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ON THE DUALITY OF MIMO TRANSMISSION TECHNIQUES FOR MULTI-USER COMMUNICATIONS

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ABSTRACT

Since the downlink has a difficult algebraic structure, it is more convenient to switch to the dual uplink problem which has better algebraic properties. We consider the uplink/downlink duality with respect to the *mean square error* (MSE), where our system model is as general as possible, i.e., we allow not only for correlations of the symbols and noise, but also model the precoders, the channels, and the equalizers as compact linear operators. We show that a duality with respect to the MSE per user is preferable to the state-of-the-art stream-wise MSE duality, since the uplink/downlink transformation of the user-wise MSE duality has a considerably lower complexity than the one of the stream-wise MSE duality. Interestingly, the uplink/downlink transformation for the total MSE duality is trivial, i.e., a simple weighting with a scalar common for all filters has to be computed. We apply the uplink/downlink duality to derive the operator form of the well-known *transmit Wiener filter* (TxWF).

1. INTRODUCTION

In the broadcast setup [1], e.g., the downlink of a cellular system, one transmitter communicates with several receivers. If the *broadcast channel* (BC) is non-degraded [1], e.g., the downlink with *multiple-input multiple output* (MIMO) channels to the multiple users, optimizing the system, e.g., maximizing the sum rate, is difficult in general, since most problems for the BC are non-convex.

A powerful tool to circumvent the difficulties with the BC is the uplink/downlink duality, i.e., the achievable region of a suitably defined dual *multiple access channel* (MAC) is the same as the achievable region of the original BC under the same total transmit power constraint. Such a duality was reported for the vector Gaussian BC capacity region in [2, 3] (with non-linear *dirty paper coding*; DPC [4]), for the MIMO Gaussian BC capacity region in [5] (with DPC), for the *signal-to-interference-and-noise ratio* (SINR) region with linear beamforming in [2, 6] (vector BC) and [7] (MIMO BC with fixed receivers), and recently, for the MIMO BC MSE region with linear precoding and with DPC in [8, 9] and [10, 11], respectively. The aforementioned dualities enabled efficient algorithmic solutions to non-convex BC optimizations, because the dual MAC problems are convex, e.g., sum rate maximization [12].

To reach the whole capacity region of the MIMO BC, the non-linear DPC has to be applied [5]. However, we restrict ourselves to linear precoding to avoid the complexity of DPC. Since the rate related problems (e.g., maximization of the sum rate and rate balancing) for linear precoding are also non-convex in the dual MAC, we have to resort to alternative quantities to be optimized. We choose the MSE, since the MSE gives a lower bound to the sum rate and the most popular MSE problems either turn out to be convex (minimization of the sum MSE, [8, 9]) or can be solved via the KKT conditions which are sufficient (balancing of the MSEs, [9]) in the dual MAC.

We aim at a duality which is as general as possible. To this end, we consider an $N_{\mathscr{H}}$-dimensional ($N_{\mathscr{H}}$ is possibly infinite) Hilbert space $\mathscr{H}$ with the inner product[1] denoted by $\langle \bullet, \bullet \rangle$ and restrict ourselves neither to uncorrelated symbols and uncorrelated noise processes nor to finite operators, i.e., matrix operators. We only have to impose the assumptions that the channel operators and the filter operators (precoder and equalizer) are bounded, that is, any input with finite norm is transformed to a bounded output. Together with the assumption that the correlation operators of the symbols and the noise are nuclear, i.e., the sum of their singular values is finite, the combination of any correlation operator with some precoding operator, channel operator, and/or equalization operator is element of the trace class [13].

We present the MSE uplink/downlink duality per data symbol, per user, and for the total MSE in Section 3. Although the uplink/downlink duality can be used to find algorithmic solutions to problems without closed form solutions, we will apply the duality to obtain an expression for the well known TxWF [14] as an operator in Section 4. Thereby, we will see the advantage of our duality especially for sum MSE minimizations, i.e., the uplink/downlink transformation is a simple weighting with a common scalar.

2. SYSTEM MODEL

As depicted in Fig. 1, we consider a BC, where K users are served by one centralized transmitter. The data signal

$$s_k(t) = \sum_{i=1}^{N_{\mathscr{H}}} s_{k,i} a_{k,i}(t) \tag{1}$$

for the k-th user is transformed by the respective precoder P_k at the transmitter, where $\mathrm{E}[s_{k,i} s_{k,j}^*] = r_{k,i,j}$. The formulation of $s_k(t)$ in (1) confirms the fundamental concept of digital communications (i.e., communication by means of waveforms), if the $s_{k,i}$'s are elements of a finite alphabet. The sum of the precoded signals is transmitted over the channel H_k to user k whose received signal is perturbed by the noise $\eta_k(t)$

[1] In our notation, we have for the inner product that $\langle \alpha\varphi, \psi \rangle = \alpha \langle \varphi, \psi \rangle$ and $\langle \varphi, \alpha\psi \rangle = \alpha^* \langle \varphi, \psi \rangle$ for $\alpha \in \mathbb{C}$. We also have that $\langle \varphi, \psi \rangle = \langle \psi, \varphi \rangle^*$, where $(\bullet)^*$ denotes complex conjugation.

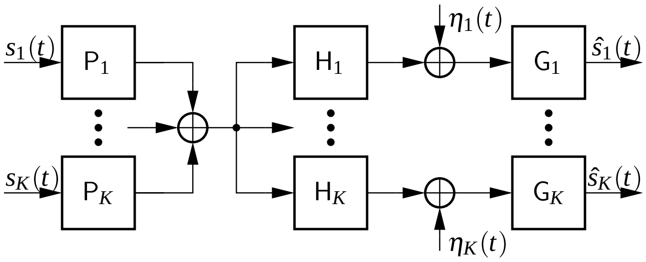

Figure 1: Broadcast Channel Model

and transformed by the equalizer G_k to get the estimation signal

$$\hat{s}_k(t) = \mathsf{G}_k\mathsf{H}_k\sum_{i=1}^{K}\mathsf{P}_i s_i(t) + \mathsf{G}_k\eta_k(t). \tag{2}$$

The estimate for the i-th data symbol $s_{k,i}$ of user k is obtained by following inner product:

$$\hat{s}_{k,i} = \langle \hat{s}_k, b_{k,i}\rangle. \tag{3}$$

Besides the assumptions that both, the $a_{k,i}(t)'s$ and the $b_{k,i}(t)'s$, are linearly independent, we do not impose any special constraints on the signatures $a_{k,i}(t)$ and $b_{k,i}(t)$, e.g., we do not assume that they form orthonormal bases of $\mathscr{H}$ or that they are the same. Popular examples for $a_{k,i}(t)$ and $b_{k,i}(t)$ are pulse shaping and its matched filter, the spreading sequences of CDMA signals, and canonical unit vectors for MIMO sytems.

The correlation operator for a random process $x(t)$ is denoted by R_x, where R_x is defined by $\mathrm{E}[x\langle\varphi,x\rangle] = \mathsf{R}_x\varphi$ for any $\varphi(t)$. As we restrict R_x to be nuclear for any process $x(t)$, we get

$$\mathrm{E}[\langle \mathsf{A}x, x\rangle] = \sum_{i=1}^{N_{\mathscr{H}}}\langle \mathsf{A}\mathrm{E}[x\langle x,\varphi_i\rangle^*],\varphi_i\rangle = \mathrm{Tr}(\mathsf{A}\mathsf{R}_x)$$

for any orthonormal basis $\varphi_i(t), i = 1,\ldots,N_{\mathscr{H}}$ of $\mathscr{H}$ and for any bounded operator A. We use the abbreviation $\mathrm{Tr}(\mathsf{A})$ for the sum $\sum_{i=1}^{N_{\mathscr{H}}}\langle \mathsf{A}\varphi_i,\varphi_i\rangle$ which is independent of the choice for the orthonormal basis $\varphi_i(t)$ and is called the trace of an operator [13, 15].[2] Note that the correlation operator R_x is positive definite by definition. Therefore, we always have that $\mathrm{Tr}(\mathsf{A}\mathsf{R}_x\overline{\mathsf{A}}) \geq 0$, where $\overline{\mathsf{A}}$ denotes the adjoint operator of A. Furthermore, we assume that the noise $\eta_k(t)$ of user k is uncorrelated with any data signal $s_i(t)$, i.e., $\mathrm{E}[\eta_k\langle\varphi,s_i\rangle] = \mathsf{R}_{\eta_k,s_i}\varphi = 0$ with $i,k = 1,\ldots,K$ for any $\varphi(t)$. Symbols of different users are assumed to be uncorrelated, that is, $\mathrm{E}[s_{k,i}s^*_{\ell,j}] = 0$ for $k \neq \ell$.

Our general system model comprises many popular special cases, e.g., flat-fading MIMO channels (the channels can be described by a matrix operator, [8, 9]) and frequency-selective MIMO channels (possibly IIR). We'd like to stress that only the case of matrix operators has been considered in the literature on uplink/downlink duality up to now. Due to our general formulation, we can show in the next section that the uplink/downlink duality also holds for many other cases,

[2] Note that $\mathrm{Tr}(\bullet)$ has following properties. First, it is linear, i.e., $\mathrm{Tr}(a\mathsf{A}+b\mathsf{B}) = a\,\mathrm{Tr}(\mathsf{A}) + b\,\mathrm{Tr}(\mathsf{B})$. Second, the operators can be rotated inside $\mathrm{Tr}(\bullet)$, that is, $\mathrm{Tr}(\mathsf{A}\mathsf{B}) = \mathrm{Tr}(\mathsf{B}\mathsf{A})$. For finite dimensional operators, it is equal to the trace of a matrix, i.e., the sum of the diagonal elements.

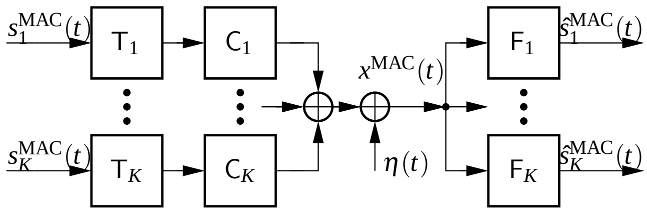

Figure 2: Multiple Access Channel Model

e.g., frequency-selective channels, FIR precoders/equalizers, and IIR precoders/equalizers. Note however that we have to assume that the chain of any precoder, channel, and equalizer has to be nuclear. Therefore, we restrict the channel operators to be nuclear and the filter operators to be bounded.

3. UPLINK/DOWNLINK DUALITY

In the BC (see Fig. 1), the MSE for the i-th data symbol of user k can be written as

$$\begin{aligned}\varepsilon_{k,i}^{\mathrm{BC}} &= \mathrm{E}\left[\left|s_{k,i} - \hat{s}_{k,i}\right|^2\right] \\ &= r_{k,i,i} - \sum_{j=1}^{N_{\mathscr{H}}} 2\,\mathrm{Re}\left(r_{k,j,i}\langle \mathsf{G}_k\mathsf{H}_k\mathsf{P}_k a_{k,j}, b_{k,i}\rangle\right) \\ &\quad + \sum_{\ell=1}^{K}\langle \mathsf{G}_k\mathsf{H}_k\mathsf{P}_\ell\mathsf{R}_{s_\ell}\overline{\mathsf{P}}_\ell\overline{\mathsf{H}}_k\overline{\mathsf{G}}_k b_{k,i}, b_{k,i}\rangle \\ &\quad + \langle \mathsf{G}_k\mathsf{R}_{\eta_k}\overline{\mathsf{G}}_k b_{k,i}, b_{k,i}\rangle.\end{aligned} \tag{4}$$

By definition, the correlation operator for the k-th data signal fulfills $\mathsf{R}_{s_k}\varphi = \sum_{i=1}^{N_{\mathscr{H}}}\sum_{j=1}^{N_{\mathscr{H}}} r_{k,i,j}a_{k,i}\langle\varphi,a_{k,j}\rangle$ for any $\varphi(t)$.

We will show in the next subsections that the MAC depicted in Fig. 2 is dual to the BC in Fig. 1 with respect to the MSE for the same sum transmit power. The transmit filter of user k for the dual MAC is denoted by T_k, the k-th user's channel is C_k, and the respective receive filter is F_k. Similar to the BC, the data signal can be written as

$$s_k^{\mathrm{MAC}}(t) = \sum_{i=1}^{N_{\mathscr{H}}} s_{k,i}b_{k,i}(t). \tag{5}$$

The corresponding correlation operator fulfills $\mathsf{R}_{s_k}^{\mathrm{MAC}}\varphi = \sum_{i=1}^{N_{\mathscr{H}}}\sum_{j=1}^{N_{\mathscr{H}}} r_{k,i,j}b_{k,i}\langle\varphi,b_{k,i}\rangle$, where $\mathrm{E}[s_{k,i}s^*_{k,j}] = r_{k,i,j}$, and the estimate for the i-th symbol of user k is $\hat{s}_{k,i}^{\mathrm{MAC}} = \langle\hat{s}_k^{\mathrm{MAC}}, a_{k,i}\rangle$. With above definitions, the MSE of the i-th data symbol for user k in the dual MAC reads as

$$\begin{aligned}\varepsilon_{k,i}^{\mathrm{MAC}} &= \mathrm{E}\left[\left|s_{k,i} - \hat{s}_{k,i}^{\mathrm{MAC}}\right|^2\right] \\ &= r_{k,i,i} - \sum_{j=1}^{N_{\mathscr{H}}} 2\,\mathrm{Re}\left(r_{k,j,i}\langle \mathsf{F}_k\mathsf{C}_k\mathsf{T}_k b_{k,j}, a_{k,i}\rangle\right) \\ &\quad + \sum_{\ell=1}^{K}\langle \mathsf{F}_k\mathsf{C}_\ell\mathsf{T}_\ell\mathsf{R}_{s_\ell}^{\mathrm{MAC}}\overline{\mathsf{T}}_\ell\overline{\mathsf{C}}_\ell\overline{\mathsf{F}}_k a_{k,i}, a_{k,i}\rangle \\ &\quad + \langle \mathsf{F}_k\mathsf{R}_\eta\overline{\mathsf{F}}_k a_{k,i}, a_{k,i}\rangle.\end{aligned} \tag{6}$$

3.1 Duality per Data Symbol

As their MSE duality is based on the SINR duality (e.g., [6]), which was naturally shown for each data symbol separately, Schubert et al. presented the MSE duality per data symbol in [8]. We prove that the symbol-wise duality also holds for infinite dimensional operators without using any SINR result. Note that we need the assumption for the symbol-wise duality that the symbols are uncorrelated ($r_{k,i,j}=0$ for $i\neq j$). Substituting this assumption into (4) and (6), we can infer that it is useful for $\varepsilon_{k,i}^{\mathrm{BC}}=\varepsilon_{k,i}^{\mathrm{MAC}}$ that following equality holds:

$$\mathrm{Re}\left(r_{k,i,i}\langle \mathsf{G}_k\mathsf{H}_k\mathsf{P}_k a_{k,i}, b_{k,i}\rangle\right)=\mathrm{Re}\left(r_{k,i,i}\langle \mathsf{F}_k\mathsf{C}_k\mathsf{T}_k b_{k,i}, a_{k,i}\rangle\right). \tag{7}$$

Fulfilling this first condition means that the desired symbol $s_{k,i}$ experiences the same (or complex conjugate) total weight in the BC and the dual MAC. As we will see later, the noise power of the BC plays the same role as the transmit power of the dual MAC and vice versa. Therefore, we set the noise power of the BC (MAC) equal to the transmit power of the MAC (BC), where we allow for a different power control in the MAC (expressed by the scalars $\xi_{k,i}\in\mathbb{R}$):

$$\xi_{k,i}^2\langle \mathsf{G}_k\mathsf{R}_{\eta_k}\overline{\mathsf{G}}_k b_{k,i}, b_{k,i}\rangle = r_{k,i,i}\langle \mathsf{T}_k b_{k,i}, \mathsf{T}_k b_{k,i}\rangle \tag{8}$$

$$\xi_{k,i}^2\langle \mathsf{F}_k\mathsf{R}_{\eta}\overline{\mathsf{F}}_k a_{k,i}, a_{k,i}\rangle = r_{k,i,i}\langle \mathsf{P}_k a_{k,i}, \mathsf{P}_k a_{k,i}\rangle. \tag{9}$$

There are infinitely many choices for the operators $\mathsf{T}_k,\mathsf{C}_k,\mathsf{F}_k$ in the dual MAC, such that $\varepsilon_{k,i}^{\mathrm{BC}}=\varepsilon_{k,i}^{\mathrm{MAC}}$, $\forall k,i$. However, we choose the operators for the dual MAC such that they have a close relationship to the operators in the BC and such that the conditions (7)–(9) are fulfilled in order to end up with a simple proof for the duality.

An obvious choice for the MAC precoders and the MAC equalizers fulfilling (8) and (9) are

$$\begin{aligned}\mathsf{T}_k &= \mathsf{R}_{\eta_k}^{1/2}\overline{\mathsf{G}}_k\Theta_k\\ \mathsf{F}_k &= \overline{\Upsilon}_k\overline{\mathsf{P}}_k\mathsf{R}_{\eta}^{-1/2}\end{aligned} \tag{10}$$

respectively. Here, we have introduced the operator Θ_k whose i-th eigenvalue is $\xi_{k,i}/\sqrt{r_{k,i,i}}$ with the eigenfunction $b_{k,i}(t)$. Likewise, $a_{k,i}(t)$ is the eigenfunction corresponding to the eigenvalue $\sqrt{r_{k,i,i}}/\xi_{k,i}$ of Υ_k. For (7), it suffices to set

$$\mathsf{C}_k = \mathsf{R}_{\eta}^{1/2}\overline{\mathsf{H}}_k\mathsf{R}_{\eta_k}^{-1/2}. \tag{11}$$

We denote the 'square root' operator of the positive correlation operator R_x as $\mathsf{R}_x^{1/2}$ which is also an adjoint operator, where $\mathsf{R}_x\varphi=\mathsf{R}_x^{1/2}\mathsf{R}_x^{1/2}\varphi$ for arbitrary $\varphi(t)$. Clearly, physically meaningful noise operator must be invertible and so their square root operators are invertible.

With the dual MAC operators in (10) and (11), the equality $\varepsilon_{k,i}^{\mathrm{BC}}=\varepsilon_{k,i}^{\mathrm{MAC}}$ can be rewritten as

$$\sum_{\ell=1}^{K}\sum_{j=1}^{N_{\mathscr{H}}} r_{\ell,j,j}w_{\ell,k,j,i}+\langle \mathsf{G}_k\mathsf{R}_{\eta_k}\overline{\mathsf{G}}_k b_{k,i}, b_{k,i}\rangle = \\ =\sum_{\ell=1}^{K}\sum_{j=1}^{N_{\mathscr{H}}}\frac{\xi_{\ell,j}^2}{\xi_{k,i}^2}r_{k,i,i}w_{k,\ell,i,j}+\frac{r_{k,i,i}}{\xi_{k,i}^2}\langle \mathsf{P}_k a_{k,i}, \mathsf{P}_k a_{k,i}\rangle.$$

Here, we introduced $w_{k,\ell,i,j}=|\langle \mathsf{G}_\ell\mathsf{H}_\ell\mathsf{P}_k a_{k,i}, b_{\ell,j}\rangle|^2$ for notational brevity. Multiplying above equation with $\xi_{k,i}^2$ and collecting all equations for $k=1,\dots,K$ and $i=1,\dots,N_{\mathscr{H}}$, we end up with the equation system

$$\boldsymbol{W}\boldsymbol{\xi}=\boldsymbol{p} \tag{12}$$

where the MAC power control parameters are put into

$$\boldsymbol{\xi}=\left[\xi_{1,1}^2,\dots,\xi_{1,N_{\mathscr{H}}}^2,\xi_{2,1}^2,\dots,\xi_{K,N_{\mathscr{H}}}^2\right]^{\mathrm{T}}$$

and $(\bullet)^{\mathrm{T}}$ denotes transposition. Let $u=(k-1)N_{\mathscr{H}}+i$ and $v=(\ell-1)N_{\mathscr{H}}+j$. Then, the u-th element of the right-hand side of (12) is

$$[\boldsymbol{p}]_u = r_{k,i,i}\langle \mathsf{P}_k a_{k,i}, \mathsf{P}_k a_{k,i}\rangle$$

and the u-th element of the v-th column of $\boldsymbol{W}$ is

$$[\boldsymbol{W}]_{u,v}=\begin{cases}-r_{k,i,i}w_{k,\ell,i,j} & u\neq v\\ \sum_{m=1}^{K}\sum_{n=1}^{N_{\mathscr{H}}} r_{m,n,n}w_{m,k,n,i}-r_{k,i,i}w_{k,k,i,i} & \\ +\langle \mathsf{G}_k\mathsf{R}_{\eta_k}\overline{\mathsf{G}}_k b_{k,i}, b_{k,i}\rangle & u=v.\end{cases}$$

Clearly, $\boldsymbol{W}$ is (column) diagonal dominant for non-vanishing noise in the BC, i.e., $[\boldsymbol{W}]_{v,v}>\sum_{u\neq v}|[\boldsymbol{W}]_{u,v}|$ for any v. Thus, the matrix $\boldsymbol{W}$ is always invertible. Moreover, $\boldsymbol{W}$ has a diagonal with positive entries and all other elements are non-positive. Consequently, all entries of $\boldsymbol{W}^{-1}$ are non-negative. Since the right-hand side $\boldsymbol{p}$ only contains non-negative numbers, the resulting $\boldsymbol{\xi}$ has always non-negative elements. This observation shows that we can always find a dual MAC with the operators in (10) and (11) which lead to the same MSE for all users and data symbols by applying the appropriate power control $\xi_{k,i}$ in the dual MAC.

When summing up the scalar equations of (12), the resulting right-hand side is the total transmit power in the BC, i.e., $\sum_{k=1}^{K}\sum_{i=1}^{N_{\mathscr{H}}} r_{k,i,i}\langle \mathsf{P}_k a_{k,i}, \mathsf{P}_k a_{k,i}\rangle$. Due to

$$[\boldsymbol{W}]_{v,v}+\sum_{u\neq v}[\boldsymbol{W}]_{u,v}=\langle \mathsf{G}_k\mathsf{R}_{\eta_k}\overline{\mathsf{G}}_k b_{k,i}, b_{k,i}\rangle$$

and (8), the resulting left-hand side is the total transmit power in the dual MAC. Thus, the dual MAC leads to the same MSEs as the BC for the same total transmit power.

With similar steps, it can be shown that the BC in Fig. 1 leads to the same MSEs for the same total transmit power as the dual MAC in Fig. 2, if the BC operators fulfill (7)–(9) and are chosen to be

$$\begin{aligned}\mathsf{P}_k &= \mathsf{R}_{\eta}^{1/2}\overline{\mathsf{F}}_k\Upsilon_k^{-1}\\ \mathsf{G}_k &= \overline{\Theta_k^{-1}}\overline{\mathsf{T}}_k\mathsf{R}_{\eta_k}^{-1/2}\\ \mathsf{H}_k &= \mathsf{R}_{\eta_k}^{1/2}\overline{\mathsf{C}}_k\mathsf{R}_{\eta}^{-1/2}.\end{aligned} \tag{13}$$

Therefore, we have proven that some MSEs for every user and data symbol can be achieved in the BC for a given total transmit power, iff the same MSEs can be achieved in the dual MAC for the same transmit power, i.e., the BC and the dual MAC have the same MSE region.

3.2 Duality per User

Obviously, the duality of the BC and the MAC per data symbol proven in the previous subsection implies that the BC and the MAC are also dual per user, i.e., a set of user MSEs (sum of the MSEs of every user's data symbols) can be achieved in the BC for some transmit power, iff the same user MSEs are possible in the dual MAC. However, the transformation between the dual MAC and the BC presented in the last subsection is very complex, since we need to solve an equation system in $KN_{\mathscr{H}}$ variables [see (12)].[3] As we will see in the following, this high complexity can be avoided, if we use the duality per user instead, since the dimensionality of the equation system reduces to K.

Contrary to the previous subsection, it is not necessary to restrict the k-th user's data symbols to be uncorrelated. For notational brevity, we use the operators Ψ_k and Γ_k which map some orthonormal basis $\varphi_i(t), i = 1, \ldots, N_{\mathscr{H}}$ of $\mathscr{H}$ to the k-th user's signatures $a_{k,i} = \Psi_k \varphi_i$ and $b_{k,i} = \Gamma_k \varphi_i$, respectively. Thus, we have that $\mathsf{R}_{s_k} = \Psi_k \mathsf{R}^{\varphi}_{s_k} \overline{\Psi}_k$ and $\mathsf{R}^{\mathrm{MAC}}_{s_k} = \Gamma_k \mathsf{R}^{\varphi}_{s_k} \overline{\Gamma}_k$, where $\mathsf{R}^{\varphi}_{s_k} y = \sum_{i=1}^{N_{\mathscr{H}}} \sum_{j=1}^{N_{\mathscr{H}}} r_{k,i,j} \varphi_i \langle y, \varphi_j \rangle$ for any $y(t) \in \mathscr{H}$.

With above definitions, the sum over the data symbol index i in the total MSEs for the BC ($\varepsilon_k^{\mathrm{BC}} = \sum_{i=1}^{N_{\mathscr{H}}} \varepsilon_{k,i}^{\mathrm{BC}}$) and for the dual MAC ($\varepsilon_k^{\mathrm{MAC}} = \sum_{i=1}^{N_{\mathscr{H}}} \varepsilon_{k,i}^{\mathrm{MAC}}$), which follow respectively from (4) and (6), leads to traces $\mathrm{Tr}(\bullet)$ of operators. The total MSE of user k in the BC can be expressed as

$$
\begin{aligned}
\varepsilon_k^{\mathrm{BC}} &= \sum_{i=1}^{N_{\mathscr{H}}} \mathrm{E}\left[\left|s_{k,i} - \hat{s}_{k,i}\right|^2\right] \\
&= \mathrm{Tr}\left(\mathsf{R}_{s_k}\right) - 2\,\mathrm{Re}\left(\mathrm{Tr}\left(\overline{\Gamma}_k \mathsf{G}_k \mathsf{H}_k \mathsf{P}_k \mathsf{R}_{s_k} \overline{\Psi_k^{-1}}\right)\right) \\
&\quad + \sum_{\ell=1}^{K} \mathrm{Tr}\left(\overline{\Gamma}_k \mathsf{G}_k \mathsf{H}_k \mathsf{P}_\ell \mathsf{R}_{s_\ell} \overline{\mathsf{P}}_\ell \overline{\mathsf{H}}_k \overline{\mathsf{G}}_k \Gamma_k\right) \\
&\quad + \mathrm{Tr}\left(\overline{\Gamma}_k \mathsf{G}_k \mathsf{R}_{\eta_k} \overline{\mathsf{G}}_k \Gamma_k\right)
\end{aligned}
$$

whereas we get for the total MSE in the MAC:

$$
\begin{aligned}
\varepsilon_k^{\mathrm{MAC}} &= \sum_{i=1}^{N_{\mathscr{H}}} \mathrm{E}\left[\left|s_{k,i} - \hat{s}_{k,i}^{\mathrm{MAC}}\right|^2\right] \\
&= \mathrm{Tr}\left(\mathsf{R}^{\mathrm{MAC}}_{s_k}\right) - 2\,\mathrm{Re}\left(\mathrm{Tr}\left(\overline{\Psi}_k \mathsf{F}_k \mathsf{C}_k \mathsf{T}_k \mathsf{R}^{\mathrm{MAC}}_{s_k} \overline{\Gamma_k^{-1}}\right)\right) \\
&\quad + \sum_{\ell=1}^{K} \mathrm{Tr}\left(\overline{\Psi}_k \mathsf{F}_k \mathsf{C}_\ell \mathsf{T}_\ell \mathsf{R}^{\mathrm{MAC}}_{s_\ell} \overline{\mathsf{T}}_\ell \overline{\mathsf{C}}_\ell \overline{\mathsf{F}}_k \Psi_k\right) \\
&\quad + \mathrm{Tr}\left(\overline{\Psi}_k \mathsf{F}_k \mathsf{R}_\eta \overline{\mathsf{F}}_k \Psi_k\right)
\end{aligned}
$$

The three conditions (7)–(9) can be rewritten as

$$
\begin{aligned}
\mathrm{Re}\left(\mathrm{Tr}\left(\overline{\Gamma}_k \mathsf{G}_k \mathsf{H}_k \mathsf{P}_k \Psi_k \mathsf{R}^{\varphi}_{s_k}\right)\right) &= \mathrm{Re}\left(\mathrm{Tr}\left(\overline{\Psi}_k \mathsf{F}_k \mathsf{C}_k \mathsf{T}_k \Psi_k \mathsf{R}^{\varphi}_{s_k}\right)\right) \\
\xi_k^2 \,\mathrm{Tr}\left(\overline{\Gamma}_k \mathsf{G}_k \mathsf{R}_{\eta_k} \overline{\mathsf{G}}_k \Gamma_k\right) &= \mathrm{Tr}\left(\mathsf{T}_k \Gamma_k \mathsf{R}^{\varphi}_{s_k} \overline{\Gamma}_k \overline{\mathsf{T}}_k\right) \\
\xi_k^2 \,\mathrm{Tr}\left(\overline{\Psi}_k \mathsf{F}_k \mathsf{R}_\eta \overline{\mathsf{F}}_k \Psi_k\right) &= \mathrm{Tr}\left(\mathsf{P}_k \Psi_k \mathsf{R}^{\varphi}_{s_k} \overline{\Psi}_k \overline{\mathsf{P}}_k\right)
\end{aligned}
$$

respectively. These conditions are fulfilled by

$$
\begin{aligned}
\mathsf{T}_k &= \xi_k \mathsf{R}_{\eta_k}^{1/2} \overline{\mathsf{G}}_k \Gamma_k \mathsf{R}^{\varphi,-1/2}_{s_k} \Gamma_k^{-1} \\
\mathsf{F}_k &= \xi_k^{-1} \overline{\Psi_k^{-1}} \mathsf{R}^{\varphi,1/2}_{s_k} \overline{\Psi}_k \overline{\mathsf{P}}_k \mathsf{R}_\eta^{-1/2} \\
\mathsf{C}_k &= \mathsf{R}_\eta^{1/2} \overline{\mathsf{H}}_k \mathsf{R}_{\eta_k}^{-1/2}.
\end{aligned}
\tag{14}
$$

[3]Note that we do not restrict $N_{\mathscr{H}}$ to be finite.

Note that above conditions and the operators in (14) only enable a user-wise power control (the $\xi_k, k = 1, \ldots, K$). Setting $\varepsilon_k^{\mathrm{BC}} = \varepsilon_k^{\mathrm{MAC}}$ leads to an equation system as in (12). Again, the properties of the resulting equation system ensure the existence of a valid solution, summing up the equations shows that the transmit powers of the BC and the dual MAC are identical, and similar steps are possible for the transformation from the dual MAC to the BC. Therefore, a duality with respect to the users' MSEs is possible with a dramatically reduced number of duality parameters compared to the duality per data symbol shown in the previous subsection. More precisely, we need to compute K instead of $N_{\mathscr{H}}$ parameters for the transformation from the BC to the dual MAC and vice versa.

3.3 Duality for Total MSE

With similar steps as in the previous two subsections, it can be shown that a duality with respect to the total sum MSE $\varepsilon = \sum_{k=1}^{K} \varepsilon_k$ can be achieved with following operators

$$
\begin{aligned}
\mathsf{T}_k &= \xi \mathsf{R}_{\eta_k}^{1/2} \overline{\mathsf{G}}_k \Gamma_k \mathsf{R}^{\varphi,-1/2}_{s_k} \Gamma_k^{-1} \\
\mathsf{F}_k &= \overline{\Psi_k^{-1}} \mathsf{R}^{\varphi,1/2}_{s_k} \overline{\Psi}_k \overline{\mathsf{P}}_k \mathsf{R}_\eta^{-1/2} \\
\mathsf{C}_k &= \mathsf{R}_\eta^{1/2} \overline{\mathsf{H}}_k \mathsf{R}_{\eta_k}^{-1/2}
\end{aligned}
\tag{15}
$$

where we set $\xi_k = \xi, k = 1, \ldots, K$ compared to (14). Consequently, the original BC problem can be solved in the dual MAC and the transformation of the solution to the BC is just a weighting of the operators with a scalar which follows from the transmit power constraint.

4. APPLICATION OF DUALITY TO TXWF

The TxWF (see [14]) minimizes the total MSE of the BC under a total transmit power constraint, where the equalization operators are constrained to be weighted identity operators and the weights have the same value $g \in \mathbb{R}$ for all receivers:

$$
\begin{aligned}
\{\mathsf{P}_{\mathrm{WF},1}, \ldots, \mathsf{P}_{\mathrm{WF},K}, g_{\mathrm{WF}}\} &= \underset{\{\mathsf{P}_1,\ldots,\mathsf{P}_K,g\}}{\operatorname{argmin}} \sum_{k=1}^{K} \sum_{i=1}^{N_{\mathscr{H}}} \varepsilon_{k,i}^{\mathrm{BC}} \\
\text{s.t.:} \quad \sum_{k=1}^{K} \mathrm{Tr}\left(\mathsf{P}_k \mathsf{R}_{s_k} \overline{\mathsf{P}}_k\right) \leq P_{\mathrm{tot}} & \qquad \mathsf{G}_k = g\mathsf{I},\ \forall k.
\end{aligned}
\tag{16}
$$

Unfortunately, the cost function of above TxWF optimization is non-convex. However, a closed form expression for finite dimensional operators has been given in [14] by solving the original BC problem. With the duality for the total MSE, the solution is obtained much easier than shown in [14].

Clearly, the dual MAC problem has precoders T_k constrained to be [see (15)]:

$$
\mathsf{T}_k = \tau \mathsf{R}_{\eta_k}^{1/2} \Gamma_k \mathsf{R}^{\varphi,-1/2}_{s_k} \Gamma_k^{-1} \tag{17}
$$

with $\tau = g\xi \in \mathbb{R}$. Thus, the total transmit power constraint (which can easily be shown to be always active) for the dual MAC can be fulfilled by the appropriate choice for the common scalar MAC weight

$$
\tau = \sqrt{\frac{P_{\mathrm{tot}}}{\sum_{k=1}^{K} \mathrm{Tr}\left(\overline{\Gamma}_k \mathsf{R}_{\eta_k} \Gamma_k\right)}}. \tag{18}
$$

With this result, the dual MAC optimization corresponding to (16) transforms into an unconstrained minimization of the total MSE with respect to the equalizers F_k. Fortunately, the problem falls apart into K separate problems, one for each F_k, which can be solved using the orthogonality principle:

$$\mathrm{E}\left[\left(s_{k,i}-\hat{s}_{k,i}^{\mathrm{MAC}}\right)^* x^{\mathrm{MAC}}\right]=0 \quad i=1,\ldots,N_{\mathscr{H}} \tag{19}$$

where $x^{\mathrm{MAC}}(t)$ denotes the received signal for the dual MAC (see Fig. 2). Since we have that

$$\mathrm{E}\left[s_{k,i}^* x^{\mathrm{MAC}}\right]=\mathsf{C}_k\mathsf{T}_k\mathsf{R}_{s_k}^{\mathrm{MAC}}\overline{\Gamma_k^{-1}}\varphi_i$$

$$\mathrm{E}\left[\hat{s}_{k,i}^{\mathrm{MAC},*} x^{\mathrm{MAC}}\right]=\sum_{\ell=1}^{K}\mathsf{C}_\ell\mathsf{T}_\ell\mathsf{R}_{s_\ell}^{\mathrm{MAC}}\overline{\mathsf{T}}_\ell\overline{\mathsf{C}}_\ell\overline{\mathsf{F}}_k\Psi_k\varphi_i+\mathsf{R}_\eta\overline{\mathsf{F}}_k\Psi_k\varphi_i$$

and the orthogonality condition (19) must hold for all i, the dual MAC equalizer reads as

$$\mathsf{F}_k=\frac{1}{\tau}\overline{\Psi_k^{-1}}\mathsf{R}_{s_k}^{\varphi,1/2}\overline{\Gamma}_k\mathsf{H}_k\left(\sum_{\ell=1}^{K}\overline{\mathsf{H}}_\ell\Gamma_\ell\overline{\Gamma}_\ell\mathsf{H}_\ell+\frac{1}{\tau^2}\mathsf{I}\right)^{-1}\mathsf{R}_\eta^{-1/2}. \tag{20}$$

Here, we took the expressions for the precoding operators T_ℓ from (17) and the channel operators C_ℓ from (15). By employing (15) again, we can find the BC precoders:

$$\mathsf{P}_k=\frac{1}{g}\left(\sum_{\ell=1}^{K}\overline{\mathsf{H}}_\ell\Gamma_\ell\overline{\Gamma}_\ell\mathsf{H}_\ell+\frac{1}{\tau^2}\mathsf{I}\right)^{-1}\overline{\mathsf{H}}_k\Gamma_k\Psi_k^{-1}. \tag{21}$$

The found precoders must fulfill the total transmit power constraint which can be used to find the receivers' weight

$$g=\sqrt{\frac{\sum_{k=1}^{K}\mathrm{Tr}\left(\sum_{\ell=1}^{K}\left(\overline{\mathsf{H}}_\ell\Gamma_\ell\overline{\Gamma}_\ell\mathsf{H}_\ell+\frac{1}{\tau^2}\mathsf{I}\right)^{-2}\overline{\mathsf{H}}_k\mathsf{R}_{s_k}^{\mathrm{MAC}}\mathsf{H}_k\right)}{P_{\mathrm{tot}}}}. \tag{22}$$

The obtained solution for the TxWF operator has some interesting properties.

- Due to the definition of Ψ_k and Γ_k, $b_{k,i}=\Gamma_k\Psi_k^{-1}a_{k,i}$ holds. Therefore, the BC data signal [see (1)] is first transformed by the BC precoders P_k to the dual MAC data signal [see (5)]. With this transformation, the signatures $b_{k,i}$ employed at the receivers perfectly match the signatures at the transmitter.
- The orthonormalized data signal $s_k^{\varphi}=\Psi_k^{-1}s_k$ propagates over the total channel $\overline{\Gamma}_k\mathsf{H}_k$. Therefore, we do not find any occurence of H_k without the corresponding $\overline{\Gamma}_k$ in above precoder solution (21).
- When creating an equivalent model for the orthonormalized data signal $s_k^{\varphi}(t)$, the operator $\overline{\Gamma}_k$ is merged with the channel operator H_k. Hence, the noise process in this equivalent model must be $\overline{\Gamma}_k\eta_k$. We follow that the regularization parameter $1/\tau^2$ [see (18)] is equal to the ratio of the total noise power in the equivalent model over the total transmit power P_{tot}, i.e., one over the signal-to-noise-ratio in this equivalent model.
- For the special case, that the signatures at the transmitter and at the receivers are identical, i.e., $a_{k,i}(t)=b_{k,i}(t)\ \forall k,i$, and the signatures are orthonormal the operators Γ_k and Ψ_k disappear. Then, the relationship to the matrix-valued TxWF solution in [14] becomes evident.

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