ABOUT MAGNITUDE INPUT DATA IN 1-D DISCRETE PHASE RETRIEVAL PROBLEM

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ABSTRACT

In this paper we consider 1-D (one dimensional) phase retrieval problem from the point of view of magnitude input data. We claim that magnitude input data should satisfy certain requirements in order to provide the acceptable minimum-phase solution. The Fejér-Riesz Theorem guarantees us that 1-D discrete phase retrieval problem has always a solution if the trigonometric polynomial is positive definite, but an arbitrary set of magnitudes does not provide always a positive definite trigonometric polynomial. Sometimes this may be the reason for iterative methods to stagnate or for direct methods to give undesired results. Finally we discuss a criterium to decide whether a set of magnitude input data can solve the 1-D phase retrieval problem.

1. INTRODUCTION

For both continuous-time and discrete-time signals, the magnitude and phase of the Fourier transform are, in general, independent functions , i.e., the signal cannot be recovered from knowledge of either alone [1]. Since the recovering problem does not have a unique solution in general, researchers have tried many ways by providing information of the signal *a priori* or constraining the properties of the signal [2]. Nevertheless, in certain cases relationships may exist between these components leading to certain signal reconstruction methods using only partial information in frequency domain [3]. The condition under which such signal reconstruction problems have unique answer is known [4]. Often these solutions are closed form expressions in terms of the given partial knowledge, but they still may be computationally intractable.

Signal reconstruction from Fourier transform magnitude has been called phase retrieval [5]. The term comes from the fact that the Fourier phase is not known and the signal should be reconstructed [6]. As an example, in a Fourier transform coding system, both the magnitude and the phase are usually coded and transmitted. However, for signals which can be recovered from only the magnitude, unnecessary redundancy is inherent in the coder. Therefore for these signals it may be possible to realize a significant bit-rate reduction by simply coding the magnitude and then reconstructing the sequence at the receiver from the coded magnitude [7]. Besides coding, the phase retrieval problem has attracted considerable interest in recent years because of its importance in a variety of applications, including optical astronomy, microscopy [8], Fourier-transform spectroscopy, x-ray crystallography, particle scattering, speckle interferometry, lens testing, single-side communication, and design of radar signals [9].

In order to find the minimum-phase solution of 1-D phase retrieval problem, the most common approaches are iterative transform algorithms [3], which alternate between time and frequency domains. This type of algorithms can implement very easily time-domain constraints like compactness of the support. It has been observed that these algorithms fail to converge to a solution as they usually stagnate [10]. Alternative for solving the 1-D phase retrieval problem are: finding the zeros of z-transform [11], Hilbert transform [12, 13], computation of cepstral coefficients [14, 15] or solving linear systems of equations [6].

In this paper we consider 1-D (one dimensional) phase retrieval problem from the point of view of magnitude input data. We claim that magnitude input data should satisfy certain requirements in order to provide the acceptable minimum-phase solution. The Fejér-Riesz Theorem guarantees us that 1-D discrete phase retrieval problem has always a solution if the trigonometric polynomial is positive definite, but an arbitrary set of magnitudes does not provide always a positive definite trigonometric polynomial. Sometimes this may be the reason for iterative methods to stagnate or for direct methods to give undesired results.

First we recall 1-D discrete phase retrieval problem (Section 2), the sampling conditions and the ambiguities of 1-D phase retrieval problem. The Fejér-Riesz Theorem and its implication in the subject are presented in Section 3. The conditions which should be satisfied by magnitude input data are discussed in Section 4. Experimental results are also shown.

We shall use the following notations:

z^*	complex conjugate of z
$r(n)$	autocorrelation of $x(n)$
$\tilde{r}(n)$	circular autocorrelation of $x(n)$
$X(z)$	z-transform of $x(n)$
$X(\omega) = \mathscr{F}\{x(n)\}$	Fourier transform of $x(n)$
$X(k)$	DFT of $x(n)$
$S(z)$	z-transform of $r(n)$
$\tilde{S}(z)$	z-transform of $\tilde{r}(n)$
$S(\omega)$	Fourier transform of $r(n)$
$\tilde{S}(\omega)$	Fourier transform of $\tilde{r}(n)$

We note that nonminimum-phase phase retrieval is an interesting subject, but in this work we shall focus only on minimum-phase phase retrieval.

2. 1-D DISCRETE PHASE RETRIEVAL PROBLEM

Although certain constraints may be added according to application, the main 1-D discrete phase retrieval problem can be stated as follows [6]:

Let $x(n)$ be a discrete signal of length N and let $X(k)$ be its N-point Discrete Fourier Transform (DFT):

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}, \quad k = 0,1,\ldots,N-1. \tag{1}$$

Given knowledge that only M consecutive values of $x(n)$ differ from zero, i.e. $x(n)$ has M-point support, and given the values of the DFT magnitudes $|X(k)|$, $k = 0,1,\ldots,N-1$, determine $x(n)$ or equivalently $X(k)$.

Certain constraints should be imposed on the type of signal $x(n)$, otherwise a zero-phase or random phase associated with given magnitudes can provide a valid signal $x(n)$.

2.1 Minimum-phase functions

Basically, a sequence is not uniquely defined by its magnitude, as is illustrated by the observation that a sequence convolved with any all-pass sequence will produce another sequence with the same magnitude [7]. Thus, without some assumptions about the sequence, the magnitude may, at best, uniquely specify a sequence to within an arbitrary all-pass factor. However, if some additional knowledge is available and under certain conditions the sequence may be uniquely defined by its magnitude.

Traditionally real and imaginary parts of Fourier transforms $X(\omega)$ are related each other when signal is causal. For the case of finite-length sequences where DFT is usually implemented to compute the spectrum, this leads to the concept of "causal" periodic sequence [12], i.e. a sequence which is zero on the second half. In such situation the number of constraints in time-domain equalizes the number of relationships between real and imaginary parts.

When all zeros of the finite length sequence are within the unit circle, the sequence is minimum-phase and thus it is uniquely defined to within a shifting factor by its magnitude. In this case in order to compute phase from magnitude, we shall focus on the logarithm of Fourier transform $\ln X(\omega)$ [11] which is a Hilbert pair with phase.

2.2 Sampling requirements

The 1-D discrete phase retrieval problem deals with sequences having finite length and finite length spectrum. Thus the z-transform of involved sequences are always polynomials. In order to satisfy the finite support requirements, any kind of time or frequency aliasing has been properly avoided previously. Note that it is imperative to know the support or some bound of the support, otherwise one cannot specify the sampling requirements of Fourier transform magnitude [8].

Let consider the signal with M-point support. Since its autocorrelation function has the support

$$[-(M-1), M-1]$$

the sampling of the Fourier transform magnitude at $\omega_k = \frac{2\pi k}{N}$, with

$$N \geq 2M-1, \tag{2}$$

will be sufficient to extract autocorrelation without time-domain aliasing. It follows that if the support of $x(n)$ does not satisfy (2), we have an ill-posed problem. Indeed, the set of squares of the DFT magnitudes is the DFT of the circular autocorrelation $\tilde{r}(n)$ of $x(n)$:

$$\tilde{r}(n) = \sum_{k=0}^{N-1} x(k)x((k-n))_N.$$

On the other hand, the autocorrelation $r(n)$ of $x(n)$:

$$r(n) = x(n) * x(-n),$$

has $2M-1$ length, if $x(n)$ has M-point support. If (2) is not satisfied, then $\tilde{r}(n)$ will be corrupted because of time-aliasing, and $r(n)$ cannot be recovered from $\tilde{r}(n)$.

Furthermore, even when time-aliasing has been avoided, we note that the succession of samples of $r(n)$ and $\tilde{r}(n)$ is not the same. Actually the first M samples of $r(n)$ are shifted to obtain the last M samples of $\tilde{r}(n)$, and the last M samples of $r(n)$ are equal with the first M samples of $\tilde{r}(n)$ with positive index:

$$\tilde{r}(n) = \begin{cases} r(0) & n = 0 \\ r(n) & n = 1,2,\ldots,M-1 \\ r(n-N) & n = N-1,\ldots,N-M+1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus in general, the z-transforms $S(z)$ and $\tilde{S}(z)$ will not have common zeros.

2.3 Ambiguities in 1-D phase retrieval problem

Even we constrain $x(n)$ to be a finite-length sequence, then other finite-length sequences having the same Fourier magnitude as $x(n)$ may be generated by the process of zero flipping [2]. Actually, in the discrete phase retrieval problem, we have a predetermined system of equations where the null space of the data matrix gives the desired flip coefficients [6]. Consequently, there are some ambiguities in phase-retrieval problem: if $x(n)$ is a solution, then $x^*(-n)$, $cx(n)$ and $x(n-b)$ are also solutions for any integer b and any complex number c having unity magnitude $c = 1$. If $x(n)$ is a real sequence, then $c = \pm 1$. These are called the trivial ambiguities and they are associated solutions to a given solution $x(n)$. Excluding these associated solutions [16], there are almost everywhere 2^M solutions to the discrete 1-D phase retrieval. If $x(n)$ is a real sequence, the zeros must be chosen in complex conjugates pairs, so there are only $2^{\frac{M-1}{2}}$ solutions if all zeros of $X(z)$ are complex [17]. Note that only one of these solutions is a minimum-phase sequence.

Finally we note the phase ambiguity $\angle X(k)$ when $|X(k)| = 0$. This differs from 1-D continuous phase retrieval case, where $\angle X(\Omega)$ when $|X(\Omega)| = 0$ can be determined based on continuous assumption.

3. THE FEJÉR-RIESZ THEOREM AND SEVERAL CONSEQUENCES

For the beginning we recall few results on trigonometric polynomials, then we shall relate them to 1-D discrete phase retrieval problem.

A trigonometric polynomial is an expression in one of the equivalent forms:

$$A_0 + \sum_{n=1}^{M} [A_n \cos k\omega + B_n \sin k\omega]$$

or

$$\sum_{n=-M}^{M} C_n e^{jn\omega}.$$

If a trigonometric polynomial has only real values for all real ω, then the coefficients A_n, B_n in the first form must be real; moreover coefficients C_n should satisfy

$$C_n^* = C_{-n},$$

for all indexes n.

It seems that Lipót Fejér (1880-1959) was the first to consider the class of trigonometric polynomials that assume only nonnegative real values. His conjecture of the form of such function was proved by Frigyes Riesz (1880-1956), consequently the result has been named as the Fejér-Riesz Theorem [18].

Theorem 1 (Fejér and Riesz)
If

$$X(z) = \sum_{n=-M}^{M} x(n) z^{-n}$$

and

$$X(e^{j\omega}) \geq 0,$$

then there is

$$Y(z) = \sum_{n=0}^{M} y(n) z^{-n}$$

such that

$$X(e^{j\omega}) = |Y(e^{j\omega})|^2$$

and $Y(z)$ *unique if maximum phase.*

It should be noted that Fejér-Riesz Theorem does not mention anything about the set from which coefficients $y(n)$ belong, i.e. they may have real or complex values.

Going back to 1-D discrete phase retrieval problem, the set of squares of Fourier transform magnitudes is the Fourier transform of the autocorrelation:

$$S(\omega) = |X(\omega)|^2 = \mathscr{F}\{r(n)\}.$$

On the other hand, for a M-point support sequence $x(n)$, the sequence and its circular autocorrelation can be written via $2M-1$-point DFT as follows [11, 13, 12]:

$$X(k) = \sum_{k=0}^{M-1} x(n) e^{-j\frac{2\pi kn}{2M-1}}$$
$$|X(k)|^2 = \sum_{k=0}^{2M-1} \tilde{r}(n) e^{-j\frac{2\pi kn}{2M-1}}. \quad (3)$$

If time aliasing has been avoided and taking into account the periodicity of DFT kernel, we have

$$|X(k)|^2 = \sum_{k=-(M-1)}^{M-1} r(n) e^{-j\frac{2\pi kn}{2M-1}}. \quad (4)$$

In view of (3) and (4), finding $x(n)$ from $r(n)$ means to find a trigonometric polynomial $X(k)$ from a given nonnegative trigonometric polynomial $|X(k)|^2$.

The Fejér-Riesz Theorem guarantees us that 1-D discrete phase retrieval problem has always a solution if the trigonometric polynomial is positive definite, but random magnitudes do not provide always a positive definite trigonometric polynomial.

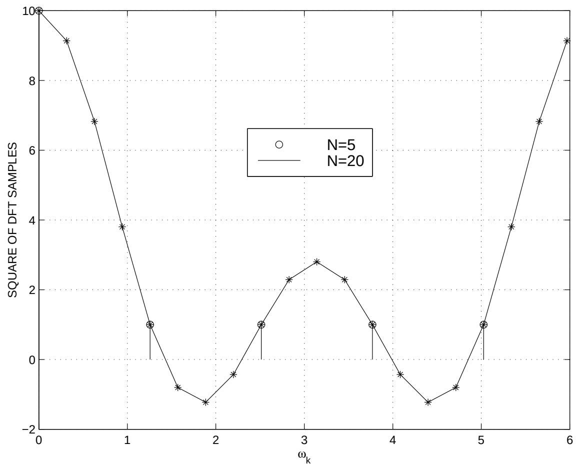

Figure 1: Square of DFT samples for N=5 (○) and N=20 (∗).

Example 1. A set of magnitude input data which does not provide positive definite trigonometric polynomial

Let $x(n)$ have 3 samples length and $N = 5$ with

$$|X(k)| = \begin{cases} \sqrt{10}, & k = 0; \\ 1 & k = 1,2,3,4. \end{cases}$$

Then

$$\tilde{r}_{xx}(n) = \text{DFT}^{-1}\{|X(k)|^2\} = \begin{cases} 2.8, & n = 0; \\ 1.8 & n = 1,2,3,4. \end{cases}$$

Taking into account the periodicity and the symmetry properties, we have

$$\tilde{r}_{xx}(n) = \begin{cases} 2.8, & n = 0; \\ 1.8 & n = \pm 1, \pm 2. \end{cases}$$

Its Fourier transform

$$\tilde{S}_{xx}(\omega) = \mathscr{F}\{\tilde{r}_{xx}(n)\} = 2.8 + 3.6\cos\omega + 3.6\cos 2\omega$$

is not always positive as $\tilde{S}_{xx}(\pi/2) < 0$. Figure 3 presents the graphic of $\tilde{S}_{xx}(\omega)$ and its samples for $N = 5$ and $N = 20$.

When $\tilde{S}_{xx}(\omega)$ is not always positive, it cannot be written as a square of modulus of a trigonometric polynomial. Consequently $\tilde{S}_{xx}(\omega) \neq S_{xx}(\omega)$, thus $\tilde{r}_{xx}(n) \neq r_{xx}(n)$ and time-aliasing should be supposed. In such a situation, when magnitude input data $|X(k)|^2$ do not provide a valid always positive $\tilde{S}_{xx}(\omega)$, any attempt to solve correctly 1-D discrete phase-retrieval problem will be unsuccessful.

Example 2. Standard algorithms for solving 1-D phase retrieval problem

In the following we shall present one direct method (finding the zeros of z-transform of the autocorrelation) and the iterative technique for solving 1-D phase retrieval problem. Hilbert transform [13], computation of cepstral coefficients [14] or solving linear systems of equations [6] can be suspected of time-aliasing, and thus they are not considered here.

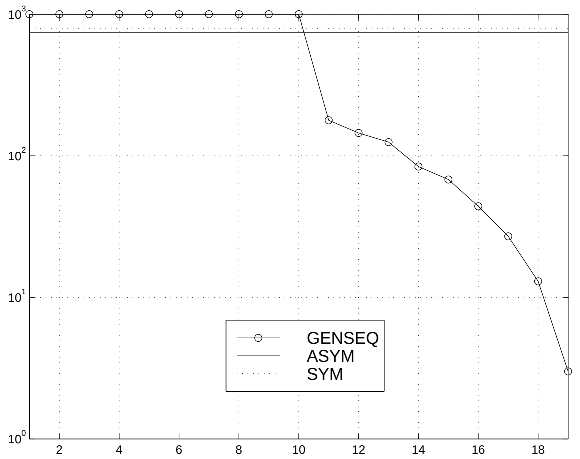

Figure 2: Number of minimum-phase sequences detected versus sequence length (symmetric, asymmetric magnitudes and magnitudes generated from sequences of different length).

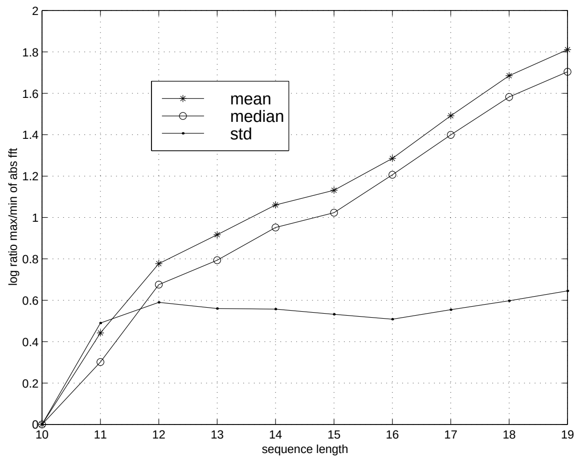

Figure 3: Mean, median and standard deviation of log ratio of maximum and minimum deviation between initial magnitudes and obtained magnitudes.

The most straightforward method for solving direct the 1-D phase retrieval problem consists of finding the zeros of z-transform of the autocorrelation. If $X(z)$ is z-transform of $x(n)$, then $X(z)X^*(1/z^*) = S(z)$ is the z-transform of $r(n)$. The zeros of $S(z)$ occur in conjugate reciprocal pairs. From every pair, one of the zeros should be selected to form $X(z)$. If the selected zeros are chosen inside the unit circle, we get a minimum-phase sequence. From the all 2^M choices, we should select only one who has all zeros inside the unit circle.

The standard iterative technique is an algorithm in which the estimate of $x(n)$ is improved in each iteration [1, 3, 4]. This algorithm belongs to the class of iterative algorithms developed by Gerchberg and Saxton [19] and Fienup [20] for reconstructing a signal from magnitude information, under the assumption that the signal is minimum phase.

To test these two methods for discrete phase retrieval, in every case we shall generate random positive sequences of certain length, as inputs of algorithms. They may represent the values $|X(k)|^2$ of DFT square magnitudes corresponding to a certain sequence $x(n)$. However, this sequence may be not real and may have certain length which is greater than M. Thus we also generate symmetric random positive magnitudes as inputs of algorithms. Moreover, we consider as inputs the magnitudes of DFT of L-point length sequences, where $L = M, M+1, \ldots, N$. For every method we reconstruct the signal $x(n)$ and we verify if its DFT magnitudes are equal with initial data. Also we shall check if the obtained sequence is minimum-phase.

Within the mentioned framework we implement the method of finding the zeros of z-transform of the autocorrelation. We run many times and the outcomes show that all the time the phase retrieval test passes, but minimum-phase sequence is not always detected. Sometimes we can have zeros on unit circle. This happens for sequences having length greater than M, then for antisymmetric magnitudes, and finally for symmetric DFT magnitudes. The minimum-phase test passes whenever the sequence length is less or equalize M. For $M = 9$ the results are presented in Figure 3. The number of runs was 1000 and it can be seen that phase retrieval has been always detected when the sequence length is less or is equal with 9.

We experienced a similar behavior in the case of iterative technique. Until the sequence length is less or equal than half of DFT length, the initial magnitudes and obtained magnitudes are equal. However, when the length exceeds half of DFT length, there are many differences between given and obtained magnitudes. We have run many times this algorithm and the results of averages show an increasing error with sequence length. Mean, median and standard deviation of log ratio of maximum and minimum deviation between initial magnitudes and obtained magnitudes are presented in Figure 3.

4. CONDITIONS ON MAGNITUDE INPUT DATA

The Fejér-Riesz Theorem is a key issue in spectral factorization and estimation, and one can find applications in approximation theory, moments problem, bounded bilinear forms, Hankel operator or Nevalinna-Pick interpolation. For our purposes it is important to find conditions to determine whether a certain set of magnitude input data can provide a positive definite trigonometric polynomial.

One way is to compute DFT of $\tilde{r}_{xx}(n)$ for a certain number of points and then to decide whether the number of DFT samples are enough to guarantee the positive definiteness of

$$\tilde{S}_{xx}(\omega) = \sum_{n=-(M-1)}^{M-1} \tilde{r}_{xx}(n) e^{-j\omega n} =$$
$$\tilde{r}_{xx}(0) + 2\sum_{n=1}^{M-1} \tilde{r}_{xx}(n) \cos n\omega$$

Note that spectral density derivative is continuous. Indeed, we have:

$$\tilde{S}'_{xx}(\omega) = 2\sum_{n=1}^{M-1} n\tilde{r}_{xx}(n) \sin n\omega$$

and

$$\left|\tilde{S}'_{xx}(\omega)\right| \leq 2\sum_{n=1}^{M-1} n|\tilde{r}_{xx}(n)|.$$

Given $|X(k)|$ and accuracy ε

STEP1: Compute $\tilde{r}_{xx}(n) = \mathrm{DFT}^{-1}_{2M-1}\{|X(k)|^2\}$;

STEP2: Compute N with (5);

STEP3: Compute $\tilde{S}_{xx} = \mathrm{DFT}_N\{\tilde{r}_{xx}(n)\}$;

Figure 4: The algorithm to compute $\tilde{S}_{xx}(\omega)$

By Mean Theorem

$$\frac{\tilde{S}_{xx}(\omega_{k+1}) - \tilde{S}_{xx}(\omega_k)}{\omega_{k+1} - \omega_k} = \tilde{S}'_{xx}(\xi), \quad \xi \in (\omega_k, \omega_{k+1})$$

we get

$$\left|\tilde{S}_{xx}(\omega_{k+1}) - \tilde{S}_{xx}(\omega_k)\right| \leq 2|\omega_{k+1} - \omega_k| \sum_{n=1}^{M-1} n|\tilde{r}_{xx}(n)|.$$

If we want to compute $\tilde{S}_{xx}(\omega)$ with ε accuracy we need a DFT in N samples, where

$$N \geq \frac{4\pi}{\varepsilon} \sum_{n=1}^{M-1} n|\tilde{r}_{xx}(n)|. \qquad (5)$$

The algorithm to compute $\tilde{S}_{xx}(\omega)$ with ε accuracy is shown in Figure 4. Based on this algorithm one can find if $\tilde{S}_{xx}(\omega)$ is positive definite and can decide whether 1-D discrete phase retrieval problem can be solved properly. Indeed, if ε is selected much smaller than $\min[\mathrm{DFT}_N\{\tilde{r}_{xx}(n)\}]$ and this minimum is positive, then $\tilde{S}_{xx}(\omega)$ can be considered positive.

5. CONCLUSIONS

We can conclude that if magnitude input data do not provide a positive definite trigonometric polynomial, solving 1-D discrete phase retrieval problem will be difficult. We add that such special situation cannot appear in the case of 1-D continuous phase retrieval problem, as there from the beginning it is known that the input data is always positive and thus input data provide a positive definite trigonometric polynomial. Moreover, aliasing of autocorrelation and phase ambiguity for zero samples are not present.

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