HYPERCOMPLEX ANALYTIC SIGNALS : EXTENSION OF THE ANALYTIC SIGNAL CONCEPT TO COMPLEX SIGNALS

Stephen J. Sangwine†, *Nicolas Le Bihan*‡

†Department of Computing and Electronic Systems, University of Essex, Wivenhoe Park, Colchester CO4 3SQ, UK
email: S.Sangwine@IEEE.org
‡Gipsa-lab UMR 5216, Département Images et Signal
961 rue de la Houille Blanche, Domaine universitaire - BP 46, F - 38402 Saint Martin d'Hères cedex, France
email: nicolas.le-bihan@gipsa-lab.inpg.fr

ABSTRACT

The analytic signal is a complex signal derived from a real signal such that its real part is identical to the original real signal, and its imaginary part is in quadrature (orthogonal) to the original signal. The analytic signal permits the envelope of the original signal to be computed, and it also admits the definition of an instantaneous frequency and phase.

In this paper we present some initial results on extending this idea to the case of a complex signal using a hypercomplex analytic signal. We show that using the hypercomplex analytic signal it is possible to calculate a *complex envelope* of the original complex signal and that the modulus of this complex envelope is the envelope of the modulus of the original signal.

1. INTRODUCTION

Given a real signal $f(t)$, the corresponding analytic signal $a(t)$ is a complex signal with real part identical to $f(t)$:

$$a(t) = f(t) + i\hat{f}(t)$$

where $\hat{f}(t)$ is the Hilbert transform of $f(t)$. The imaginary part of the analytic signal is orthogonal to the real part (this is also described as being *in quadrature*). The modulus of the analytic signal is an *envelope* of the original signal (also known as an instantaneous amplitude) and the phase of the analytic signal (the argument at each time point) may be used to derive an instantaneous frequency (the derivative of the phase gives the instantaneous frequency).

The analytic signal was first defined by Ville in 1948 [1]. A modern account is given by Bracewell [2, pp359–364].

In the frequency domain, the analytic signal is simply defined in terms of the Hermitian symmetry of the Fourier transform of a real signal. Since the Fourier coefficients of a real signal exhibit Hermitian symmetry (the negative frequency coefficients are the complex conjugates of the positive frequency coefficients) it is obvious that only half of the Fourier coefficients are needed to represent the original signal. If the negative frequency coefficients are suppressed, then the inverse transform of the modified Fourier spectrum gives the analytic signal[1].

In this paper we consider the problem of extending the analytic signal concept to the case of a signal $f(t)$ which is complex, so that we can construct orthogonal complex signals, and a *complex* envelope. In order to do this, it is necessary to use Fourier transforms based on a higher-order algebra than the complex numbers and in this paper we use a newly-defined biquaternion (or complexified quaternion) Fourier transform published in 2006 [3]. We have also made use of a freely-available Matlab library which implements this transform [4] without which we would not have been able to investigate this idea nearly so easily.

Complex signals occur where amplitude and phase are measured simultaneously. Synthetic Aperture Radar (SAR) imaging is a classic case, but there are many other possibilities where complex signals may occur.

In what follows, we limit ourselves to the case of a band-limited discrete-time signal and consider the discrete-time analytic signal obtained by simple Fourier methods which assume that the original signal and its analytic signal is periodic. The full theory of continuous-time analytic signals in the complex case, and the generalization of the Hilbert transform are left for a later paper. The results we present here demonstrate the validity of the approach.

2. HYPERCOMPLEX SIGNAL PROCESSING

The idea of extending signal processing beyond complex signals to signals with more than two components is not new. Various authors have studied the use of hypercomplex algebras including quaternions and Clifford algebras, and have defined and studied Fourier transforms based on such algebras. Sommer, Bülow and Felsberg, in particular, have considered the extension of the concept of the analytic signal to 2-dimensions or higher [5, 6], but not to the case of complex samples as is done in this paper. In this paper our extension of the analytic signal concept to hypercomplex analytic signals is based on the use of a complexified quaternion (or biquaternion) Fourier transform [3] which is an extension of earlier work on quaternion transforms [7]. The complexified quaternions are a relatively straightforward development of the quaternions. A general introduction is given by Ward [8]. A complexified quaternion has four complex components based on a complex operator different from all three of the quaternion operators i, j and k. In this paper we follow the example of [3] and denote the complex operator by I. It commutes with the three quaternion operators, and therefore all complex numbers commute with the three quaternion operators. Therefore a complexified quaternion can be represented in the following form:

$$q = w + xi + yj + zk$$

[1] In the discrete-time case, the positive frequency coefficients must be doubled, the DC and Nyquist coefficients must be left unchanged and the negative frequency coefficients must be zeroed.

where $w = \Re(w) + I\Im(w)$ and similarly for the other three complex components.

Complexified quaternion Fourier transforms are a straightforward extension of the quaternion Fourier transforms, although some questions remain about their behaviour in the presence of samples or coefficients with vanishing semi-norms[2] (the complexified quaternions are not a division algebra, and there exists a well-defined set of non-zero complexified quaternions with zero semi-norm). The complexified quaternion Fourier transform used in this paper is defined as follows:

$$\begin{aligned} F[u] = \mathrm{BiQFT}\left(f[n]\right) &= \sum_{n=0}^{N-1} \exp\left(-2\pi\mu\frac{nu}{N}\right) f[n] \\ f[n] = \mathrm{BiQFT}^{-1}\left(F[u]\right) &= \frac{1}{N} \sum_{u=0}^{N-1} \exp\left(2\pi\mu\frac{nu}{N}\right) F[u] \end{aligned} \tag{1}$$

where the signal $f[n]$ and its 'spectrum' $F[u]$ have N samples. The key to the existence of this transform is that μ is a complexified quaternion root of -1 and therefore the exponential has complexified quaternion values [10]. Full details of the transform are given in [3] including an algorithm for computing the transform using decomposition into four complex Fourier transforms.

The QTFM toolbox for Matlab by Sangwine and Le Bihan [4] includes functions for computing with complexified quaternions, including a complexified quaternion FFT, and it is this toolbox which we have used to obtain the results in this paper.

3. THE *HYPERANALYTIC* SIGNAL

In this section, we present a definition, computation and some properties of the hyperanalytic signal. (It may be that this is not the only possible definition of a hyperanalytic signal and therefore we use the indefinite article in referring to it, because at this stage we cannot prove that our definition is unique.)

3.1 Definition

The hyperanalytic signal defined here is based on the classical complex analytic signal: given a *non-analytic* complex signal (in the classical sense defined by Ville [1]), it is possible to construct a one-sided Fourier spectral representation. This is achieved as follows:

Definition 1 *Given a* non-analytic *complex-valued discrete-time signal $f[n]$, we define its complexified quaternion representation $q[n] = f[n](1+0i+0j+0k)$, and its Biquaternion Fourier Transform $F[u] = \mathrm{BiQFT}(q[n])$. The hyperanalytic signal $a[n]$ associated with $f[n]$ is given by:*

$$a[n] = \mathrm{BiQFT}^{-1}\left(F'[u]\right) \tag{2}$$

where the $F'[u]$ is simply derived from $F[u]$ by scaling the positive frequency coefficients by two and the negative frequency coefficients by zero (the DC and Nyquist frequency components are not modified).

[2] A semi-norm is a generalization of the concept of a norm, with no requirement that the norm be zero only at the origin [9]. It is possible for the norm of a complexified quaternion $||q||$ to be zero, even though $q \neq 0$. The set of complexified quaternions with vanishing semi-norms is well-defined: they have real and imaginary parts that are of equal modulus, and are orthogonal.

This definition ensures that $a[n]$ is a signal with a single-sided spectrum. The hyperanalytic signal $a[n]$ defined in this way is a complexified quaternion valued signal.

In the classical case, with a real signal and a complex analytic signal, the analytic signal is defined for a continuous-time signal $f(t)$ in terms of the Hilbert transform, as discussed in section 1. We see no reason why a similar definition should not exist in the hyperanalytic case, but at the time of writing we have not yet identified the hyperanalytic equivalent of the Hilbert transform.

The complexified quaternion spectrum of the signal $f[n]$ will consist of complexified quaternion coefficients. Their arrangement in terms of frequency representation follows that of standard discrete-time Fourier transforms. Thus the first element of the spectrum as computed by [4] is the DC coefficient, which we leave unmodified; the next $N/2-1$ coefficients represent the complexified quaternion 'amplitudes' of positive frequencies, and we scale them by a factor of 2; the next coefficient (if N is even) represents the Nyquist frequency, and we leave it unmodified; and the remaining $N/2-1$ coefficients correspond to negative frequencies, and we zero them. (If N is odd there is no Nyquist coefficient, but otherwise the algorithm is unchanged.)

3.2 Some properties of the hyperanalytic signal

We give here a non-exhaustive list of properties of the hyperanalytic signal without proofs. The complexified quaternion valued hyperanalytic signal $a[n]$, calculated from a complex valued signal $f[n]$, is expressible in the form: $a[n] = w_a[n] + x_a[n]i + y_a[n]j + z_a[n]k$, where $w_a[n], x_a[n], y_a[n]$ and $z_a[n]$ are complex-valued signals.

Property 1 *The scalar part of the hyperanalytic signal $a[n]$ is identical to the original signal $f[n]$, i.e.*

$$w_a[n] = f[n]$$

Property 2 *The three components of the vector part of the hyperanalytic signal are orthogonal to the scalar part, and thus to the original signal, $f[n]$:*

$$\langle w_a[n], x_a[n]\rangle = \langle w_a[n], y_a[n]\rangle = \langle w_a[n], z_a[n]\rangle = 0$$

where $\langle a[n], b[n]\rangle = \sum_{i=1}^{N} a[i]b[i]$.

Property 3 *The semi-norm of the hyperanalytic signal $a[n]$ is a complex valued signal called the* complex envelope.

Some details of this envelope are developed in Section 3.3, where we show an example.

Property 4 *The phase of the hyperanalytic signal $a[n]$ is complex valued and called the* complex phase.

This property is based on the existence of the Euler formula for biquaternions. It is not yet known what significance the complex phase has, if any. It should also be noted that a complexified quaternion has an axis or direction in 3-space with complex components (analogous to the axis of a quaternion). Again, it is not yet known what significance this has.

3.3 The complex envelope

A striking property of the analytic signal is that its modulus gives the envelope of the original signal, and the most significant finding that we present in this paper is that the hyperanalytic signal as defined here does indeed yield a complex envelope by a fairly simple process mathematically. Numerical implementation depends on phase unwrapping which is done with the standard Matlab `unwrap` function. In simple mathematical terms the complex envelope $c[n]$ is the absolute value or modulus of the analytic signal $a[n]$, but since $a[n]$ is a complexified quaternion signal, its modulus is complex. The QTFM toolbox implements the `abs` function for complexified quaternions, but the implementation uses the `sqrt` function which results in complex values which are always in the right half-plane (because the `sqrt` function halves the argument of its complex parameter). We believe a complex envelope with values in only half the complex plane cannot be a correct generalization of the concept of the envelope, even though in the case of the standard analytic signal, the envelope is always positive-valued because again, the square root function returns the positive root. We therefore use the idea of phase-unwrapping to yield a true complex envelope which is not limited to half the complex plane. The algorithm simply requires us to compute the complex modulus of the analytic signal ourselves in three steps:

1. Compute the semi-norm of the analytic signal samples, using the following formula:

$$||a[n]|| = w_a^2[n] + x_a^2[n] + y_a^2[n] + z_a^2[n]$$

   where the squaring applies to individual samples (*i.e.* elementwise). The samples of the semi-norm will be complex, of course.

2. Construct the samples of the complex envelope as the square root of the samples of the semi-norm using phase unwrapping and polar form to implement the square root:
   (a) Compute the amplitude of the samples of the complex envelope using the square root of the modulus of the semi-norm samples. This is a straightforward calculation using real numbers and yielding a real result, which we denote by $A[n]$.
   (b) Calculate the arguments of the samples of the semi-norm using a standard complex function (known in Matlab as `angle` or `atan2`). The argument will exhibit discontinuities which must be eliminated by phase unwrapping, which adds multiples of 2π to the samples in order to yield a smoothly varying phase (which is no longer limited to the range $(0, 2\pi)$). Let the unwrapped angle signal be $\Phi[n]$. Then the complex envelope is simply computed as:

$$c[n] = A[n] \exp\left(I \frac{\Phi[n]}{2}\right)$$

4. ILLUSTRATIVE RESULTS

We show some results graphically using the following example complex function calculated over $N = 1000$ sample points:

$$f[n] = \sin(16\pi n/N)\,[\sin(4\pi n/N) + I \sin(6\pi n/N)]$$

This signal can be described as a 'carrier' of 8 cycles modulated with two sinusoids of 2 and 3 cycles respectively to give the real and imaginary parts. It is plotted in Fig 1 as real and imaginary parts. Figure 2 shows the four complex components of the hyperanalytic signal. Notice that the scalar component is the same as the original signal. Figure 3 is the most significant illustration in the paper. It shows the original signal as a three-dimensional plot with discrete time as the left-to-right axis and the real and imaginary parts plotted perpendicular to this axis. As can be seen, the signal samples trace out a complicated trajectory. The green signal in the figure is the complex envelope. It traces out a much less complicated trajectory which 'encloses' the original signal, and touches it at several points, roughly on the points where the original signal has a peak in modulus. This can also be seen in Fig 4 where the modulus of the complex envelope is superimposed on the modulus of the original signal. Finally, Fig 5 shows the real and imaginary parts of the complex envelope, demonstrating its simplicity, despite the apparent complexity in Fig 3. The envelope has two cycles in the real part, and three in the imaginary part, as would be expected from the construction of the original signal.

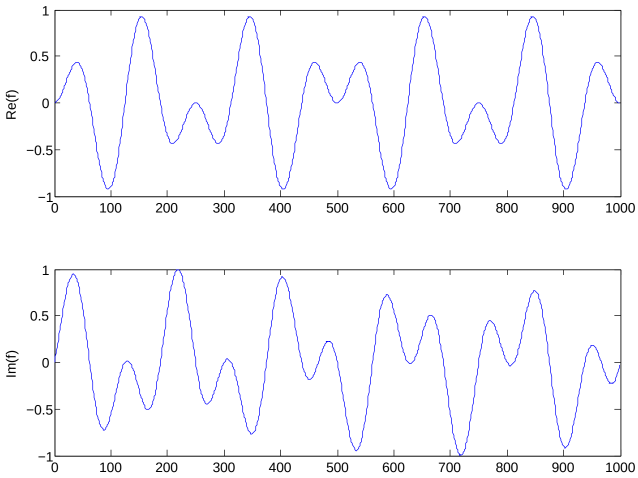

Figure 1: Original signal, showing real and imaginary parts.

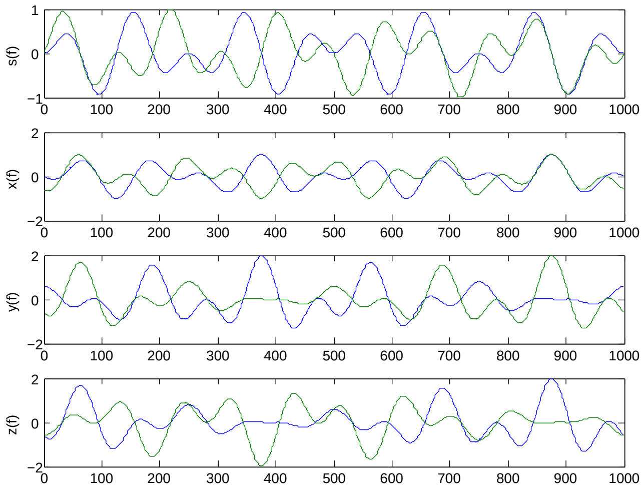

Figure 2: Hypercomplex (biquaternion) analytic signal showing four complex components with real parts (blue) and imaginary parts (green).

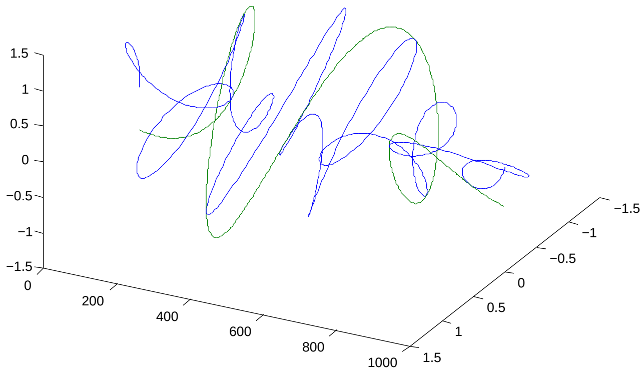

Figure 3: Original complex signal (blue) and complex envelope (green).

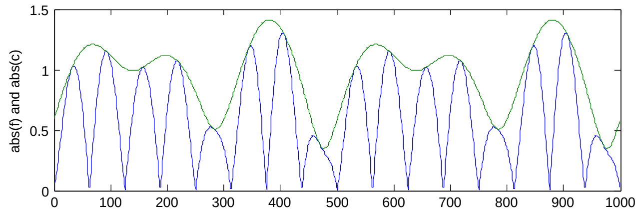

Figure 4: Modulus of original complex signal and modulus of the complex envelope.

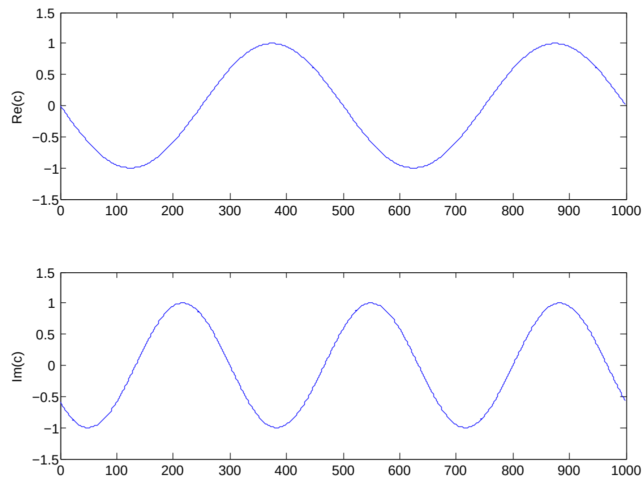

Figure 5: Real and imaginary parts of the complex envelope.

5. DISCUSSION AND CONCLUSIONS

The results presented in this paper demonstrate that the concept of the analytic signal can be extended to the case of a complex signal. The analytic signal corresponding to the original complex signal can be constructed by methods analogous to those used to construct the analytic signal from an original real signal. The generalization depends on complexified quaternion Fourier transforms.

There remain some important questions to be resolved by further work. Firstly, we have not yet established the analogue of the Hilbert transform, that is how to define the 'quadrature' components directly from the original signal. We have also not established whether the use of a complexified quaternion transform is essential. In using the complexified Fourier transform we have constructed a hyperanalytic signal of 8 dimensions when perhaps 4 would be enough. However, we cannot be certain that because the analytic signal of a real signal requires two dimensions, the hyperanalytic signal of a complex signal requires only 4.

Secondly, we could extend the concept further to the case of an original signal with quaternion samples (of which pure quaternion, or vector samples would be a special case). If this requires a 16-dimensional or 32-dimensional analytic signal then it is probably not worthwhile to pursue, but if this can be done with an 8-dimensional complexified quaternion analytic signal it could be both possible and useful.

Thirdly, the extension of the analytic signal to two-dimensional signals (that is signals which are functions of two independent variables, such as images) has been studied: this is known as the monogenic signal [6]. Whether the monogenic signal can be combined with complex samples, to give a hypermonogenic signal is unknown, but it would be an important generalization as it would permit the definition of the envelope of a vector (*e.g.* colour) image, which could have important applications in image segmentation.

We conclude overall therefore, that the idea of extending the concept of analytic signals to hyperanalytic signals of complex signals is valid, and worthy of further research.

REFERENCES

[1] J. Ville, "Théorie et applications de la notion de signal analytique," *Cables et Transmission*, vol. 2A, pp. 61–74, 1948.

[2] R. N. Bracewell, *The Fourier Transform and its Applications*, 3rd ed. Boston: McGraw–Hill, 2000.

[3] S. Said, N. Le Bihan, and S. J. Sangwine, "Fast complexified quaternion Fourier transform," Preprint, Mar. 2006. [Online]. Available: http://www.arxiv.org/abs/math.NA/0603578

[4] S. J. Sangwine and N. Le Bihan, "Quaternion Toolbox for Matlab®," [Online], 2005, software library available at: http://qtfm.sourceforge.net/.

[5] T. Bülow and G. Sommer, "Hypercomplex signals – a novel extension of the analytic signal to the multidimensional case," *IEEE Trans. Signal Process.*, vol. 49, no. 11, pp. 2844–2852, Nov. 2001.

[6] M. Felsberg and G. Sommer, "The monogenic signal," *IEEE Trans. Signal Process.*, vol. 49, no. 12, pp. 3136–3144, Dec. 2001.

[7] T. A. Ell and S. J. Sangwine, "Hypercomplex Fourier transforms of color images," *IEEE Trans. Image Process.*, vol. 16, no. 1, pp. 22–35, Jan. 2007.

[8] J. P. Ward, *Quaternions and Cayley Numbers: Algebra and Applications*, ser. Mathematics and Its Applications. Dordrecht: Kluwer, 1997, vol. 403.

[9] E. J. Borowski and J. M. Borwein, Eds., *Collins Dictionary of Mathematics*, 2nd ed. Glasgow: HarperCollins, 2002.

[10] S. J. Sangwine, "Biquaternion (complexified quaternion) roots of -1," *Advances in Applied Clifford Algebras*, vol. 16, no. 1, pp. 63–68, Feb. 2006, publisher: Birkhäuser Basel.