MSE LOWER BOUNDS CONDITIONED BY THE ENERGY DETECTOR

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ABSTRACT

A wide variety of processing incorporates a binary detection test that restricts the set of observations for parameter estimation. This statistical conditioning must be taken into account to compute the Cramér-Rao bound [1] (CRB) and more generally, lower bounds on the Mean Square Error (MSE) [2]. Therefore, we propose a derivation of some lower bounds - including the CRB - for the deterministic signal model conditioned by the energy detector [3] widely used in signal processing applications.

1. INTRODUCTION

Lower bounds on the MSE in estimating a set of deterministic parameters [1] from noisy observations provide the best performance of any estimators in terms of the MSE. They allow to investigate fundamental limits of a parameter estimation problem or to assess the relative performance of a specific estimator. Historically the first MSE lower bound for deterministic parameters to be derived was the CRB [1], which has been the most widely used since. Its popularity is largely due to its simplicity of calculation, the fact that in many cases, it can be achieved asymptotically (high SNR [4] and/or large number of snapshots [1]) by maximum likelihood estimators (MLEs), and last but not least, its noticeable property of being the lowest bound on the MSE of locally unbiased estimators. This initial characterization of locally unbiased estimators has been extended first by Bhattacharyya's work, and significantly generalized by Barankin's work which allows the derivation of the highest lower bound on MSE since it takes into account the unbiasedness over the parameter space [1][2][5][6]. Unfortunately the Barankin bound (BB) is generally incomputable [6]. Numerous works (see references in [2][6] and [7]) devoted to the computing and tightness of bounds on MSE have shown that the CRB and BB can be regarded as key representative of two general classes of bounds, respectively the Small-Error bounds and the Large-Error bounds. Indeed, in non-linear estimation problems three distinct regions of operation can be observed. In the asymptotic region, the MSE of estimators is small and, in many cases, close to the Small-Error bounds. In the *a priori* performance region where the number of independent snapshots and/or the SNR are very low, the observations provide little information and the MSE is close to that obtained from the prior knowledge about the problem. Between these two extremes, there is an additional ambiguity region, also called the transition region. In this region, the MSE of MLEs deteriorates rapidly with respect to Small-Error bounds and generally exhibits a threshold behavior corresponding to a "performance breakdown" [8] highlighted by Large-Error bounds. As a result, the search for an easily computable but tight approximation of the BB is still a subject worth investigation. Therefore, Quinlan-Chaumette-Larzabal [6] have suggested a new approximation (QCLB) of the BB that allows a better prediction of the SNR value at the start of the transition region than existing approximations with a comparable computational complexity (CRB, Hammersley-Chapman-Robbins bound (HCRB), McAulay-Seidman bound (MSB), Abel bound of order 1 (AB_1)).
However, in nearly all fields of science and engineering, a wide variety of processing requires a binary detection step designed to decide if a signal is present or not in noise. As a detection step restricts the set of observations available for parameter estimation, any accurate MSE lower bound must take into account this initial statistical conditioning. As a contribution to the theoretical characterization of the joint detection and estimation problem, we propose in the present paper the derivation of above mentioned approximations of the BB (CRB, HCRB, MSB, AB_1, QCLB) for the deterministic signal model conditioned by the energy detector, which is a simple *realizable* test widely used in signal processing applications [3]. We therefore complete the characterization obtained for the CRB in [9].

2. DETERMINISTIC SIGNAL AND ENERGY DETECTOR

In many practical problems of interest, the received data samples is a vector $\mathbf{x}$ consisting of a bandpass signal that can be modelled as a mixture of a complex signal $\mathbf{s}_{\boldsymbol{\theta}}$ and a complex circular zero mean Gaussian noise $\mathbf{n}$: $\mathbf{x} = \mathbf{s}_{\boldsymbol{\theta}} + \mathbf{n}$. We consider the case where the signal of interest $\mathbf{s}_{\boldsymbol{\theta}}$ is dependent upon the vector of unknown deterministic parameters $\boldsymbol{\theta}$. The noise covariance matrix $\mathbf{C_n}$ does not depend upon $\boldsymbol{\theta}$. Therefore $\mathbf{x} \sim \mathcal{CN}_L(\mathbf{m_x}, \mathbf{C_x})$, i.e. is complex circular Gaussian of dimension L with mean $\mathbf{m_x} = \mathbf{s}_{\boldsymbol{\theta}}$ and covariance matrix $\mathbf{C_x}$ ($\mathbf{C_x} = \mathbf{C_n}$), with p.d.f. [3, §13]:

$$f_{\boldsymbol{\theta}}(\mathbf{x}) = f_{\mathcal{CN}_L}(\mathbf{x}; \mathbf{m_x}(\boldsymbol{\theta}), \mathbf{C_x}) = \frac{e^{-(\mathbf{x}-\mathbf{s}_{\boldsymbol{\theta}})^H \mathbf{C_x}^{-1}(\mathbf{x}-\mathbf{s}_{\boldsymbol{\theta}})}}{\pi^L |\mathbf{C_x}|} \quad (1)$$

In practical problems, the signal of interest $\mathbf{s}_{\boldsymbol{\theta}}$ is not always present. Such problems require first a binary detection step (decision rule) to decide if the signal of interest $\mathbf{s}_{\boldsymbol{\theta}}$ is present or not in the noise before running an estimation scheme [2]. Let us recall that optimal decision rules are based on the exact statistics of the observations [3, §3]. Their expressions require knowledge of the p.d.f. of observations under each hypothesis and the *a priori* probability of each hypothesis, if known (Bayes criterion). If no *a priori* probability of hypotheses is available, then the likelihood ratio test (LRT) is often used for binary hypothesis testing. Unfortunately these optimal detection tests are generally not *realizable* since they almost always depend at least on one of the unknown parameters $\boldsymbol{\theta}$. The LRTs are intended for providing the best attainable performance of any decision rule for a given problem [3, §3]. Therefore, a common approach to designing *realizable* tests is to replace the unknown parameters by estimates, the detection problem becoming a composite hypothesis testing problem (CHTP) [3, §6]. Although not necessarily optimal for detection performance, the estimates are generally chosen in the maximum likelihood sense, thereby obtaining the generalized likelihood ratio test (GLRT). If $\mathbf{C}_{\mathbf{x}}$ is known and $\mathbf{s}_{\boldsymbol{\theta}}$ supposed to be completely unknown, then the GLRT reduces to the energy detector [3, §7.3]:

$$\left\|\mathbf{W}_{\mathbf{x}}^{-1}\mathbf{x}\right\|^2 = \mathbf{x}^H\mathbf{C}_{\mathbf{x}}^{-1}\mathbf{x} \gtrless T, \quad \mathbf{C}_{\mathbf{x}} = \mathbf{W}_{\mathbf{x}}\mathbf{W}_{\mathbf{x}}^H \tag{2}$$

where T is the detection threshold. It is a simple practical *realizable* detection test that can be used in any application. Additionally from a theoretical standpoint, one can expect the detection performance of the GLRT derived from the parametric model of $\mathbf{s}_{\boldsymbol{\theta}}$ to be somewhere between that of the Neyman-Pearson detector and the energy detector [3, §7.3].

3. BACKGROUND ON THE QCLB

The general approach lately introduced in [6] allows to revisit existing bounds by exploring the unbiasedness assumptions, from its *weakest* formulation (CRB) to its *strongest* formulation (BB). This approach has suggested a new approximation (QCLB) of the BB that allows a better prediction of the SNR threshold value than existing approximations (CRB, HCRB, MSB, AB_1), with a comparable computational complexity. Indeed, all mentioned lower bounds can be computed via the QCLB. This versatility will be used in §4 to take into account the detection test. For the sake of simplicity, we focus on the estimation of a single real function $g(\theta)$ of a single unknown real deterministic parameter θ. Ω denotes the observation space, Θ the parameter space, $\mathbb{F}_\Omega$ the real vector space of square integrable functions over Ω and $f_\theta(\mathbf{x})$ the p.d.f. of observations. A fundamental property of the MSE of a particular estimator $\widehat{g(\theta_0)}(\mathbf{x}) \in \mathbb{F}_\Omega$ of $g(\theta_0)$, where θ_0 is a selected value of the parameter θ, is that it is a norm associated with a particular scalar product $\langle \mid \rangle_\theta$:

$$MSE_{\theta_0}\left[\widehat{g(\theta_0)}\right] = \left\|\widehat{g(\theta_0)}(\mathbf{x}) - g(\theta_0)\right\|_{\theta_0}^2$$

where:

$$\begin{aligned}\langle g(\mathbf{x}) \mid h(\mathbf{x})\rangle_{\theta_0} &= E_{\theta_0}\left[g(\mathbf{x})\,h(\mathbf{x})\right] \\ &= \int_\Omega g(\mathbf{x})\,h(\mathbf{x})\,f_{\theta_0}(\mathbf{x})\,d\mathbf{x}\end{aligned}$$

In the search for a lower bound on the MSE, this property allows the use of two equivalent fundamental results: the generalization of the Cauchy-Schwartz inequality to Gram matrices (generally referred to as the "covariance inequality") and the minimization of a norm under linear constraints introduced hereinafter. Let $\mathbb{U}$ be an Euclidean vector space of any dimension (finite or infinite) on the body of real numbers $\mathbb{R}$ which has a scalar product $\langle \mid \rangle$. Let $(\mathbf{c}_1, \ldots, \mathbf{c}_K)$ be a free family of K vectors of $\mathbb{U}$ and $\mathbf{v} = (v_1, \ldots, v_K)^T$ a vector of $\mathbb{R}^K$. The problem of the minimization of $\|\mathbf{u}\|^2$ under the K linear constraints $\langle \mathbf{u} \mid \mathbf{c}_k\rangle = v_k$, $k \in [1, K]$ then has the solution:

$$\begin{aligned}\min\left\{\|\mathbf{u}\|^2\right\} &= \mathbf{v}^T\mathbf{G}^{-1}\mathbf{v} \text{ for } \mathbf{u}_{opt} = \sum_{k=1}^{K}\alpha_k\mathbf{c}_k \\ (\alpha_1, \ldots, \alpha_K)^T &= \boldsymbol{\alpha} = \mathbf{G}^{-1}\mathbf{v},\ \mathbf{G}_{n,k} = \langle \mathbf{c}_k \mid \mathbf{c}_n\rangle\end{aligned} \tag{3}$$

As formulated by Barankin [5], the ultimate constraint that an unbiased estimator $\widehat{g(\theta_0)}(\mathbf{x})$ of $g(\theta_0)$ should verify is to be unbiased for all possible values of the unknown parameter:

$$E_\theta\left[\widehat{g(\theta_0)}(\mathbf{x})\right] = g(\theta),\ \forall\theta \in \Theta \tag{4}$$

In this case the problem of interest becomes:

$$\min\left\{MSE_{\theta_0}\left[\widehat{g(\theta_0)}\right]\right\} \text{ under } E_\theta\left[\widehat{g(\theta_0)}(\mathbf{x})\right] = g(\theta), \tag{5}$$

$\forall\theta \in \Theta$ and corresponds to the search for the locally-best unbiased estimator. Unfortunately, it is generally impossible to find an analytical solution of (5) providing the BB. Nevertheless the BB can be approximated by discretization of Barankin unbiasedness constraint (4). A general approach introduced in [6] consists in partitioning the parameter space Θ in N real sub-intervals $I_n = [\theta_n, \theta_{n+1}[$ where (4) is piecewise approximated by the constraints, $\theta_n + d\theta \in I_n$:

$$E_{\theta_n+d\theta}\left[\widehat{g(\theta_0)}(\mathbf{x})\right] = g(\theta_n + d\theta) + o\left(d\theta^{L_n}\right) \tag{6}$$

Provided that both $f_\theta(\mathbf{x})$ and $g(\theta)$ can be developed in piecewise series expansions of order L_n, then $\min\left\{MSE_{\theta_0}\left[\widehat{g(\theta_0)}\right]\right\}$ under (6) is easily obtained using (3) [6]. Designating the BB approximations obtained as N-piecewise BB approximation of *homogeneous* order L, if on all sub-intervals I_n the series expansions are of the same order L, and of *heterogeneous* orders $\{L_1, ..., L_N\}$ if otherwise, this approach suggests a straightforward practical BB approximation: the QCLB based on a $N+1$-piecewise BB approximation of *homogeneous* order 1 defined by the constraints:

- $E_{\theta_n+d\theta}\left[\widehat{g(\theta_0)}(\mathbf{x})\right]=g(\theta_n+d\theta)+o(d\theta),\theta_n+d\theta\in I_n$

The QCLB is therefore a generalization of the CRB based on a 1-piecewise BB approximation of *homogeneous* order 1:

- $E_{\theta_0+d\theta}\left[\widehat{g(\theta_0)}(\mathbf{x})\right]=g(\theta_0+d\theta)+o(d\theta),\theta_0+d\theta\in\Theta$

is as well a generalization of the usual BB approximation used in the open literature, i.e. the MSB, based on an $N+1$-piecewise BB approximation of *homogeneous* order 0:

- $E_{\theta_n+d\theta}\left[\widehat{g(\theta_0)}(\mathbf{x})\right]=g(\theta_n+d\theta)+O(d\theta),\theta_n+d\theta\in I_n$

and a generalization of the AB_1 based on a $N+1$-piecewise BB approximation of *heterogeneous* order $\{1,0,...,0\}$:

- $\left\{\begin{array}{l} E_{\theta_0+d\theta}\left[\widehat{g(\theta_0)}(\mathbf{x})\right]=g(\theta_0+d\theta)+o(d\theta) \\ E_{\theta_n+d\theta}\left[\widehat{g(\theta_0)}(\mathbf{x})\right]=g(\theta_n+d\theta)+O(d\theta)\end{array}\right.$

where $\theta_0+d\theta\in I_0,\ \ \theta_n+d\theta\in I_{n>1}$.

For any set of $N+1$ test points $\{\theta_n\}_{[1,N+1]}=\{\theta_0\}\cup\{\theta_n\}_{[1,N]}$ (or set of $N+1$ sub-intervals I_n), the QCLB verify $QCLB\geq AB_1\geq\max\{MSB,CRB\}$ and is given by:

$$QCLB=\mathbf{v}^T\left[\begin{array}{cc}\mathbf{MS} & \mathbf{C}\\ \mathbf{C}^T & \mathbf{EFI}\end{array}\right]^{-1}\mathbf{v} \tag{7}$$

where:

$$\begin{aligned}
\mathbf{v} &= \left(\Delta\mathbf{g}^T,\left(\ldots,\frac{\partial g(\theta_n)}{\partial\theta},\ldots\right)\right)^T\\
\Delta\mathbf{g}^T &= (\ldots,g(\theta_n)-g(\theta_0),\ldots)\\
\mathbf{MS}_{n,l} &= E_{\theta_0}\left[\frac{f_{\theta_n}(\mathbf{x})\,f_{\theta_l}(\mathbf{x})}{f_{\theta_0}(\mathbf{x})^2}\right]\\
\mathbf{C}_{n,l} &= E_{\theta_0}\left[\frac{\partial\ln f_{\theta_l}(\mathbf{x})}{\partial\theta}\frac{f_{\theta_n}(\mathbf{x})\,f_{\theta_l}(\mathbf{x})}{f_{\theta_0}(\mathbf{x})^2}\right]\\
\mathbf{EFI}_{n,l} &= E_{\theta_0}\left[\frac{\partial\ln f_{\theta_n}(\mathbf{x})}{\partial\theta}\frac{\partial\ln f_{\theta_l}(\mathbf{x})}{\partial\theta}\frac{f_{\theta_n}(\mathbf{x})\,f_{\theta_l}(\mathbf{x})}{f_{\theta_0}(\mathbf{x})^2}\right]
\end{aligned}$$

MS is the Mac-Aulay Seidman matrix, **EFI** stands for the Extended Fisher Information matrix, as it reduces to the FI (Fisher Information) when the set of test points is reduced to θ_0 only. **C** is a kind of "hybrid" matrix. An immediate generalization consists of taking their supremum over sub-interval definitions (set of test points).

4. CONDITIONAL LOWER BOUNDS

In this section, we provide an extension of QCLB analytical expression - and therefore of the CRB, HCRB, MSB and AB_1- by taking into account the energy detector. Indeed, if $\mathcal{D}$ is a *realizable* conditioning event, conditional bounds are obtained by substituting $\mathcal{D}$ and $f_\theta(\mathbf{x}\mid\mathcal{D})$ for Ω and $f_\theta(\mathbf{x})$ in the various expressions [2]:

$$\mathbf{MS}_{n,l}=E_{\theta_0}\left[\frac{f_{\theta_n}(\mathbf{x}\mid\mathcal{D})\,f_{\theta_l}(\mathbf{x}\mid\mathcal{D})}{f_{\theta_0}(\mathbf{x}\mid\mathcal{D})^2}\mid\mathcal{D}\right]$$

$$\mathbf{C}_{n,l}=E_{\theta_0}\left[\frac{\partial\ln f_{\theta_l}(\mathbf{x}\mid\mathcal{D})}{\partial\theta}\frac{f_{\theta_n}(\mathbf{x}\mid D)\,f_{\theta_l}(\mathbf{x}\mid\mathcal{D})}{f_{\theta_0}(\mathbf{x}\mid D)^2}\mid\mathcal{D}\right]$$

$$\mathbf{EFI}_{n,l}=E_{\theta_0}\left[\frac{\partial\ln f_{\theta_n}(\mathbf{x}\mid\mathcal{D})}{\partial\theta}\frac{\partial\ln f_{\theta_l}(\mathbf{x}\mid\mathcal{D})}{\partial\theta}\right.\left.\frac{f_{\theta_n}(\mathbf{x}\mid\mathcal{D})\,f_{\theta_l}(\mathbf{x}\mid\mathcal{D})}{f_{\theta_0}(\mathbf{x}\mid\mathcal{D})^2}\mid\mathcal{D}\right]$$

If $f_{\boldsymbol\theta}(\mathbf{x})$ is given by (1) and $\mathcal{D}=\left\{\mathbf{x}\mid\mathbf{x}^H\mathbf{C}_\mathbf{x}^{-1}\mathbf{x}\geq T\right\}$ is the event of the energy detector (2), then [9]:

$$P_\mathcal{D}(\mathbf{s}_\theta)=\int_\mathcal{D}f_\theta(\mathbf{x})\,d\mathbf{x}=\int_{t\geq T}f_{\mathcal{X}^2_{2L}}\left(t;\mathbf{s}_\theta^H\mathbf{C}_\mathbf{x}^{-1}\mathbf{s}_\theta\right)dt \tag{8}$$

where $f_{\mathcal{X}^2_{2L}}(t;\lambda)$ is the p.d.f. of a non central chi-squared random variable with $2L$ degrees of freedom and noncentrality parameter λ:

$$f_{\mathcal{X}^2_{2L}}(t;\lambda)=e^{-(t+\lambda)}I_{L-1}\left(2\sqrt{\lambda t}\right)\left(\sqrt{\frac{t}{\lambda}}\right)^{(L-1)} \tag{9}$$

$I_L(z)$ being the modified Bessel functions of the first kind [3, p 26]. Then a few lines of algebra leads to:

$$\begin{aligned}
\frac{f_{\theta_n}(\mathbf{x}\mid\mathcal{D})\,f_{\theta_l}(\mathbf{x}\mid\mathcal{D})}{f_{\theta_0}(\mathbf{x}\mid\mathcal{D})} &= (\mathbf{MS}_{n,l})\,f_{\mathcal{CN}_L}(\mathbf{x}\mid\mathcal{D};\mathbf{m}_\mathbf{x},\mathbf{C}_\mathbf{x})\\
\mathbf{m}_\mathbf{x} &= \mathbf{s}_{\theta_n}+\mathbf{s}_{\theta_l}-\mathbf{s}_{\theta_0}
\end{aligned}$$

$$\mathbf{MS}_{n,l}=e^{2\operatorname{Re}\left\{\left(\mathbf{s}_{\theta_n}-\mathbf{s}_{\theta_0}\right)^H\mathbf{C}_\mathbf{x}^{-1}\left(\mathbf{s}_{\theta_l}-\mathbf{s}_{\theta_0}\right)\right\}}\frac{P_\mathcal{D}(\mathbf{s}_{\theta_0})\,P_\mathcal{D}(\mathbf{s}_{\theta_n}+\mathbf{s}_{\theta_l}-\mathbf{s}_{\theta_0})}{P_\mathcal{D}(\mathbf{s}_{\theta_n})\,P_\mathcal{D}(\mathbf{s}_{\theta_l})} \tag{10}$$

Let us denote $E[\mathbf{x}\mid\mathcal{D}]=\int_\mathcal{D}\mathbf{x}f_{\mathcal{CN}_L}(\mathbf{x}\mid\mathcal{D};\mathbf{m}_\mathbf{x},\mathbf{C}_\mathbf{x})\,d\mathbf{x}$.

Since $\frac{\partial\ln f_\theta(\mathbf{x}\mid\mathcal{D})}{\partial\theta}=2\operatorname{Re}\left\{\frac{\partial\mathbf{s}_\theta^H}{\partial\theta}\mathbf{C}_\mathbf{x}^{-1}(\mathbf{x}-\mathbf{s}_\theta)\right\}-\frac{\partial\ln P_\mathcal{D}(\mathbf{s}_\theta)}{\partial\theta}$, then:

$$\begin{aligned}
\mathbf{EFI}_{n,l} &= (\mathbf{MS}_{n,l})\,E\left[\frac{\partial\ln f_{\theta_n}(\mathbf{x}\mid\mathcal{D})}{\partial\theta}\frac{\partial\ln f_{\theta_l}(\mathbf{x}\mid\mathcal{D})}{\partial\theta}\mid\mathcal{D}\right]\\
\mathbf{EFI}_{n,l} &= (\mathbf{MS}_{n,l})\left[2\operatorname{Re}\left\{\frac{\partial\mathbf{s}_{\theta_n}^H}{\partial\theta}\mathbf{C}_\mathbf{x}^{-1}\mathbf{A}_{n,l}\mathbf{C}_\mathbf{x}^{-1}\frac{\partial\mathbf{s}_{\theta_l}}{\partial\theta}\right\}\right.\\
&\quad+2\operatorname{Re}\left\{\frac{\partial\mathbf{s}_{\theta_n}^H}{\partial\theta}\mathbf{C}_\mathbf{x}^{-1}\mathbf{B}_{n,l}\left(\mathbf{C}_\mathbf{x}^{-1}\right)^T\frac{\partial\mathbf{s}_{\theta_l}^*}{\partial\theta}\right\}\\
&\quad+\frac{\partial\ln P_\mathcal{D}(\mathbf{s}_{\theta_n})}{\partial\theta}\frac{\partial\ln P_\mathcal{D}(\mathbf{s}_{\theta_l})}{\partial\theta}\\
&\quad-2\frac{\partial\ln P_\mathcal{D}(\mathbf{s}_{\theta_l})}{\partial\theta}\operatorname{Re}\left\{\frac{\partial\mathbf{s}_{\theta_n}^H}{\partial\theta}\mathbf{C}_\mathbf{x}^{-1}\left(E[\mathbf{x}\mid\mathcal{D}]-\mathbf{s}_{\theta_n}\right)\right\}\\
&\quad\left.-2\frac{\partial\ln P_\mathcal{D}(\mathbf{s}_{\theta_n})}{\partial\theta}\operatorname{Re}\left\{\frac{\partial\mathbf{s}_{\theta_l}^H}{\partial\theta}\mathbf{C}_\mathbf{x}^{-1}\left(E[\mathbf{x}\mid\mathcal{D}]-\mathbf{s}_{\theta_l}\right)\right\}\right]
\end{aligned} \tag{11}$$

$$\mathbf{C}_{n,l} = (\mathbf{MS}_{n,l})\, E\left[\frac{\partial \ln f_{\theta_l}(\mathbf{x} \mid \mathcal{D})}{\partial \theta} \mid \mathcal{D}\right]$$

$$\mathbf{C}_{n,l} = (\mathbf{MS}_{n,l}) \left[2\,\mathrm{Re}\left\{ \frac{\partial \mathbf{s}_{\theta_l}^H}{\partial \theta} \mathbf{C}_{\mathbf{x}}^{-1} \left(E[\mathbf{x} \mid \mathcal{D}] - \mathbf{s}_{\theta_l} \right) \right\} - \frac{\partial \ln P_{\mathcal{D}}(\mathbf{s}_{\theta_l})}{\partial \theta} \right] \quad (12)$$

where:

$$\begin{aligned} \mathbf{A}_{n,l} &= E\left[(\mathbf{x} - \mathbf{s}_{\theta_l})(\mathbf{x} - \mathbf{s}_{\theta_n})^H \mid \mathcal{D} \right] \\ &= E\left[\mathbf{x}\mathbf{x}^H \mid \mathcal{D}\right] - E[\mathbf{x} \mid \mathcal{D}]\, \mathbf{s}_{\theta_n}^H - \mathbf{s}_{\theta_l} E[\mathbf{x} \mid \mathcal{D}]^H \\ &\quad + \mathbf{s}_{\theta_l}\mathbf{s}_{\theta_n}^H \\ \mathbf{B}_{n,l} &= E\left[(\mathbf{x} - \mathbf{s}_{\theta_l})(\mathbf{x} - \mathbf{s}_{\theta_n})^T \mid \mathcal{D} \right] \\ &= E\left[\mathbf{x}\mathbf{x}^T \mid \mathcal{D}\right] - E[\mathbf{x} \mid \mathcal{D}]\, \mathbf{s}_{\theta_n}^T - \mathbf{s}_{\theta_l} E[\mathbf{x} \mid \mathcal{D}]^T \\ &\quad + \mathbf{s}_{\theta_l}\mathbf{s}_{\theta_n}^T \end{aligned}$$

and [9]:

$$E[\mathbf{x} \mid \mathcal{D}] = \frac{1 - P_{L+1}(\mathbf{m_x})}{1 - P_L(\mathbf{m_x})} \mathbf{m_x}$$

$$E\left[\mathbf{x}\mathbf{x}^H \mid \mathcal{D}\right] = \frac{1 - P_{L+1}(\mathbf{m_x})}{1 - P_L(\mathbf{m_x})} \mathbf{C_x} + \frac{1 - P_{L+2}(\mathbf{m_x})}{1 - P_L(\mathbf{m_x})} \mathbf{m_x}\mathbf{m_x}^H$$

$$E\left[\mathbf{x}\mathbf{x}^T \mid \mathcal{D}\right] = \frac{1 - P_{L+2}(\mathbf{m_x})}{1 - P_L(\mathbf{m_x})} \mathbf{m_x}\mathbf{m_x}^T$$

$$\frac{\partial \ln P_{\mathcal{D}}(\mathbf{s}_\theta)}{\partial \theta} = \left(\frac{P_L(\mathbf{s}_\theta) - P_{L+1}(\mathbf{s}_\theta)}{1 - P_L(\mathbf{s}_\theta)} \right) \frac{\partial \left(\mathbf{s}_\theta^H \mathbf{C}_{\mathbf{x}}^{-1} \mathbf{s}_\theta \right)}{\partial \theta}$$

$$P_{\mathcal{D}}(\mathbf{s}_\theta) = 1 - P_L(\mathbf{s}_\theta)$$

$$P_L(\mathbf{s}) = \int_0^T f_{\mathcal{X}_{2L}^2}\left(t; \mathbf{s}_\theta^H \mathbf{C}_{\mathbf{x}}^{-1} \mathbf{s}_\theta\right) dt$$

Finally the conditional QCLB is given by (7) computed according to (10)(11)(12) and the conditional MSB, AB_1, CRB are given by:

$$MSB = \Delta\mathbf{g}^T [\mathbf{MS}]^{-1} \Delta\mathbf{g}$$

$$AB_1 = \mathbf{v}^T \begin{bmatrix} \mathbf{MS} & \mathbf{c} \\ \mathbf{c}^T & \mathbf{EFI}_{0,0} \end{bmatrix}^{-1} \mathbf{v}, \left\{ \begin{array}{l} \mathbf{c} = (\ldots, \mathbf{C}_{n,0}, \ldots)^T \\ \mathbf{v} = \left(\Delta\mathbf{g}^T, \frac{\partial g(\theta_0)}{\partial \theta} \right)^T \end{array} \right.$$

$$CRB = \frac{\partial g(\theta_0)}{\partial \theta} [\mathbf{EFI}_{0,0}]^{-1} \frac{\partial g(\theta_0)}{\partial \theta}$$

where [9]:

$$\mathbf{EFI}_{0,0} = 2\,\mathrm{Re}\left\{ \frac{\partial \mathbf{s}_{\theta_0}^H}{\partial \theta} \mathbf{C}_{\mathbf{x}}^{-1} \frac{\partial \mathbf{s}_{\theta_0}}{\partial \theta} \right\} \left(\frac{1 - P_{L+1}(\theta_0)}{1 - P_L(\theta_0)} \right) + w_L(\theta_0) \left(\frac{\partial \left(\mathbf{s}_{\theta_0}^H \mathbf{C}_{\mathbf{x}}^{-1} \mathbf{s}_{\theta_0} \right)}{\partial \theta} \right)^2$$

$$w_L(\theta) = \frac{2P_{L+1}(\theta) - P_L(\theta) - P_{L+2}(\theta)}{1 - P_L(\theta)} - \left(\frac{P_{L+1}(\theta) - P_L(\theta)}{1 - P_L(\theta)} \right)^2$$

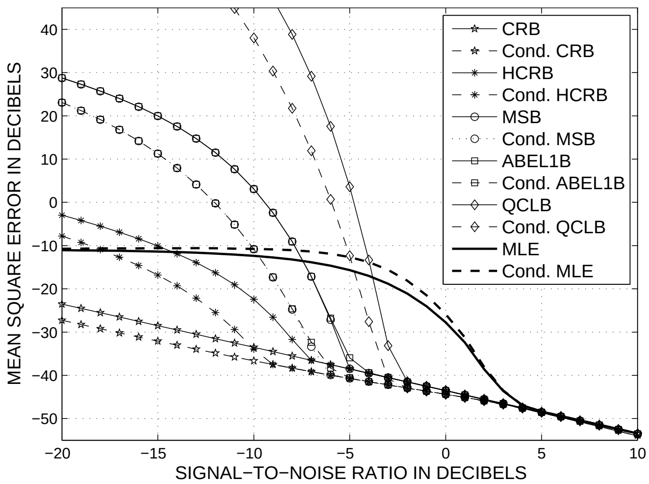

Figure 1: MSE of MLE and MSE Lower Bounds conditioned or not by the Energy Detector versus SNR, $L = 10$, $P_{FA} = 10^{-3}$

5. SINGLE TONE THRESHOLD ANALYSIS

Let us consider the reference estimation problem where the vector $\mathbf{x}$ is modelled by:

$$\begin{aligned} \mathbf{x} &= a\boldsymbol{\psi}(\theta) + \mathbf{n} \\ \boldsymbol{\psi}(\theta) &= \left[1, e^{j2\pi\theta}, \ldots, e^{j2\pi(L-1)\theta}\right]^T, \ \theta \in \left]-0.5, 0.5\right[\end{aligned}$$

i.e. $\mathbf{s}_\theta = a\boldsymbol{\psi}(\theta)$ and $\mathbf{C_x} = \mathbf{Id}$, a^2 being the SNR ($a > 0$). Then $\frac{\partial \ln P_{\mathcal{D}}(\mathbf{s}_\theta)}{\partial \theta} = 0$ and $\widehat{\theta}_{ML} = \max_\theta \left\{ \mathrm{Re}\left[\boldsymbol{\psi}(\theta)^H \mathbf{x} \right] \right\}$

For any set of $N+1$ test points $\{\theta_n\}_{[1,N+1]}$, only the MSB, the AB_1 and the QCLB are of a comparable complexity. Nevertheless, we also include in the comparison the HCRB as it is the simplest representative of Large Errors bounds. For the sake of fair comparison with the HCRB which is the supremum of the MSB where $\{\theta_n\}_{[1,2]} = \{\theta_0, \theta_0 + d\theta\}$, the MSB, AB_1, QCLB are also computed as supremum over the possible values of $\{\theta_n\}_{[1,N+1]}$. For the sake of simplicity $\{\theta_n\}_{[1,3]} = \{\theta_0, \theta_0 + d\theta, \theta_0 - d\theta\}$. We consider the reference estimation case where $\theta_0 = 0$.

Figure (1) compares the various bounds, conditioned or not by the Energy Detector, as a function of SNR in the case of $L = 10$ samples and $P_{FA} = 10^{-3}$. The MSE of the MLE is also shown in order to compare the threshold behaviour of the bounds (10^6 trials). As expected, the QCLB keeps providing a significant improvement in the prediction of the SNR threshold value, whatever the observations are conditioned or not (same results can be observed for $L = 2, 4, \ldots, 32$ and $P_{FA} = 10^{-1}, 10^{-2}, \ldots, 10^{-6}$).

A more unexpected and non intuitive result is the increase of the MSE of the MLE in the transition region as the detection threshold increases (as the P_{FA} decreases) highlighted by figure (2). Indeed, intuitively, a detection step is expected to decrease the MSE of the MLE by selecting instances with relatively high signal

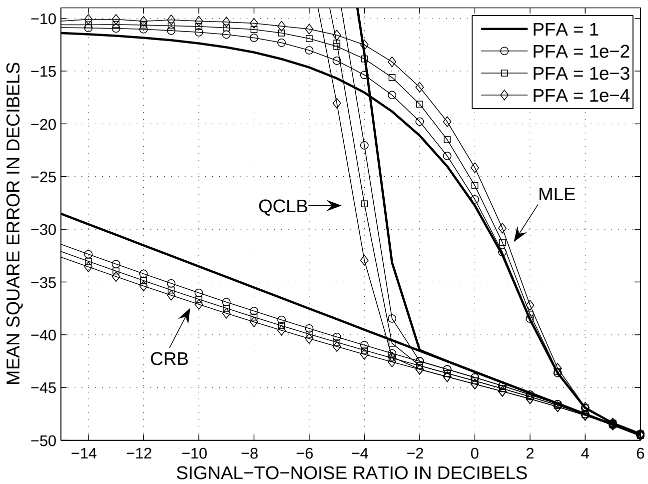

Figure 2: MSE of MLE, CRB and QCLB conditioned or not by the Energy Detector versus SNR, $L = 10$, $P_{FA} = 10^{-2}, 10^{-3}, 10^{-4}$

energy - sufficient to exceed the detection threshold - and disregarding instances belonging to the *a priori* region that deteriorate the MSE. The former analysis is reinforced theoretically by the lower bounds behavior (CRB and QCLB) in figure (2) and has also been reinforced so far practically by results obtained in [2] for the monopulse ratio estimation problem under a stochastic signal model. Again, if we consider the stochastic case, i.e. $a \sim \mathcal{CN}_1(0, snr)$, then $\widehat{\theta}_{ML} = \max_{\theta} \left\{ \left| \psi(\theta)^H \mathbf{x} \right|^2 \right\}$ and one can check that the behavior of its MSE is the opposite and true to the common intuition.

This paradoxical result clearly addresses a challenging theoretical issue that will have to be the subject of further research.

6. CONCLUSION

In the present paper, we have derived lower bounds on MSE (CRB, HCRB, MSB, AB_1, QCLB) for the deterministic signal model conditioned by the Energy Detector. This results will be useful to update the estimation performance analysis for a wide variety of processing including the Energy Detector. Additionally, we have shown that the QCLB keeps providing a significant improvement in the prediction of the SNR threshold value when the observations are conditioned, in comparison with the MSB (the usual BB approximation in the open literature [7]).

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