APPLICATION OF MIXED MODELING STRATEGIES FOR THE SIMULATION OF THE 2D WAVE EQUATION FOR ARBITRARY GEOMETRIES

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ABSTRACT

For the simulation of multi-dimensional systems, currently so called *block based* methods are under investigation. A physical model is split into a number of blocks, each corresponding to a specific spatial region, which are modeled and realized separately. The correct interaction of these blocks is guaranteed by interaction laws, which can be derived from the assumption of a global model. In doing so, this paper presents an application of mixed modeling strategies for the 2D wave equation, where different blocks are realized with different methods. On the one hand the Finite Difference Time Domain (FDTD) approach can model arbitrary geometries, but suffers from numerical dispersion. The Functional Transformation Method (FTM) on the other hand, is completely free of numerical dispersion, but is restricted to simple geometries. Via a combination of both methods, it is possible to model arbitrary geometries, while large parts of the modeling region are realized free of dispersion with the FTM.

1. INTRODUCTION

Block based modeling is frequently used in modeling and simulation for two main reasons: either to represent a complex system as a combination of simpler subsystems or to subdivide an irregular spatial domain into more regularly shaped building blocks. This second application is considered here for the solution of the wave equation in irregular geometries. Subdividing an irregular spatial domain allows to apply different numerical methods to different spatial subregions. If used in a clever way, such a mixed modeling approach may combine the advantages of the several methods while avoiding their disadvantages.

Two numerical methods for the simulation of wave propagation are considered here: the Finite Difference Time Domain (FDTD) method and the Functional Transformation Method (FTM). The FDTD method is widely applied for the numerical solution of partial differential equations (PDEs). It can be easily adapted to irregular domains by forming a suitable spatial grid and by establishing the relations between neighboring grid points. A severe disadvantage of the FDTD is that numerical dispersion is introduced through spatial discretization [1, 2]. The FTM has first been developed for physical modeling sound synthesis [3]. It represents the solution of a multidimensional system through its eigenfunctions. Unfortunately, the calculation of these eigenfunctions becomes rather involved for irregular domains. However, it is a great advantage of the FTM that it does not exhibit dispersion.

Considering these two methods, the following mixed modeling approach becomes apparent: Subdivide an irregularly shaped spatial domain into rectangular regions, where the FTM is applied, and into the remaining irregular regions, where the FDTD method is applied. The calculation of eigenfunctions for the FTM is easy in for rectangular domains, while the total dispersion effects resulting from the FDTD method are reduced if it is only applied in the irregular border regions. The crucial point in such a spatial subdivision is the physically correct description of the block interfaces for a smooth transition between the different numerical methods. A general approach for establishing these interface conditions is presented here. The wave equation in two spatial dimensions is chosen as an example, although the general method is applicable to other PDEs and to higher dimensions.

This mixed modeling approach has first been presented in a simplified form for spatially one-dimensional (1D) problems in [4]. An extension to two-dimensional (2D) block based physical modeling with the FTM only has been shown in [5, 6]. This contribution presents 2D mixed modeling with the FTM and FDTD method based on a more general interface condition. Sec. 2 introduces the mathematical tools for the wave equation. The interface conditions for block based modeling are established in Sec. 3 and a simple example is presented in Sec. 4.

2. WAVE EQUATION

Under certain simplifications, the propagation of sound in the air is described by the following two physical principles

$$
\begin{aligned}
-\frac{\partial}{\partial t} p(\vec{x},t) &= \rho c^2 \nabla \vec{v}(\vec{x},t) \qquad \text{equ. of continuity,} \\
-\nabla p(\vec{x},t) &= \rho \frac{\partial}{\partial t} \vec{v}(\vec{x},t) \qquad \text{equ. of motion ,}
\end{aligned}
\tag{1}
$$

where $\vec{v}(\vec{x},t) = v_1(\vec{x},t)\vec{e}_1 + v_2(\vec{x},t)\vec{e}_2$ is the particle velocity and $p(\vec{x},t)$ is the sound pressure. Time is denoted by t and the 2D space coordinates by $\vec{x} = [x_1,\, x_2]^T$. The spatial unit vectors are $\vec{e}_1$ and $\vec{e}_2$. The nabla operator is given by ∇ and the superscript T denotes transposition. The mass density of air ρ and the speed of sound c are assumed to be constant.

The pair of PDEs (1) can be given in more general form as a system of coupled PDEs

$$
\left[\mathbf{B}_1 \frac{\partial}{\partial x_1} + \mathbf{B}_2 \frac{\partial}{\partial x_2} + \mathbf{B}_3 \frac{\partial}{\partial t}\right] \mathbf{y} = \mathbf{0} \tag{2}
$$

with the specific system matrices and the vector of outcomes $\mathbf{y}$

$$\mathbf{B}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3}$$

$$\mathbf{B}_3 = -\begin{bmatrix} 0 & 0 & c^{-2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} v_1(\vec{x},t) \\ v_2(\vec{x},t) \\ -\frac{1}{\rho}p(\vec{x},t) \end{bmatrix}. \tag{4}$$

The identity between (1) and (2-4) can be shown e.g. by introducing the velocity potential as an intermediate quantity [7]. The assignment of $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, and $\mathbf{y}$ according to (3,4) is not unique, but it is suitable for the definition of the interface conditions.

3. BLOCK BASED MODELING

In most practical cases, a PDE is defined within a finite spatial region denoted by V. For the purpose of block based modeling, V may be divided into subregions. Fig. 1 shows a subdivision of the 2D region which is considered as an example in Sec. 4. This section introduces the interface conditions between the subregions in general form and derives the port connections for mixed modeling with different numerical methods.

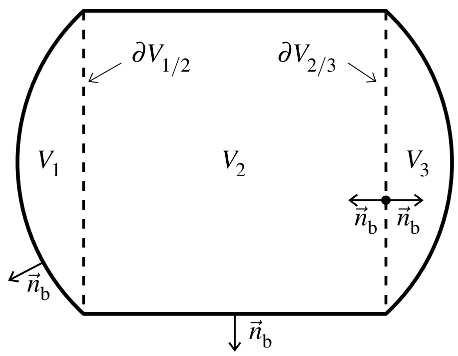

Figure 1: Irregular domain V composed from a simple rectangle V_2 and two segments of a circle V_1 and V_3.

3.1 Interface Conditions

The interface conditions between subregions like the ones shown in Fig. 1 are introduced here for PDEs of the general form (2). The following assumptions are required:

- The PDE (2) holds for the complete spatial region

$$V = V_1 \cup \partial V_{1/2} \cup V_2 \cup \partial V_{2/3} \cup V_3 \,.$$

- The solutions of the PDE in V are elements of a Sobolev space which ensures the weak differentiability in V with respect to the differential operator in (2).

Now define the normal component of the differential operator of (2) by

$$\mathbf{B}_\mathrm{n} = n_1\mathbf{B}_1 + n_2\mathbf{B}_2 + n_3\mathbf{B}_3 \tag{5}$$

where $\vec{n}_\mathrm{b} = [n_1,\, n_2,\, n_3]^T$ is the normal vector on the interface region.

Then it can be shown [8, 9, 10] that the following continuity condition holds at the interface $\partial V_{1/2}$

$$\lim_{\substack{\vec{x}\to\vec{x}_\mathrm{b} \\ \vec{x}\in V_1}} \mathbf{B}_\mathrm{n}\mathbf{y} + \lim_{\substack{\vec{x}\to\vec{x}_\mathrm{b} \\ \vec{x}\in V_2}} \mathbf{B}_\mathrm{n}\mathbf{y} = \mathbf{0}. \tag{6}$$

Similar conditions hold for the interface $\partial V_{2/3}$. The normal components $\mathbf{B}_\mathrm{n}$ in each region are defined with respect to the corresponding orientation of the normal vector $\vec{n}_\mathrm{b}$ (see Fig. 1).

These very general interface conditions are now specialized to the wave equation, i.e. to the case where (3,4) hold. Furthermore, a rectangular region like V_2 with $[0,l_1]\times[0,l_2]$ is considered (see Fig. 2). Depending on each of the four sides of the rectangle, the normal vector $\vec{n}_\mathrm{b}$ is given by either $\pm\vec{e}_1$ or $\pm\vec{e}_2$ and $\mathbf{B}_\mathrm{n}$ results in either

$$\mathbf{B}_\mathrm{n} = \pm\mathbf{B}_1 \qquad \text{or} \qquad \mathbf{B}_\mathrm{n} = \pm\mathbf{B}_2\,. \tag{7}$$

Evaluating the continuity condition (6) for (7) with $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{y}$ according to (3,4) results in conditions for

$$\mathbf{y}_\mathrm{n}(\vec{x},t) = \mathbf{B}_\mathrm{n}\mathbf{y}(\vec{x},t), \tag{8}$$

where

$$\mathbf{y}_\mathrm{n}(\vec{x},t) = \begin{bmatrix} v_1(\vec{x},t) \\ -p(\vec{x},t)/\rho \\ 0 \end{bmatrix} \tag{9}$$

on the right hand side of vertical boundaries ($\vec{x} = [\mathrm{const}, x_2]^T$) and

$$\mathbf{y}_\mathrm{n}(\vec{x},t) = \begin{bmatrix} v_2(\vec{x},t) \\ 0 \\ -p(\vec{x},t)/\rho \end{bmatrix} \tag{10}$$

on the upper side of horizontal boundaries ($\vec{x} = [x_1, \mathrm{const}]^T$). The orientation of x_1 and x_2 is assumed as in Fig. 2. These conditions require the continuity of the normal component of the velocity (either v_1 or v_2) and the continuity of the sound pressure (for constant density ρ).

This result is physically intuitive for this simple example. However the general formulation in (6) holds also for other shapes of the spatial region V, other divisions of V into subregions, and other PDEs, as long as the initial assumptions are fulfilled.

3.2 Boundary Conditions

Since the vectors $\mathbf{y}_\mathrm{n}(\vec{x},t)$ from (8) describe the behavior of the solution of (2) at the boundary, they appear also in the definition of the boundary conditions. In homogenous form, they can be written as

$$\mathbf{f}_\mathrm{b}^T\mathbf{y}_\mathrm{n}(\vec{x},t) = 0\,. \tag{11}$$

where the boundary operator $\mathbf{f}_\mathrm{b}$ defines the kind of boundary condition. Simple examples are

$$\mathbf{f}_\mathrm{b}^T = [0 \quad 1 \quad 0], \tag{12}$$

which sets the pressure at the boundary to zero and

$$\mathbf{f}_\mathrm{b}^T = [R \quad -1 \quad 0], \tag{13}$$

which introduces a fixed relation between pressure and velocity.

3.3 Port Connections

The continuity condition (6) corresponds to the Kirchhoff laws at the ports of an electrical network. It is fulfilled at every instant of the continuous space and time variables. For a discrete realization, two major problems arise:

- time discretization
  Any discrete time model has to be computable. To this end, the variables in the continuity condition (here velocity and pressure) have to be organized into input and output variables. A meaningful connection of two modeling blocks requires, that their input values in one time step can be computed from their output values in preceeding time steps.
- space discretization
  A discrete space realization ensures the validity of the interface conditions only at discrete points of the interface regions. The input and output variables are thus restricted to a discrete set of locations. The numerical methods for each spatial subregion have to specify their input and output variables at these locations.

These problems are now discussed in more detail.

3.3.1 Time Discretization

The organization of the port variables into input and output variables and their discretization is most easily accomplished if all blocks are implemented for a specific geometry. Then the block models can be custom designed to meet the interface conditions (6) where required. However it is much more practical to design a set of suitable blocks beforehand and store them in a block library. Then many different spatial regions can be assembled from the library elements. Solving a new problem with a new spatial region requires only to rearrange existing blocks, not to redesign the blocks from scratch. For example, the block library for a problem like in Fig. 1 would contain two different elements, a rectangle and a segment of a circle. These library elements have to be designed independently from each other. Interface conditions to other blocks cannot yet be considered in the design phase. However, the implementation of these blocks by a suitable numerical method (FDTD, FTM, etc.) requires a properly posed problem in each subregion. The best one can do in this situation, is to design these blocks individually for standard boundary conditions (e.g. Dirichlet, Neumann, or free field conditions). Whenever these blocks are assembled to form a specific spatial region, the resulting interface conditions have to be fulfilled by suitable block interfaces. Here, the aspect of computability of output values from input values is more important than the discrete time values themselves. Therefore the notation of continuous time signals is kept with the tacit assumption, that all time-dependent variables are represented by their respective samples.

3.3.2 Space Discretization

The connection of modeling blocks requires the exchange of variables at discrete spatial positions. For the rectangular region considered above, possible spatial sampling points are indicated in Fig. 1. For FDTD and other methods based on regular grids, these sampling points are simply the grid points on the boundary. For non grid based methods like the FTM, the solution has to be evaluated at these points. In the sequel, only sampling points on the vertical or horizontal boundaries are considered. They are denoted by $\vec{x}_\nu,\ \nu = 1 \ldots n$.

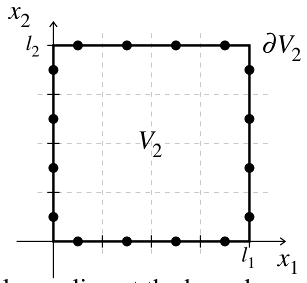

Figure 2: Spatial sampling at the boundary of region V_2. The sampling points $\vec{x}_\nu$ are denoted by the black dots •.

3.3.3 Port Impedance

The spatial sampling points described above are the locations where numerical values are exchanged between the (possibly different) models in adjacent spatial regions. Regarding these locations as ports in the sense of electric circuit theory allows to formulate the interface conditions in terms of port impedances. The requirement of computability demands that a part of the port values are input values and the other part are output values. Here we choose velocity as input and sound pressure as output variables. Then the input and output values can be assembled into the vectors $\mathbf{v}_\mathrm{b}(t)$ and $\mathbf{y}_\mathrm{b}(t)$, respectively

$$\mathbf{v}_\mathrm{b}(t) = [v_\mathrm{b}(\vec{x}_1,t) \quad \ldots \quad v_\mathrm{b}(\vec{x}_n,t)]^T \tag{14}$$

$$\mathbf{y}_\mathrm{b}(t) = [y_\mathrm{b}(\vec{x}_1,t) \quad \ldots \quad y_\mathrm{b}(\vec{x}_n,t)]^T \tag{15}$$

with

$$v_\mathrm{b}(\vec{x}_\nu,t) = v(\vec{x}_\nu,t)\,, \qquad \nu = 1,\ldots,n \tag{16}$$

$$y_\mathrm{b}(\vec{x}_\nu,t) = -p(\vec{x}_\nu,t)/\rho\,. \tag{17}$$

Through (14, 16) and (4), $\mathbf{v}_\mathrm{b}(t)$ and $\mathbf{y}_\mathrm{b}(t)$ are defined in terms of $\mathbf{y}_\mathrm{n}(\vec{x}_\nu,t)$. A relation between $\mathbf{v}_\mathrm{b}(t)$ and $\mathbf{y}_\mathrm{b}(t)$ is easily established in the frequency domain by Laplace transformation with respect to time, i.e. $\mathbf{V}_\mathrm{b}(s) = \mathscr{L}\{\mathbf{v}_\mathrm{b}(t)\}$ and similar for $\mathbf{Y}_\mathrm{b}(s)$. Then the input-output behavior at the boundary points can be expressed in terms of the port impedance matrix $\mathbf{Z}_\mathrm{b}(s)$ as

$$\mathbf{Y}_\mathrm{b}(s) = \mathbf{Z}_\mathrm{b}(s)\mathbf{V}_\mathrm{b}(s)\,. \tag{18}$$

The port impedance matrix describes the relation between a pair of input and output values at one port. It is mainly determined by the boundary conditions of the continuous system (2), but it also includes spatial sampling effects. The dependence on the boundary conditions is demonstrated now by two simple examples.

Boundary conditions of the form (12) set the pressure at the boundary to zero and result in the port impedance $\mathbf{Z}_\mathrm{b}(s) = \mathbf{0}$. Boundary conditions of the form (13) introduce a fixed relation between $v_\mathrm{b}(\vec{x}_\nu,t)$ and $y_\mathrm{b}(\vec{x}_\nu,t)$ and result in the port impedance $\mathbf{Z}_\mathrm{b}(s) = R \cdot \mathbf{I}$. The real value R of the impedance may be adjusted for free field conditions suitable for connecting spatial regions. For free field propagation of acoustic waves, R is equal to speed of sound c.

3.3.4 Block Connections through Ports

The description of interface and boundary conditions by port impedances has important consequences for block based modeling. It concerns especially the design of standardized modeling blocks which can later be assembled to complex models (see Sec. 3.3.1). These consequences are listed below.

- The design of modeling blocks with standard boundary conditions leads to spatially discretized models with simple port impedances.
- The problem of fulfilling the interface conditions (6) is transformed to the problem of connecting ports with possibly different impedances.
- The problem of compatibility between blocks which are designed according to different numerical paradigms (mixed modeling) is transformed to the compatibility of their respective ports.

Matching of port impedances, however, is a well studied problem, for 1D as well as for MD systems. The so-called wave digital principle provides proven and general techniques for this purpose [11, 12]. They have shown to be applicable also for numerical models in state space and other structures using so-called Kirchhoff-to-wave-variable (KW) converters [13, 14].

4. EXAMPLE

As an example consider the propagation of acoustic waves in the 2D spatial region shown in Fig. 1. It can be decomposed into subdomains of two different kinds: a rectangle (V_2) and two segments of a circle (V_1, V_3). The rectangular region V_2 has been realized with a FTM model and is thus free of numerical dispersion. The segments V_1 and V_3 have been realized with the FDTD method, which introduces numerical dispersion in their respective domains, but can be easily adapted to curved boundaries. Both kinds of models have been connected at the interfaces by suitable adaptors, which are 2D extensions of the ones described in [4].

The simulation results in Fig. 3 demonstrate that this mixed modeling approach faithfully reproduces the propagation of waves in an irregular structure. The regional subdivision, introduced for modeling reasons only, is not visible in the simulation result. More complex geometries can be easily constructed from these block models, as demonstrated in [6] for the FTM only.

5. CONCLUSIONS

A mixed modeling strategy for multidimensional systems described by linear PDEs has been presented. Its application has been shown for the 2D wave equation, where different numerical models were used in separate spatial regions. The interface conditions between these regions constitute the requirements for the connection of the numerical models. They can be formulated in a general way through the normal component of the differential operator of the underlying PDE.

By considering the spatial sampling points at the interface regions, these interface conditions are interpreted as ports where input and output values are exchanged between the numerical models. Then the interface conditions are formulated in terms of port impedances. Thus the problem of fulfilling interface conditions is reduced to matching port impedances, which is a well-known topic in one- and multidimensional circuit theory.

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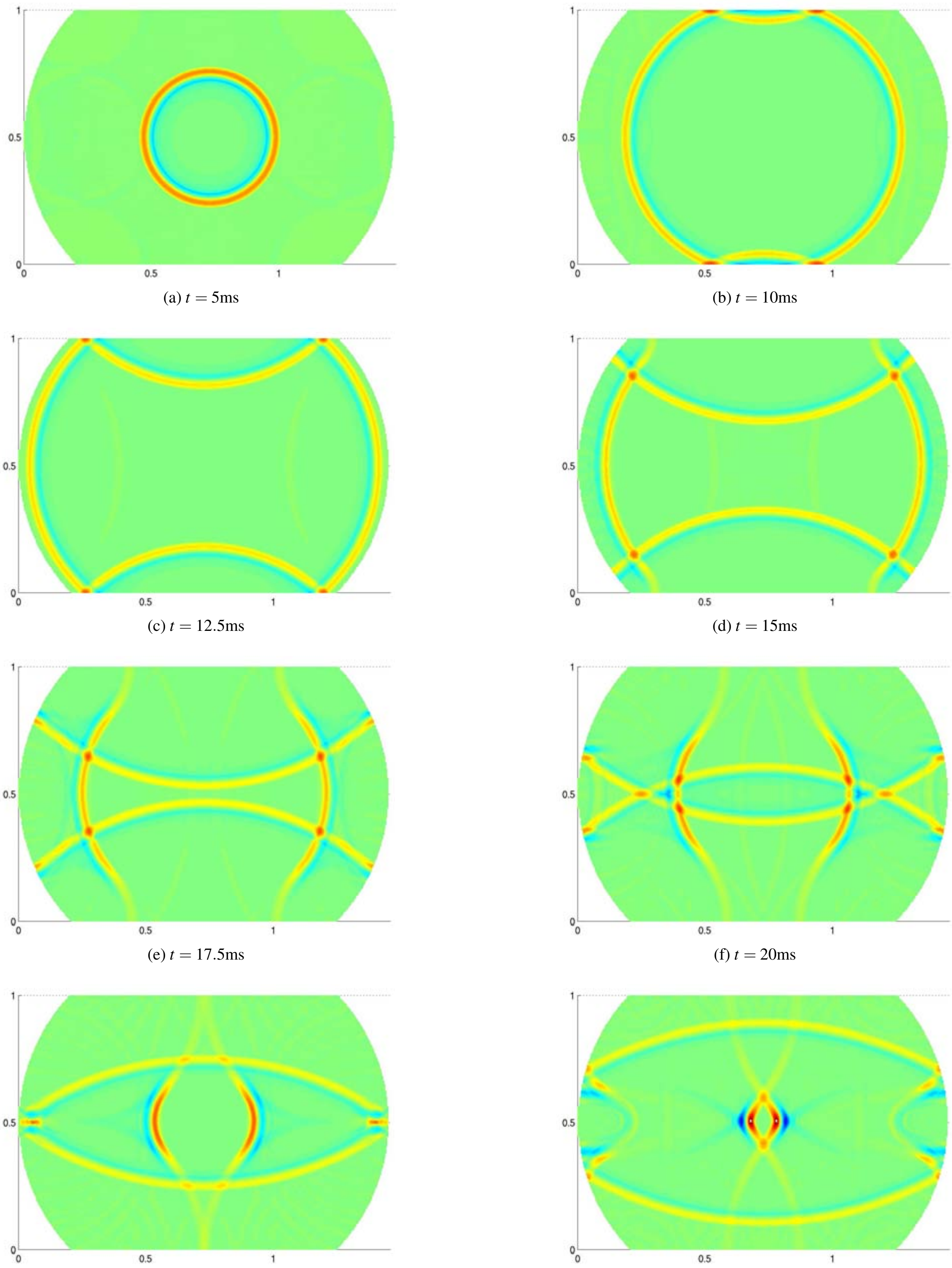

(a) $t = 5$ms (b) $t = 10$ms

(c) $t = 12.5$ms (d) $t = 15$ms

(e) $t = 17.5$ms (f) $t = 20$ms

(g) $t = 22.5$ms (h) $t = 25$ms

Figure 3: Simulation of the wave equation for the irregular domain from Fig. 1. The model is excited in the center with a band-limited Gaussian impulse and Neumann boundary conditions are applied, resulting in perfectly reflecting walls. The snapshots are taken at several time instances t, demonstrating the correct behavior of the block based model.