible input in L_2 (17) and in H_2 (27). Both signals are consistent with the known samples and are depicted in Figure 4. Figure 5 depicts the maximum potential approximation error as a function of the sampling interval for both the L_2 (dots) and the H_2 (x-marks) cases, where the error values are normalized by $\|x\|_{L_2}$ and $\|x\|_{H_2}$, respectively. As expected, the constrained minmax objective yields smaller upper-bounds values than does the non-constrained objective. This is also reflected in Figure 3 where $P_{\widetilde{\mathsf{S}}} x$ is much similar to x than $P_{\mathsf{S}} x$.

5.2 Example 2

Another sampling model to be considered is the RC circuit

$$s_n(t) = \begin{cases} (RC)^{-1} e^{\frac{t-nT}{RC}}, & t \le nT \\ 0, & otherwise \end{cases}, \tag{36}$$

where T is the sampling interval. For such a sampling scheme $\tilde{A} > 0$ regardless of T. Figure 6 depicts $P_{\mathsf{S}} x$ and $P_{\widetilde{\mathsf{S}}} x$ for the case of $T = 0.15$ and $RC = 0.05$. A worst-case scenario is shown in Figure 7 for a worst L_2 (solid) and a worst

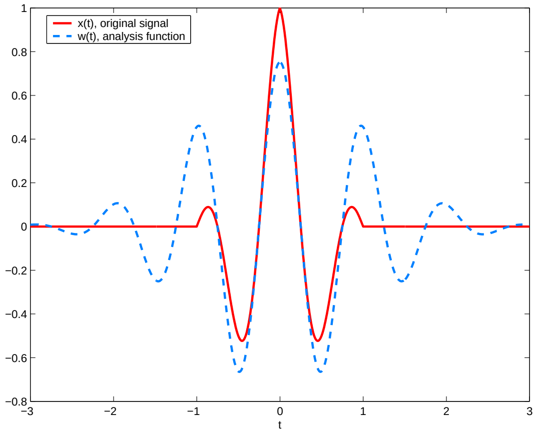

Figure 2: The problem. Given $x(t)$ (solid), known only by its generalized samples, what is the best approximation for $\langle x, w \rangle$? $w(t)$ is an analytically known analysis function (dashed).

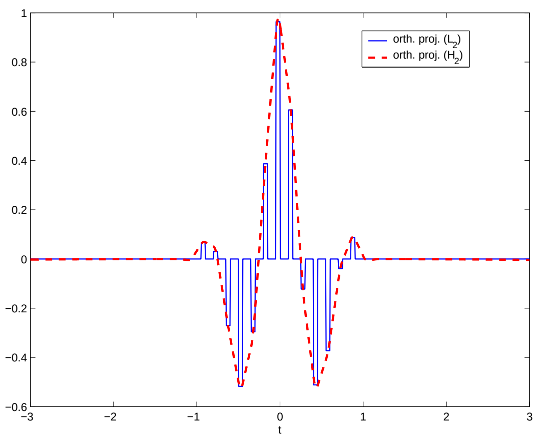

Figure 3: ZOH sampling scheme. The generalized samples of $x(t)$ give rise to its orthogonal projections. $P_S x$ (solid) enables one to find the minmax solution in L_2. $P_{\tilde{S}} x$ (dashed) enables one to find the minmax solution in H_2.

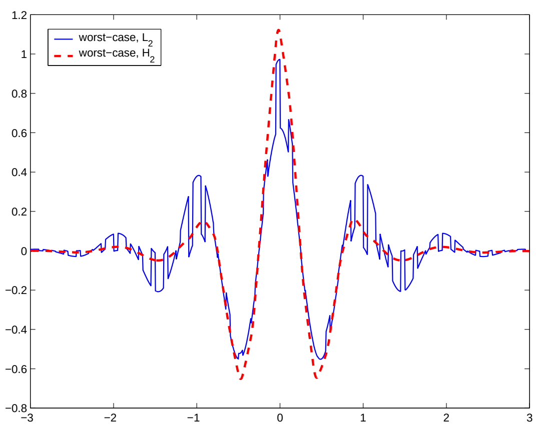

Figure 4: ZOH sampling scheme. The worst case scenario corresponds to the maximum approximation error possible. Shown are functions for a worst case scenario in L_2 (solid) and in H_2 (dashed) according to (17) and (27), respectively.

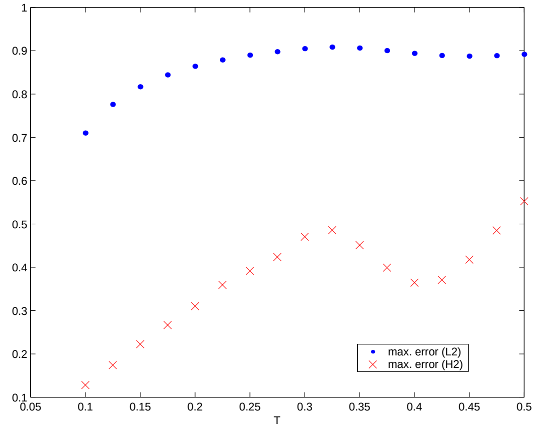

Figure 5: Shown is the maximum potential approximation error for the ZOH sampling scheme as a function of the sampling interval for both the L_2 (dots) and H_2 (x-marks) cases. The values are normalized by $\|x\|_{L_2}$ and $\|x\|_{H_2}$, respectively.

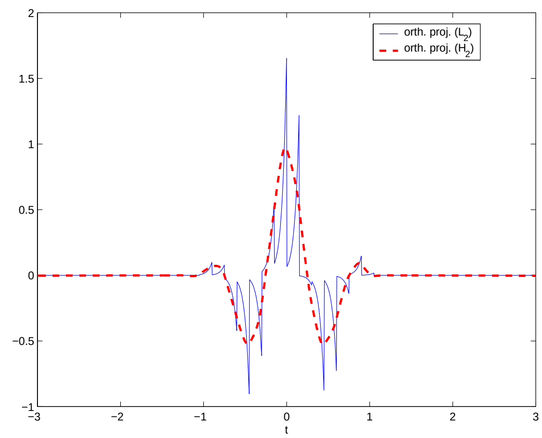

Figure 6: RC sampling scheme. Similar to Figure 3.

H_2 (dashed) signals. Figure 8 depicts the maximum potential approximation error as a function of the sampling interval for both the L_2 and the H_2 cases, where the error values are normalized by $\|x\|_{L_2}$ and $\|x\|_{H_2}$, respectively. Similar to the ZOH example, the constrained minmax objective yields smaller upper-bounds values than does the non-constrained objective. This relatively large difference of upper-bounds values originates from the fact that the sampling functions are not smooth, giving rise to a non-smooth worst-case function in the L_2 setup.

6. CONCLUSIONS

The task of approximating representation coefficients of an analog signal has been considered, being its generalized samples the only available data. Relying on recent results, a minmax approach was applied, while incorporating regularity constraints on the original signal. This was done by recasting the minmax objective into a proper Hilbert space. An upper bound on the maximum potential representation error was then given for the case of a single analysis function. The ensued solution was shown to correspond with an orthogonal projection onto a certain sampling space within the frame-

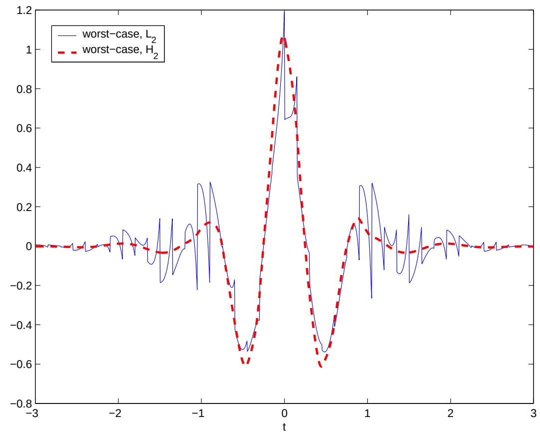

Figure 7: RC sampling scheme. Similar to Figure 4.

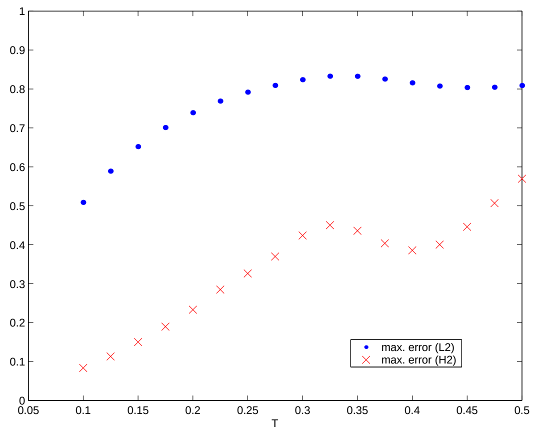

Figure 8: RC sampling scheme. Similar to Figure 5.

work of a Sobolev space. The proposed approach exploits the knowledge one has on the acquisition process, making it applicable to digital signal processing systems having the samples of an analog signal as their input data.

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