A JOINTLY OPTIMAL PRECODER AND BLOCK DECISION FEEDBACK EQUALISER DESIGN WITH LOW REDUNDANCY

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ABSTRACT

In this paper we propose a filter bank based design for jointly optimum precoding and block decision feedback equalisation. Precoding and equalisation using filter banks typically is block based, and redundancy needs to be injected into the transmission in order to avoid inter-block interference (IBI). We target the case where spectral efficiency demands low levels of redundancy such that IBI remains. For our proposed system, we combine two recently reported idea — one on equalisation in the presence of IBI, and one on jointly optimal design of the overall system in the absence of IBI. The result is a jointly optimal design in terms of both zero-forcing and minimum mean square error that can operate in the presence of IBI, i.e. at low levels of redundancy and high spectral efficiency. We show by means of simulation results, that the proposed system can provide significantly better performance than a benchmark design.

1. INTRODUCTION

Block transmission has been shown to be a very effective method to combat inter-symbols interference (ISI) caused by finite impulse response (FIR) frequency selective channels. However, in order to eliminate inter-block interference (IBI), block transmission systems always require an amount of redundancy in form of either cyclic prefix or zero padded intervals whose length must be equal or larger than the channel order. This requirement makes it difficult for block transmission systems to be applied to channels with a long impulse response since a long guard interval will decrease the bandwidth efficiency.

An approach to cope with long channel impulse responses (CIR) is channel shortening \[ h(t) = \sum_{n=0}^{L} h_n \delta(t-nT) \] where \( T \) is the symbol interval, rather than inverting the channel, reduces the effective channel length to a very short support. The shortened support permits the deployment of complex detectors such as the Viterbi algorithm, although part of the channel energy (and therefore capacity) is lost \[ C_{\text{IBI}} = \sum_{n=1}^{L} h_n^2 \] The problem of long CIRs has also been approached in \cite{6}, where a Wiener filter is employed as equaliser and a precoder minimises the minimum mean square error (MMSE) of the system. In \cite{6} Stamoulis et al. have proposed a block decision feedback equaliser (BDFE) for the case where IBI is present, e.g. if the redundancy in the transmission does not allow a guard interval that is longer than the CIR order. These BDFEs can work well even with small transmit redundancy, however the precoder in \cite{6} has been chosen independently from the equaliser and the problem of joint optimisation of precoder and equaliser is still open. In \cite{6}, Xu et al. have proposed jointly optimal designs for precoder and BDFE in the absence of IBI, which can achieve much better performance than linear designs in \cite{6} but still require sufficient redundancy to suppress IBI.

Combining the designs in \cite{5} and \cite{6} we propose in this paper a precoding and BDFE scheme which can work in the case of insufficient redundancy to suppress IBI. Due to the joint optimisation of the precoder and equaliser, the proposed design can perform better than the designs in \cite{5} even when the latter use optimal linear zero-forcing (ZF) or minimum mean-square error (MMSE) precoders proposed in \cite{5}.

The paper is organised as follows. In Sec. 2 the system model and its components are described. Sec. 3 addresses the proposed jointly optimal precoder and BDFE design, while Sec. 4 considers the designs of BDFEs as proposed in \cite{5} as well as the optimal linear precoders proposed in \cite{5}. The simulations of the latter two designs form the benchmark for a numerical example provided in Sec. 5 while conclusions are drawn in Sec. 6.

In our notation, we use lower- and uppercase boldface font for vector and matrix quantities, respectively. The operator \( E \{ \cdot \} \) denotes expectation, \( \left( \cdot \right)^T \) the Hermitian transpose, \( \left( \cdot \right)^{+} \) pseudo-inversion.

2. SYSTEM MODEL

We consider a block transmission system over an FIR channel as illustrated in Fig. 1. The channel is assumed to be stationary with CIR coefficient \[ h[0], \ldots, h[L] \] where \( L \) is the channel order. With the input symbol stream, \( s[n] \) and the sampled version of received signal, \( r[n] \), we define the input symbol blocks as \( s[i] = [s[iN], \ldots, s[iN+N-1]]^T \), the symbol blocks at the receiver input as \( r[i] = [r[iP], \ldots, r[iP+N-1]]^T \), and the transmitted symbols as \( z[i] = [z[iP], \ldots, z[iP+N-1]]^T \).

The result is a jointly optimal design in terms of both zero-forcing and minimum mean square error that can operate in the presence of IBI, i.e. at low levels of redundancy and high spectral efficiency. We show by means of simulation results, that the proposed system can provide significantly better performance than a benchmark design.

Figure 1: System model comprising of (a) precoder, channel and (b) equaliser.
The symbol blocks \( s[i] \) at the input of the feed-forward filter bank are given by

\[
y[i] = H_0Fss[i] + H_1Fs[i-1] - G_1\hat{s}[i-1] + v[i]
\]

where \( H_0 \) and \( H_1 \) are \( P \times M \) matrices,

\[
H_0 = \begin{bmatrix} h[0] & 0 & 0 & \cdots & 0 \\ \vdots & h[0] & 0 & \cdots & 0 \\ h[L] & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & h[L] & \cdots & h[0] \end{bmatrix},
\]

\[
H_1 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & h[L] & \cdots & h[0] \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}.
\]

For the case \( P > L \), the first feedback filter bank \( G(z) = \sum_{n=1}^{\infty} G_nz^{-n} \) suffices to be of non-polynomial form and removes the remaining IBI from the received data stream by setting \( G_1 = H_1F \). Assuming that the past decisions are correct, we can re-write \( 4 \) as

\[
y[i] = H_0Fs[i] + v[i] = HMFs[i] + v[i],
\]

where \( H_M \) contains the first \( M \) columns of \( H_0 \), and obtain

\[
\hat{s}[i] = W_0H_MFv[i] + W_0v[i] - B_0s[i].
\]

The feed-forward filter is initially set \( W(z) = W_0 \). Similarly, the inner feedback filter bank \( B(z) = B_0 \) is here initially set to non-polynomial form and aims to cancel intra-block IBI. The feedback filter bank \( B_0 \) works such that the symbols in each block \( \hat{s}[i] \) are detected sequentially, starting from the \( n \)th symbol, whereby the detected symbols are weighted by the feedback filter bank and removed from \( z[i] \) prior to the detection of the next symbol \( 5,6 \).

With the assumption that the past decisions are correct, the error between the symbols at the input of the decision device, \( \hat{s}[i] \), and the output symbols, \( s[i] \), is

\[
e[i] = \hat{s}[i] - s[i] = (W_0H_MF_0 - B_0 - I)s[i] + W_0v[i]. \tag{7}
\]

The covariance matrix of the error, \( R_{ee} = E\{e[i]e[i]^H\} \), is given by

\[
R_{ee} = (W_0H_MF_0 - B_0 - I)(W_0H_MF_0 - B_0 - I)^H + W_0R_vW_0^H, \tag{8}
\]

where the input symbol \( s[i] \) is assumed to be uncorrelated with unit variance, and the noise covariance matrix is given by \( R_v \).

3. JOINT PRECODING AND BDFE WITH LOW REDUNDANCY

After the remaining IBI has been removed by the first feedback loop, we apply the design of joint optimal precoder and the BDFE proposed in \( 6 \) to remove intra-block IBI. With \( 7 \) being exact for the non-IBI case, and together with \( H_M \), we can now derive the precoder matrix \( F_0 \) as well as the feed-forward and feedback matrices \( W_0 \) and \( B_0 \), such that the system can achieve its minimised lower MSE bound.

3.1 ZF Joint Optimal Precoding and BDFE

The design problem for an MSE precoder and BDFE equaliser that is jointly optimal in the zero-forcing (ZF) sense can be stated as

\[
\begin{align*}
\min_{F_0, W_0, B_0} & \quad \text{trace}(R_{ee}) \\
\text{subject to} & \quad \text{trace}(F_0F_0^H) = P_0 \\
& \quad W_0H_MF_0 = B_0 + I \\
& \quad B_0 \text{ is strictly upper triangular},
\end{align*}
\]

where \( P_0 \) is the transmit power.

Considering the following eigenvalue decomposition (EVD)

\[
H_0^H\Gamma_0^{-1}H_M = VAV^H,
\]

we denote the first \( N \) columns of \( V \) as \( \tilde{V}_N \), the upper left \( N \times N \) block of the diagonal matrix \( \Gamma \) holding the eigenvalues \( \lambda_i, t = 0 \ldots (N-1) \), as \( \lambda_N \), and set \( \Gamma = \sqrt{\lambda_N} \). Based on the geometric mean decomposition \( 11 \), we can find a unitary matrix \( \Theta \) which satisfies the condition

\[
\Gamma\Theta = UR \tag{10}
\]

where \( U \) is a unitary matrix and \( R \) is an upper-triangular matrix with equal diagonal elements identical to the geometric mean of the eigenvalues \( \sqrt{\lambda_N} \Gamma \). The lower MSE bound can be minimised to the value of \( (N/P_0)\Gamma_0^{(N-1)/N} \) by appropriate power control, which results in an optimal precoder matrix of the form

\[
F_0 = \sqrt{P_0/N}\tilde{V}_N\Theta \tag{11}
\]

The feedback and feedforward matrices are then given by

\[
B_0 = \left( \prod_{i=1}^{N} \lambda_i \right)^{-\frac{1}{N}} R - I \tag{12}
\]

\[
W_0 = (B_0 + I)(H_MF_0)^T. \tag{13}
\]
3.2 MMSE Joint Optimal Precoding and BDFE

The MSE precoder and BDFE equaliser are required to fulfill the following design problem for the case of joint optimality in the MMSE sense [6]:

\[
\min_{F_0, W_0, B_0} \text{trace}(R_{ee})
\]

subject to

\[
\text{trace}(F_0^H F_0) = P_0
\]

\[
W_0 = (B_0 + I) R_y R_y^{-1}
\]

\[B_0\] is strictly upper triangular.

where

\[
R_{yy} = (H_M F_0)^H (H_M F_0) + R_{vv}
\]

(14)

According to [6], the minimisation of the lower MSE bound will maximise the mutual information between transmitter and receiver for Gaussian input. Therefore from the EVD in [6], a water-filling algorithm with a single water level is applied to \(\Lambda\) in order to obtain a \((q \times q)\) diagonal matrix \(\Phi\) with

\[
|\phi_i|^2 = \frac{P_0 + \sum_{j=1}^q \lambda_j^{-1}}{q} - \frac{1}{\lambda_i}
\]

(16)

whereby \(q = \min(N, N)\), \(N\) is the maximum integer satisfying \(1/\lambda_N < (P_0 + \sum_{j=1}^N \lambda_j^{-1})/N\).

From \(\Phi\), we construct \(\Phi^* = [\Phi \ 0_{q \times (N-q)}]\) and thus the optimal precoder that will minimise the MSE lower bound takes the form

\[
F_0 = V_q \Phi \Theta
\]

(17)

where matrix \(V_q\) contains the first columns of \(V\) and \(\Theta\) is a unitary matrix satisfying the geometric mean decomposition

\[
\left(I_q + \Phi^T \Lambda_q^{-1} \Phi\right)^{1/2} \Theta = UR
\]

(18)

whereby \(A_q\) is the upper left \(q \times q\) block of \(\Lambda\), \(U\) is a unitary and \(R\) an upper-triangular matrix with equal diagonal elements. The feedback and feedforward matrices that help to achieve the minimised MSE lower bound are given by

\[
B_0 = R - I
\]

(19)

\[
W_0 = \sigma_y R_y R_y^{-1}
\]

(20)

where

\[
\sigma_y^2 = q^{q/(N)} (P_0 + \sum_{j=1}^q \lambda_j^{-1}) - q/N \prod_{j=1}^q \lambda_j^{-1/N}
\]

(21)

4. EXISTING BDFE SYSTEMS WITH OPTIMAL LINEAR PRECODING

A BDFE which can work in the presence of remaining IBI has been proposed by Stamoulis et. al. in [5], which is referred to as IBI-BDFE and classified into zero-forcing (ZF-) IBI-BDFE and MMSE-IBI-BDFE. On the transmitter side, a precoder can be operated; below we utilise (locally) optimal ZF and MMSE linear precoders as proposed in [4], which are similar to those used by [5] except for an additional power constraint in order to be compatible with our approach developed in Sec. 4.

4.1 ZF-IBI-BDFE

The ZF-IBI-BDFE system in [4] has a structure similar to the one in Fig. 1(b), where a first feedback loop with a filter bank \(G\) aims to cancel IBI. The feed-forward and feedback filter banks are designed to satisfy the ZF requirement \(W_0 H_0 F = B_0 + I\) and the noise-whitening requirement \(W_0 R_y W_0^H = \Sigma\), where \(\Sigma\) is diagonal and \(B_0\) upper-triangular in order to permit sequential detection. Based on the Cholesky decomposition

\[
(H_0 F)^H R_y^{-1} (H_0 F) = A^H \Sigma A
\]

(22)

where \(A\) is upper triangular with a unit diagonal, we have

\[
B_0 = A - I
\]

(23)

\[
W_0 = \Sigma^{-1} A^{-H} (H_0 F)^H R_y^{-1}
\]

(24)

4.2 MMSE-IBI-BDFE

Instead of using a feedback loop to remove ISI, the MMSE-IBI-BDFE system sets \(G(z) = 0\), but has more complex feed-forward and feedback filter banks \(W(z)\) and \(B(z)\) to combat ISI. With \(P \geq L\), the feed-forward filter bank is in [4] set to have three taps \(W_1, W_0, W_1\) and the feedback filter bank two taps, \(B_0\) and \(B_1\). Equation 4 is now replaced by

\[
y(i) = H_v F s(i) + H_1 F s(i - 1) + v(i)
\]

(25)

Assuming the input signal is white with unit variance, we define the following matrices

\[
S = \begin{bmatrix} H_v F & H_1 F \\ 0 & H_2 F & H_3 F \end{bmatrix}
\]

(26)

\[
R_{yy} = \begin{bmatrix} R_{yy} & 0 \\ 0 & R_{yy} & 0 \end{bmatrix}
\]

(27)

\[
R_{yy} = S S^H + R_{yy}
\]

(28)

The tap weights of the feed-forward and feedback filter banks of the IBI-MMSE-BDFE are given by [4]

\[
[W_{-1} W_0 W_1] = [Q_{22} Q_{23}] S^H R_{yy}^{-1}
\]

(29)

\[
B_0 = Q_{22} - I
\]

(30)

\[
B_1 = Q_{23}
\]

(31)

where \(Q_{22}, Q_{23}\) are sub-matrices of the matrix \(Q \in \mathbb{C}^{(M \times 3M)}\) which is derived from the following Cholesky decomposition

\[
I + S^H R_{yy}^{-1} S = Q^H \Sigma Q
\]

(32)

\[
Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ 0_{N \times N} & Q_{22} & Q_{23} \\ 0_{N \times N} & 0_{N \times N} & Q_{33} \end{bmatrix}
\]

(33)

4.3 ZF Optimal Linear Precoder

A ZF optimal linear precoder has been proposed in [4], which is designed in conjunction with a linear equaliser such that the signal-to-noise ratio at the receiver output is maximised. Linear equalisation is however only viable in the absence of IBI, hence we replace the linear equaliser and combine the
precoder with the ZF-IBI-BDFE to form a benchmark for the ZF case. The precoder design is accomplished via the EVD in (2), whereby the ZF linear precoder is given by $F_{0,ZF} = V_N \Phi$, where $\Phi$ is a $N \times N$ diagonal matrix with on-diagonal elements $|\phi_{ii}|^2 = (P_0 / \sum_{k=1}^{M} \lambda_k^{-1}) \lambda_i^{-1}$.

### 4.4 MMSE Optimal Linear Precoder

Similar to the ZF linear precoder, the MMSE linear precoder proposed in [4] is meant to operate with a linear equaliser in the absence of IBI, but here is combined as a locally optimised precoder with the MMSE-IBI-BDFE of Sec. 4.2. From the EVD in (9), the MMSE precoder is given by $F_{0,MMSE} = V_N \Phi$, where the on-diagonal elements of the diagonal $N \times N$ matrix $\Phi$ are

$$|\phi_{ii}|^2 = \max \left( P_0 + \sum_{k=1}^{M} \lambda_k^{-1}, \lambda_i^{-1/2}, \lambda_i^{-1} \right),$$

with $M$ the number of $|\phi_{ii}|^2 > 0$.

### 5. SIMULATION AND RESULT

In order to assess and compare the performance of the proposed design, we consider a channel of order $L = 5$ with coefficients drawn from a complex Gaussian distribution with zero mean and unit variance. With a transmit block length of $P = 18$, the input block length of $N = M = 16$ admits a very limited amount of redundancy that is insufficient to permit the suppression of IBI by design of the precoder/equalisation system. The transmit power is constrained to $P_0 = 10$.

The results in terms of BER performance for QPSK transmission over the proposed jointly optimal system comprising a linear precoder and a BDFE for the ZF and MMSE case are shown in Fig. 2 averaged over 50 randomised channel realisations. Jointly optimised linear precoding and linear equalisation is unsuitable, since linear equalisation such as in [5] offers no capability to combat IBI. Therefore, we have considered the two composite schemes outlined in Sec. 4 as a benchmark:

1. optimal ZF linear precoding [4] combined with ZF-IBI-BDFE [5], and

Although ZF and MMSE precoders are referred to as optimal, these are locally, i.e. with view of the transmitter only, optimised components. Considering the benchmark results in Fig. 3, it is evident that the proposed design can achieve a considerably lower BER performance than the benchmark systems.

### 6. CONCLUSION

In this paper, we have proposed a design which can work with low levels of redundancy where linear block transmission schemes such as in [4] will suffer from inter-block interference. The proposed approach utilises a non-linear block decision feedback equaliser suggested in [5]. We have used this receiver structure to create a jointly optimal design of both precoder and BDFE, overcoming the required absence of IBI in previous work [5].

Simulation results have demonstrated the advantage of the system in terms of BER when compared to a design akin

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**REFERENCES**


