STATISTICAL ANALYSIS OF THE DEFICIENT LENGTH AFFINE PROJECTION ADAPTIVE ALGORITHM

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ABSTRACT

This paper presents a statistical analysis of the Affine Projection (AP) adaptive algorithm for the insufficient order case. Deterministic recursive equations are derived for the mean weight and mean-square error behavior. The analysis assumes a large number of adaptive coefficients when compared to the algorithm's order, autoregressive input signals and unity step-size. Monte Carlo simulations show excellent agreement with the theoretically predicted behavior. It is shown that the AP coefficients converge in the mean to the initial plant coefficients, producing an unbiased solution even for the correlated input signal case. It is also shown that the steady-state mean square error has a term that is proportional to the power of the unpredictable part of the input signal filtered by the un-modeled part of the unknown impulse response.

1. INTRODUCTION

The least mean square (LMS) is probably the most widely used algorithm in adaptive signal processing applications due to its low computational cost, robustness and tracking capability [1]-[2]. Several authors have studied the behavior of the LMS algorithm. Basically, for a given step-size value, the algorithm's performance is determined by two factors [1]-[3]: (a) the number of adaptive coefficients; and (b) the input signal statistics.

Most theoretical studies on adaptive filter performance consider the sufficient order case, in which it is assumed that the number of adaptive filter coefficients is sufficient to exactly represent the signal to be estimated (the desired signal) except for an additive noise that is uncorrelated with the input signal [1]-[2]. The results of such studies do not necessarily apply to the deficient order case, which is very common in practice [4]. If the sufficient order is overestimated, convergence slows down due to the added extra stochastic gradient noise. On the order hand, adaptive filters of insufficient orders may not reach acceptable estimation performance [5]-[6].

Designers of adaptive systems often have to deal with computational limitations, or lack the necessary information to accurately determine the sufficient order. For instance, most of the energy of voice band telephone channel impulse responses is concentrated in a relatively short period of time [7]. In network echo cancellation, the impulse responses to be identified usually have long and exponentially decaying tails [5]. In brainstem auditory evoked potential studies, the clinical interest is concentrated only on the first 10ms of an impulse response that may last more than 100ms [8]-[9].

A recent work [10] studied the properties of the deficient length LMS algorithm for correlated stationary Gaussian input signals. The results in [10] show that a deficient LMS algorithm leads to a biased solution for the adaptive weights for correlated input signals. As a consequence, the excess mean-square error (EMSE) is increased. Thus, it may render LMS performance insufficient for important practical applications. The reason for the increased LMS EMSE is that the un-modeled coefficients lead to an additive noise which is correlated with the input signal whenever this input signal is correlated in time. Thus, it is of interest to study the performance of other adaptive algorithms that have implicit mechanisms to reduce this effect in their optimum solution. Such algorithms are natural candidates to replace LMS in applications where the use of a deficient length adaptive filter is unavoidable or simply convenient.

The Affine Projection (AP) algorithm [11] is known to implicitly decorrelate the input signal during weight vector updating. This property has been demonstrated in [12] for the important case of unity step size and autoregressive (AR) inputs. Because of this property, it is expected that the AP algorithm can lead to unbiased adaptive coefficient solutions in the deficient order case even for correlated input signals. However, this AP characteristic was not theoretically demonstrated until now.

The AP algorithm behavior has been analyzed in [13] for AR inputs and unity step size. The analysis in [13] assumed a sufficient order adaptive filter. It was shown that the behavior of the AP algorithm for AR inputs is similar to the behavior of a normalized LMS algorithm with white inputs.

This work extends the analysis in [13] to the deficient length AP adaptive algorithm. For an adaptive filter with M coefficients in a system identification setup, it is shown that the mean weight vector converges to the first M samples of the unknown impulse response. Moreover, it is shown that under-modeling leads to an additional steady-state mean-square error (MSE) which is proportional to the power of the unpredictable part of the input signal filtered by the un-modeled part of the unknown impulse response.

This paper is organized as follows. Section 2 introduces the input signal model and the used notation. Section 3 presents the AP weight update equation. Section 4 presents the statistical analysis leading to the deterministic analytical models for the mean weight error vector, the MSE and the second order moments. Section 5 presents Monte Carlo simulation results to verify the accuracy of the theoretical models. Finally, Section 6 presents the main conclusions of this work. In this paper scalars are denoted by plain lowercase or uppercase letters, vectors are denoted by lowercase boldface letters and matrices by uppercase boldface letters. The superscript T denotes transposition. The letter n denotes discrete time.

2. DEFICIENT LENGTH ADAPTIVE FILTER

Consider an input signal $u(n)$ described by an AR process of order P. Then,

$$u(n) = \sum_{i=1}^{P} a_i u(n-i) + z(n) \quad (1)$$

where $z(n)$ is zero-mean, white and Gaussian with variance σ_z^2; and the a_i are the AR coefficients.

Consider $d(n)$ (the desired signal) related to $u(n)$ through the linear model of the form

$$d(n) = \mathbf{w}_N^{\mathbf{o}T}\mathbf{u}_N(n) + r(n) \quad (2)$$

where $\mathbf{w}^{\mathbf{o}}_N=[w^o_0\ w^o_1\ w^o_2\ \dots\ w^o_{N-1}]^T$ can be regarded as the impulse response of the unknown system with length N; $\mathbf{u}_N(n)=[u(n)\ u(n-1)\ u(n-2)\ \dots\ u(n-N+1)]^T$ is an input vector; and $r(n)$ is a zero-mean white Gaussian noise, independent of $u(n)$. The output of an adaptive filter with length M is given by

$$y(n) = \mathbf{w}^T(n)\mathbf{u}(n) \quad (3)$$

where $\mathbf{w}(n)=[w_0(n)\ w_1(n)\ w_2(n)\ \dots\ w_{M-1}(n)]^T$; $\mathbf{u}(n)=[u(n)\ u(n-1)\ u(n-2)\ \dots\ u(n-M+1)]^T$ and $M<N$ (deficient length case). The instantaneous error is given by

$$\begin{aligned} e(n) &= d(n) - y(n) \\ &= \mathbf{w}_N^{\mathbf{o}T}\mathbf{u}_N(n) - \mathbf{w}^T(n)\mathbf{u}(n) + r(n) \\ &= r(n) - \mathbf{v}^T(n)\mathbf{u}(n) + \bar{\mathbf{w}}^{\mathbf{o}T}\bar{\mathbf{u}}(n) \end{aligned} \quad (4)$$

where $\bar{\mathbf{w}}^{\mathbf{o}}=[w^o_M\ w^o_{M+1}\ w^o_{M+2}\ \dots\ w^o_{N-1}]^T$; $\bar{\mathbf{u}}(n)=[u(n-M)\ u(n-M-1)\ u(n-M-2)\ \dots\ u(n-N+1)]^T$; and $\mathbf{v}(n)=\mathbf{w}^{\mathbf{o}}_M-\mathbf{w}(n)$ is the weight-error vector ($\mathbf{w}^{\mathbf{o}}_M=[w^o_0\ w^o_1\ w^o_2\ \dots\ w^o_{M-1}]^T$). The last term in (4) describes the part of the channel output that is due to the exceeding N-M coefficients in $\mathbf{w}^{\mathbf{o}}_N$.

3. WEIGTH UPDATE EQUATION

The weight-error update equation of the AP algorithm with AR input can be written as [12], [13]

$$\mathbf{v}(n+1) = \mathbf{v}(n) + \frac{\mathbf{\Phi}(n)}{\mathbf{\Phi}^T(n)\mathbf{\Phi}(n)} e(n) \quad (5)$$

where

$$\mathbf{\Phi}(n) = \mathbf{u}(n) - \mathbf{U}(n)\hat{\mathbf{a}}(n) \quad (6)$$

and $\hat{\mathbf{a}}(n)$ is the least squares estimate of the AR coefficients:

$$\hat{\mathbf{a}}(n) = \left[\mathbf{U}^T(n)\mathbf{U}(n)\right]^{-1}\mathbf{U}^T(n)\mathbf{u}(n) \quad (7)$$

where $\mathbf{U}(n)=[\mathbf{u}(n-1)\ \mathbf{u}(n-2)\ \mathbf{u}(n-3)\ \dots\ \mathbf{u}(n-P)]$; $\mathbf{U}(n)^T\mathbf{U}(n)$ is assumed of rank P; and $\hat{\mathbf{a}}(n)=[\hat{a}_1(n)\ \hat{a}_2(n)\ \hat{a}_3(n)\dots\ \hat{a}_P(n)]^T$. Using (4) and (6) in (5) we have

$$\begin{aligned} \mathbf{v}(n+1) &= \mathbf{v}(n) - \frac{\mathbf{\Phi}(n)\mathbf{\Phi}^T(n)}{\mathbf{\Phi}^T(n)\mathbf{\Phi}(n)}\mathbf{v}(n) \\ &- \mathbf{\Phi}(n)\left[\mathbf{\Phi}^T(n)\mathbf{\Phi}(n)\right]^{-1}\hat{\mathbf{a}}^T(n)\mathbf{U}^T(n)\mathbf{v}(n) \\ &+ \mathbf{\Phi}(n)\left[\mathbf{\Phi}^T(n)\mathbf{\Phi}(n)\right]^{-1}\left(r(n) + \bar{\mathbf{w}}^{\mathbf{o}T}\bar{\mathbf{u}}(n)\right) \end{aligned} \quad (8)$$

Note from (8) that the effect of $\bar{\mathbf{w}}^{\mathbf{o}}$ is the increase of the additive noise $r(n)$ by a term equal to $\bar{\mathbf{w}}^{\mathbf{o}T}\bar{\mathbf{u}}(n)$, which is zero-mean, correlated in time and correlated with $u(n)$.

4. ANALYSIS

The following statistical assumptions are used in the analysis, and were initially presented and fully discussed in [13]:

- Assumption A1: The order P of the AP algorithm is assumed sufficient to model the input AR process.
- Assumption A2: The statistical dependence between $z(n)$ and $\mathbf{U}(n)$ can be neglected for $M>>P$.
- Assumption A3: The vector $\mathbf{\Phi}(n)$ is orthogonal to the columns of $\mathbf{U}(n)$.
- Assumption A4: The vectors $\mathbf{\Phi}(n)$ and $\mathbf{w}(n)$ are statistically independent.

4.1 Mean Weight Behavior

Using Assumption 3 ($\mathbf{U}^T(n)\mathbf{\Phi}(n)=\mathbf{0}$) it can be shown that

$$\begin{cases} \mathbf{u}^T(n)\mathbf{v}(n+1) = r(n) + \bar{\mathbf{u}}^T(n)\bar{\mathbf{w}}^{\mathbf{o}} \\ \mathbf{U}^T(n)\mathbf{v}(n+1) = \mathbf{U}^T(n)\mathbf{v}(n) \end{cases} \quad (9)$$

Combining these results we obtain

$$\mathbf{U}^T(n)\mathbf{v}(n) = \mathbf{r}(n-1) + \bar{\mathbf{U}}^T(n)\bar{\mathbf{w}}^{\mathbf{o}} \quad (10)$$

where $\mathbf{r}(n-1)=[r(n-1)\ r(n-2)\ r(n-3)\ \dots\ r(n-P)]^T$ and $\bar{\mathbf{U}}(n)=[\bar{\mathbf{u}}(n-1)\ \bar{\mathbf{u}}(n-2)\ \bar{\mathbf{u}}(n-3)\ \dots\ \bar{\mathbf{u}}(n-P)]$. Using (10) in (8) we obtain

$$\begin{aligned} \mathbf{v}(n+1) &= \mathbf{v}(n) \\ &- \frac{\mathbf{\Phi}(n)}{\mathbf{\Phi}^T(n)\mathbf{\Phi}(n)}\left[\mathbf{\Phi}^T(n)\mathbf{v}(n) - r_a(n) - b_a(n)\right] \end{aligned} \quad (11)$$

where

$$\begin{cases} r_a(n) = r(n) - \hat{\mathbf{a}}^T(n)\mathbf{r}(n-1) \\ b_a(n) = \left(\bar{\mathbf{u}}^T(n) - \hat{\mathbf{a}}^T(n)\bar{\mathbf{U}}^T(n)\right)\bar{\mathbf{w}}^{\mathbf{o}} \end{cases} \quad (12)$$

Note that $r_a(n)$ is the white noise component of $d(n)$ filtered by the inverse of the all-pole filter that generates $u(n)$. Likewise, $b_a(n)$ corresponds to the correlated additive noise filtered by the same all-pole filter. This is in agreement with the results shown in [12], [13] that the AP algorithm replaces the noise floor by its filtered version.

Taking the expected value of (11) and using Assumption 4 we obtain

$$\begin{aligned} E\{\mathbf{v}(n+1)\} &= \left[\mathbf{I} - E\left\{\frac{\mathbf{\Phi}(n)\mathbf{\Phi}^T(n)}{\mathbf{\Phi}^T(n)\mathbf{\Phi}(n)}\right\}\right]E\{\mathbf{v}(n)\} \\ &+ E\left\{\frac{\mathbf{\Phi}(n)r_a(n)}{\mathbf{\Phi}^T(n)\mathbf{\Phi}(n)}\right\} + E\left\{\frac{\mathbf{\Phi}(n)b_a(n)}{\mathbf{\Phi}^T(n)\mathbf{\Phi}(n)}\right\} \end{aligned} \quad (13)$$

The first expected value in (13) was already solved in [13]:

$$E\left\{\frac{\boldsymbol{\Phi}(n)\boldsymbol{\Phi}^T(n)}{\boldsymbol{\Phi}^T(n)\boldsymbol{\Phi}(n)}\right\}=\frac{1}{G-2} \quad (14)$$

where $G=M-P$.

The second expected value in (13) is a null vector since $r(n)$ and $\boldsymbol{\Phi}(n)$ are zero-mean and uncorrelated. The last expectation is evaluated in the following way

$$\begin{aligned} &E\{\boldsymbol{\Phi}(n)b_a(n)\} \\ &= E\left\{\boldsymbol{\Phi}(n)\left(\bar{\mathbf{u}}^T(n)-\hat{\mathbf{a}}^T(n)\bar{\mathbf{U}}^T(n)\right)\bar{\mathbf{w}}^{\mathbf{o}}\right\} \\ &= E\left\{\boldsymbol{\Phi}(n)\left[\mathbf{a}-\hat{\mathbf{a}}(n)\right]^T\bar{\mathbf{U}}^T(n)\right\}\bar{\mathbf{w}}^{\mathbf{o}} \\ &+ E\left\{\boldsymbol{\Phi}(n)\bar{\mathbf{z}}^T(n)\right\}\bar{\mathbf{w}}^{\mathbf{o}}=\mathbf{0} \end{aligned} \quad (15)$$

where it has been assumed that $\hat{\mathbf{a}}(n)\cong\mathbf{a}$ (Assumption A1) and (from (1)) $\bar{\mathbf{u}}(n)=\bar{\mathbf{U}}(n)\mathbf{a}+\bar{\mathbf{z}}(n)$. Using these results in (13) we obtain a deterministic recursive equation to the mean weight error vector of the deficient length AP algorithm

$$E\{\mathbf{v}(n+1)\}=\frac{G-3}{G-2}E\{\mathbf{v}(n)\} \quad (16)$$

Assuming convergence, the steady-state mean weight error vector can be obtained from (16), resulting in

$$\lim_{n\to\infty}E\{\mathbf{v}(n)\}=\mathbf{0} \quad (17)$$

Eq. (17) demonstrates the mean weights of the deficient length AP algorithm converge to the actual first M plant coefficients.

4.2 Mean-Square Behavior

Using (6) in (4) and (10) in the resulting expression yields

$$e(n)=-\boldsymbol{\Phi}^T(n)\mathbf{v}(n)+b_a(n)+r_a(n) \quad (18)$$

Squaring (18) and taking its expected value we have an expression for the MSE

$$\begin{aligned} E\{e^2(n)\} &= \sigma_\Phi^2 tr\{\mathbf{K}(n)\}+E\{b_a^2(n)\}+E\{r_a^2(n)\} \\ &-2E\{\boldsymbol{\Phi}^T(n)b_a(n)\}E\{\mathbf{v}(n)\} \\ &-2E\{\boldsymbol{\Phi}^T(n)r_a(n)\}E\{\mathbf{v}(n)\}+2E\{r_a(n)b_a(n)\} \end{aligned} \quad (19)$$

where $E\{\boldsymbol{\Phi}(n)\boldsymbol{\Phi}^T(n)\}=\sigma_\Phi^2\mathbf{I}$; $\sigma_\Phi^2=\sigma_z^2(M-P)/M$ [13]; $\mathbf{I}$ is the identity matrix and $\mathbf{K}(n)=E\{\mathbf{v}(n)\mathbf{v}^T(n)\}$.

The first expected value in (19) can be evaluated using $\bar{\mathbf{u}}(n)=\bar{\mathbf{U}}(n)\mathbf{a}+\bar{\mathbf{z}}(n)$ in (12) and assuming $\hat{\mathbf{a}}(n)\cong\mathbf{a}$, resulting in

$$\begin{aligned} E\{b_a^2(n)\} &= E\left\{\left(\bar{\mathbf{u}}^T(n)\bar{\mathbf{w}}^{\mathbf{o}}-\hat{\mathbf{a}}^T(n)\bar{\mathbf{U}}^T\bar{\mathbf{w}}^{\mathbf{o}}\right)^2\right\} \\ &= \bar{\mathbf{w}}^{\mathbf{o}T}E\{\bar{\mathbf{z}}(n)\bar{\mathbf{z}}^T(n)\}\bar{\mathbf{w}}^{\mathbf{o}} \\ &= \sigma_z^2\bar{\mathbf{w}}^{\mathbf{o}T}\bar{\mathbf{w}}^{\mathbf{o}} \end{aligned} \quad (20)$$

Eq. (20) shows that the effect of the correlated additive noise on the MSE is a function only of the power of $z(n)$, the unpredictable part of the input signal, filtered by the unmodeled part of the unknown impulse response.

The second expected value in (19) was already solved in [13], Eq.(32).

$$E\{r_a^2\}=\left(1+\mathbf{a}^T\mathbf{a}+\sigma_z^2 tr\left\{E\left\{\left[\mathbf{U}^T(n)\mathbf{U}(n)\right]^{-1}\right\}\right\}\right)\sigma_r^2 \quad (21)$$

where $tr\{\cdot\}$ is the trace operation.

The third expected value was already developed in (15).

The last two expected values in (19) are null since $r(n)$ is independent of $u(n)$ and $\boldsymbol{\Phi}(n)$.

Using all results in (19) we obtain

$$E\{e^2(n)\}=\left(1+\mathbf{a}^T\mathbf{a}\right)\sigma_r^2+\sigma_\Phi^2T_K(n)+\sigma_z^2\bar{\mathbf{w}}^{\mathbf{o}T}\bar{\mathbf{w}}^{\mathbf{o}} \quad (22)$$

where $T_K(n)=tr\{\mathbf{K}(n)\}$ and from [13] we know that σ_z^2 $tr\{E\{[\mathbf{U}^T(n)\mathbf{U}(n)]^{-1}\}\}<<1$ for $M>>P$. A recursive equation to the second order moments ($T_K(n)$) will be derived in the next section.

4.3 Second Order Moments

The second order moments for the deficient length case can be obtained in the same way as in [13]. Post-multiplying (11) by its transpose, taking its expected value and applying the statistical assumptions A1-A4 yields, after calculations,

$$\begin{aligned} \mathbf{K}(n+1) &= \mathbf{K}(n)-\frac{2}{(G-2)}\mathbf{K}(n) \\ &+\frac{1}{M(G+2)}tr\{\mathbf{K}(n)\}\mathbf{I} \\ &+\frac{(M-G)}{GM(G+2)}E\{\mathbf{v}^T(n)\}E\{\mathbf{v}(n)\}\mathbf{I} \\ &+\frac{M\left(1+\mathbf{a}^T\mathbf{a}\right)\sigma_r^2}{G(G-2)(G-4)\sigma_z^2}\mathbf{I} \\ &+E\{b_a^2(n)\}E\left\{\frac{\boldsymbol{\Phi}(n)\boldsymbol{\Phi}^T(n)}{\left[\boldsymbol{\Phi}^T(n)\boldsymbol{\Phi}(n)\right]^2}\right\} \end{aligned} \quad (23)$$

where terms dependent on the correlation between $b_a(n)$ and $\boldsymbol{\Phi}(n)$ are equal to zero.

The last expected value in (23) can be approximated by

$$\begin{aligned} &E\left\{\frac{b_a^2(n)\boldsymbol{\Phi}(n)\boldsymbol{\Phi}^T(n)}{\left[\boldsymbol{\Phi}^T(n)\boldsymbol{\Phi}(n)\right]^2}\right\} \\ &= E\{b_a^2(n)\}E\left\{\left[\boldsymbol{\Phi}^T(n)\boldsymbol{\Phi}(n)\right]^{-2}\right\}E\{\boldsymbol{\Phi}(n)\boldsymbol{\Phi}^T(n)\} \quad (24) \\ &= \frac{M\bar{\mathbf{w}}^{\mathbf{o}T}\bar{\mathbf{w}}^{\mathbf{o}}}{G(G-2)(G-4)}\mathbf{I} \end{aligned}$$

Substituting (24) in (23) we have

$$\begin{aligned} \mathbf{K}(n+1) &= \mathbf{K}(n)-\frac{2}{(G-2)}\mathbf{K}(n) \\ &+\frac{1}{M(G+2)}tr\{\mathbf{K}(n)\}\mathbf{I} \\ &+\frac{M-G}{GM(G+2)}E\{\mathbf{v}^T(n)\}E\{\mathbf{v}(n)\}\mathbf{I} \\ &+\frac{M\left[\left(1+\mathbf{a}^T\mathbf{a}\right)\sigma_r^2+\bar{\mathbf{w}}^{\mathbf{o}T}\bar{\mathbf{w}}^{\mathbf{o}}\sigma_z^2\right]}{G(G-2)(G-4)\sigma_z^2}\mathbf{I} \end{aligned} \quad (25)$$

Eq. (25) is a deterministic recursive model for the behavior of the second order moments of the deficient length AP algorithm with an AR input signal. However, only the trace of $\mathbf{K}(n)$ is required to determine the MSE. Taking the trace of (25) we obtain

$$T_K(n+1) = \frac{G^2 - G - 10}{(G-2)(G+2)} T_K(n) + \frac{(M-G)}{G(G+2)} E\{\mathbf{v}^T(n)\} E\{\mathbf{v}(n)\} + \frac{M^2\left[(1+\mathbf{a}^T\mathbf{a})\sigma_r^2 + \bar{\mathbf{w}}^{\mathbf{o}T}\bar{\mathbf{w}}^{\mathbf{o}}\sigma_z^2\right]}{G(G-2)(G-4)\sigma_z^2} \quad (26)$$

where $T_K(n)=tr\{\mathbf{K}(n)\}$.

Eqs. (22) and (26) constitute a general model to the mean square error of the AP adaptive algorithm, valid for both the deficient and the sufficient length cases. The results provided in [13] can now be considered a particular case of these expressions.

4.4 Steady-State MSE

Assuming convergence, the steady-state MSE can be obtained from (22) as

$$\lim_{n\to\infty} E\{e^2(n)\} = (1+\mathbf{a}^T\mathbf{a})\sigma_r^2 + \sigma_z^2\bar{\mathbf{w}}^{\mathbf{o}T}\bar{\mathbf{w}}^{\mathbf{o}} + \sigma_\Phi^2 \lim_{n\to\infty} T_K(n) \quad (27)$$

where, from (26)

$$\lim_{n\to\infty} T_K(n) = \frac{M^2(G+2)\left[(1+\mathbf{a}^T\mathbf{a})\sigma_r^2 + \bar{\mathbf{w}}^{\mathbf{o}T}\bar{\mathbf{w}}^{\mathbf{o}}\sigma_z^2\right]}{G(G+6)(G-4)\sigma_z^2} \quad (28)$$

Using (28) in (27) we obtain

$$\lim_{n\to\infty} E\{e^2(n)\} = k(1+\mathbf{a}^T\mathbf{a})\sigma_r^2 + k\bar{\mathbf{w}}^{\mathbf{o}T}\bar{\mathbf{w}}^{\mathbf{o}}\sigma_z^2 \quad (29)$$

where

$$k = \left[1 + \frac{M(G+2)}{(G+6)(G-4)}\right] \quad (30)$$

5. SIMULATION RESULTS

This section presents simulation results to verify the accuracy of the analytical models given by (22), (26) and (29). Monte Carlo simulations of (5) have been verified to be in excellent agreement with the theoretical expression (17). These results will not be shown here due to space limitation. All examples assume a plant that emulates an echo path with a constant decay envelope [14] with 256 coefficients (Figure 1) and a signal to noise ratio (SNR) of 60dB. The innovation $z(n)$ of the input signal has a power σ_z^2=0.19 for the first order AR model (Figure 2) and σ_z^2=0.6 for the second and third order cases (Figure 3 and Figure 4). The additive noise ($r(n)$) is $\sigma_r^2 = 10^{-6}$ for all examples. Initialization of the adaptive coefficients was in the origin $\mathbf{w}(0)=\mathbf{0}$ and a regularization factor of 10^{-4} was added to $\mathbf{U}^T(n)\mathbf{U}(n)$ before obtaining its inverse. Monte Carlo simulations correspond to the average of 400 runs.

Figure 2 to Figure 4 present the MSE obtained for different sets of parameters, which are informed in the upper right corner of each figure. Here, N is the number of coefficients in the plant, M is the number of adaptive filter taps, P is the order of the AP algorithm and $\mathbf{a}$ is the vector with AR process coefficients. Each figure shows a comparison between simulations (ragged yellow curves), theoretical model ((22)) and the theoretical steady-state model to the MSE (horizontal line – Eq. (29)). Values of M equal to 30, 40, 50, 60 and 70 in each figure correspond, respectively, to the concentration of 61, 68, 84, 87 and 88 percent of the total impulse response power in the first M samples ($\mathbf{w}^{\mathbf{o}T}\mathbf{w}^{\mathbf{o}}$=1 in all cases). All examples present an excellent match between the theoretical results and the Monte Carlo simulations.

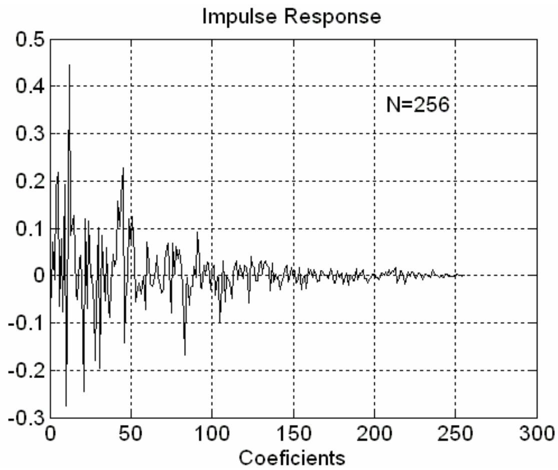

Figure 1 – Impulse response of the plant

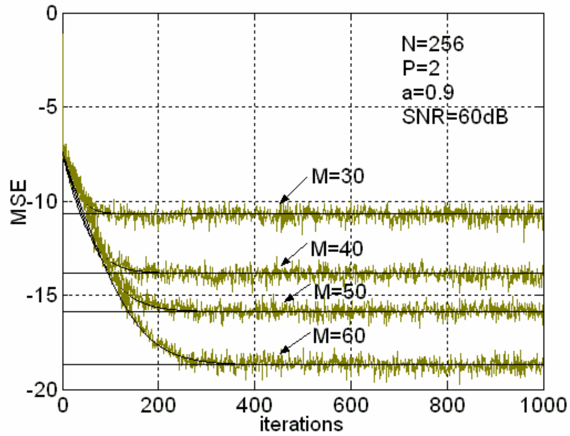

Figure 2 – MSE evolution for the deficient AP algorithm

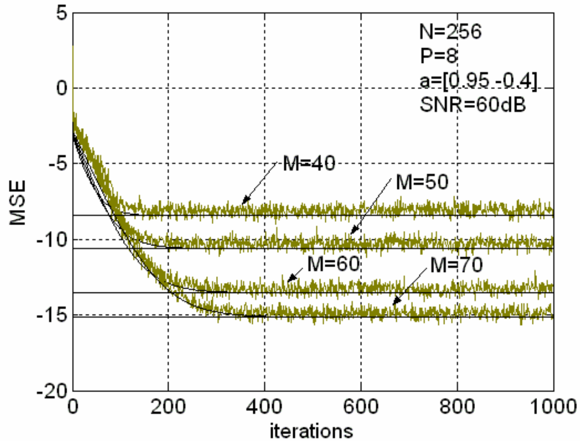

Figure 3 – MSE evolution for the deficient AP algorithm

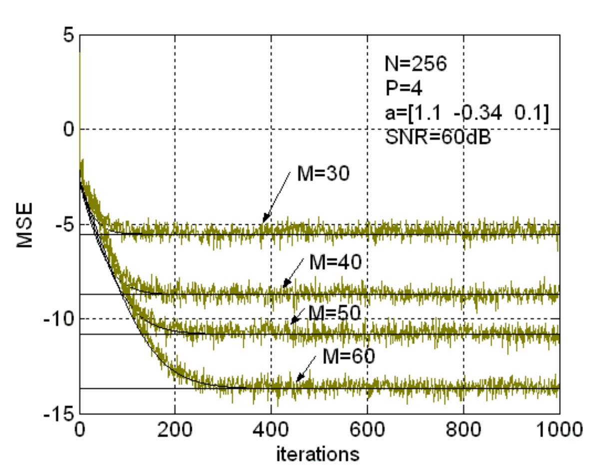

Figure 4 – MSE evolution for the deficient AP algorithm

6. CONCLUSIONS

This work presented a theoretical analysis of the Affine Projection adaptive algorithm that includes the possibility of a deficient length adaptive vector. Deterministic recursive equations were derived for the mean weight and the mean square error behaviors assuming large number of adaptive coefficients (compared to the algorithm's order), autoregressive input signals and unity step-size. Two main results were obtained. First, it has been shown that the AP coefficients converge in the mean to the initial plant coefficients, producing an unbiased solution even for the correlated input signal case. Second, it is shown that the steady-state mean square error has a term that is proportional to the power of the unpredictable part of the input signal filtered by the un-modeled part of the unknown impulse response. Monte Carlo simulations results were shown to be in excellent agreement with theoretical predictions. These results corroborate the conceptual idea that the AP algorithm may be a good alternative to LMS in identification problems where under-modeling is anticipated.

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