PARAMETER ESTIMATION FOR SINUSOIDAL SIGNALS WITH DETERMINISTIC AMPLITUDE MODULATION

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ABSTRACT

The problem of estimating the phase and frequency of an amplitude modulated sinusoidal signal is well-known. Estimators and corresponding performance bounds both for noise-like modulating functions and for purely deterministic amplitude modulating functions of different parametric models, i.e. polynomial or auto-regressive, are available in the literature. In this paper, we consider the case of known and unknown deterministic modulating functions with unspecified model. It will be shown that the asymptotic Cramér-Rao lower bounds (ACRBs) for known and unknown modulating functions are equal. Some similarities to the case of multiplicative noise are shown. Furthermore, the variance and bias of the standard distributions are equal. Some similarities to the case of multiplicative noise or obeys a paramet-

2. DERIVATIONS

For all following derivations, we assume generally that \( N \) is large enough and \( a[n] \) is smooth enough (i.e. the continuous-time counterpart of \( a[n] \) is a continuous function) so that the following asymptotic expressions hold for \( i = 0, 1, 2 \) in good approximation:

\[
\frac{1}{N^{i+1}} \sum_{n=0}^{N-1} a[n]^i \cos(4\pi \psi_1 n + 2\theta_1) n^i \approx 0 \quad (7)
\]

\[
\frac{1}{N^{i+1}} \sum_{n=0}^{N-1} a[n]^i \sin(4\pi \psi_1 n + 2\theta_1) n^i \approx 0 \quad (8)
\]

\[
\sum_{n=0}^{N-1} n^i \approx \frac{N^{i+1}}{i+1} \quad (9)
\]

see [6] and also [7]. The case of \( a[n] \equiv 0 \) is excluded for obvious reasons in all derivations.

2.1 ACRB, Known Modulating Function

2.1.1 Complex Data

Assuming \( a[n] \) is known up to a multiplicative constant \( A_1 \), that is \( a[n] = A_1 a'[n] \), we have \( \theta = [A_1 \ 1 \ \psi_1]^T \). To calculate the ACRB, that account for the unknown modulation, with \( \text{Re}\{\cdot\} \) and \( \text{Im}\{\cdot\} \) denoting the real and the imaginary part operator, respectively, to the ACRB and to the standard DFT-based estimators

\[
\hat{\psi}_1^{(b)} = \arg \max_{\psi_1} \left| \sum_{n=0}^{N-1} \tilde{z}[n] \exp(-j4\pi \psi_1 n) \right| \quad (3)
\]

\[
\hat{\phi}_1^{(a)} = \frac{1}{2} \arctan \left( \frac{\text{Im} \left( \sum_{n=0}^{N-1} \tilde{z}[n] \exp(-j4\pi \psi_1^{(a)} n) \right)}{\text{Re} \left( \sum_{n=0}^{N-1} \tilde{z}[n] \exp(-j4\pi \psi_1^{(a)} n) \right)} \right) \quad (4)
\]
can be applied due to the AWGN assumption to calculate the elements of the Fisher information matrix (FIM) [8], with * denoting complex conjugation. We obtain

\[
[F(\theta)]_{11} = \frac{2}{\sigma^2} \text{Re} \left( \sum_{n=0}^{N-1} a'[n] \exp(-j(2\pi \nu_1 n + \phi_1)) \right) - \frac{2}{\sigma^2} \sum_{n=0}^{N-1} a'[n]^2
\]

\[
[F(\theta)]_{22} = \frac{2}{\sigma^2} \text{Re} \left( \sum_{n=0}^{N-1} jA_1a'[n] \exp(-j(2\pi \nu_1 n + \phi_1)) \right) - \frac{2}{\sigma^2} \sum_{n=0}^{N-1} a'[n]^2
\]

\[
[F(\theta)]_{33} = \frac{2}{\sigma^2} \text{Re} \left( \sum_{n=0}^{N-1} j2\pi n A_1 a'[n] \exp(-j(2\pi \nu_1 n + \phi_1)) \right) - \frac{6\pi^2 + 4}{\sigma^2} \sum_{n=0}^{N-1} a'[n]^2 n^2
\]

and \([F(\theta)]_{12} = [F(\theta)]_{13} = 0\), as can be easily verified. All other elements can be obtained using the symmetry of the FIM. Inverting the block-diagonal FIM yields the following asymptotic bounds

\[
\text{var}\{\hat{\phi}_1^{(c)}\}_{\text{complex}} \geq \frac{\sigma^2}{2N_1} \sum_{n=0}^{N-1} a'[n]^2 \sigma^2 \sum_{n=0}^{N-1} \left( \sum_{n=0}^{N-1} a'[n]^2 \right)^2
\]

(14)

\[
\text{var}\{\hat{\psi}_1^{(c)}\}_{\text{complex}} \geq \frac{\sigma^2}{2(2\pi)^2 A_1^2} \sum_{n=0}^{N-1} a'[n]^2 \sum_{n=0}^{N-1} a'[n]^2 \sum_{n=0}^{N-1} a'[n]^2 - \frac{\left( \sum_{n=0}^{N-1} a'[n]^2 \right)^2}{N-1}
\]

(15)

As a simple validation, consider \(a'[n] = 1\), which yields, after some straightforward manipulations and using (9), the standard ACRBs for the “classical” sinusoidal parameter estimation problem without modulation. Furthermore, above bounds also hold if \(a[n]\) is completely known.

2.2.2 Real Data

Similarly to the complex case but assuming \(0 < \psi_1 < 1/2\) so that interference between the positive and the negative half of the spectrum can be neglected, application of

\[
[F(\theta)]_{km} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \frac{\partial^2 \text{Re}\{\theta(n)\}}{\partial \theta^2} \frac{\partial a'[n]}{\partial \theta} a'[n]
\]

(16)

yields the following ACRBs:

\[
\text{var}\{\hat{\phi}_1^{(c)}\}_{\text{real}} \geq \frac{2\sigma^2}{\sum_{n=0}^{N-1} a'[n]^2} \sum_{n=0}^{N-1} a'[n]^2 \sum_{n=0}^{N-1} a'[n]^2 - \frac{1}{2\sigma^2} \sum_{n=0}^{N-1} a'[n]^2 \sum_{n=0}^{N-1} a'[n]^2
\]

(17)

\[
\text{var}\{\hat{\psi}_1^{(c)}\}_{\text{real}} \geq \frac{2\sigma^2}{\sum_{n=0}^{N-1} a'[n]^2} \sum_{n=0}^{N-1} a'[n]^2 \sum_{n=0}^{N-1} a'[n]^2 - \frac{1}{2\sigma^2} \sum_{n=0}^{N-1} a'[n]^2 \sum_{n=0}^{N-1} a'[n]^2
\]

(18)

2.2 ACRB, Unknown Modulating Function

2.2.1 Complex Data

For \(\theta = [a[0], a[1], \ldots, a[N-1]]^T\), we can also apply [10], but \(F(\theta)\) is now of dimension \((N+2) \times (N+2)\). After some work, the following elements of the FIM can be obtained:

\[
[F(\theta)]_{km} = \begin{cases} \frac{2}{\sigma^2} & k = m = 1, 2, \ldots, N \\ 0 & k = m, 1, 2, \ldots, N, k \neq m \end{cases}
\]

Furthermore, \([F(\theta)]_{k1:N+1,1:N+1}, [F(\theta)]_{k1:N+1,1:N+2}\), and \([F(\theta)]_{k1:N+2,1:N+2}\) are given by (11), (12), and (13), respectively. All other elements are obtained by the symmetry of the FIM. It can be seen that \(F(\theta)\) is block-diagonal, and hence the well-known rule for inverting block-diagonal matrices can be applied. We obtain

\[
\text{var}\{\hat{a}[n]\} \geq \frac{\sigma^2}{2}
\]

and the bounds for the phase and the frequency are equal to the bounds when the modulating function is known and are given by (14) and (15). This somewhat unexpected result is further discussed in Section 2.2.2.

2.2.2 Real Data

Using the abbreviation \(\beta = 2\pi \nu_1 n + \phi_1\), some trigonometric identities, and [16], the following elements of the FIM can be obtained:

\[
[F(\theta)]_{km} = \begin{cases} \frac{1}{\sigma^2} & k = m = 1, 2, \ldots, N \\ -a[n] \sin(\beta) & k = m, 1, 2, \ldots, N, k \neq m \\ -a[2n-1] \sin(\beta) & k = N + 1, m = 1, 2, \ldots, N \end{cases}
\]

(19)

\[
[F(\theta)]_{k1:N+1,1:N+1} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} a[n]^2 \sin^2(\beta) \approx \frac{1}{2\sigma^2} \sum_{n=0}^{N-1} a[n]^2 \sin^2(\beta)
\]

\[
[F(\theta)]_{k1:N+2,1:N+2} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} a[n]^2 \sin^2(\beta) \approx \frac{1}{2\sigma^2} \sum_{n=0}^{N-1} a[n]^2 \sin^2(\beta)
\]

Using standard formulas for the determinants of partitioned matrices, it can be easily seen that the so obtained FIM is singular. This typically occurs if the number of unknown parameters is greater than \(N\), like here. Rao [9] suggested to use the Moore-Penrose Pseudoinverse of the FIM, which yields the same bounds [17] and [13].
as for the case of known modulating function. However, Stoica et al. [10] showed that in most cases, there is no unbiased estimator with finite variance for $\hat{a}$. In fact, it is intuitively clear that in case of real data, there is no unbiased counterpart to the estimator

$$\hat{a}[n] = \Re \left\{ \hat{\xi}[n] \exp \left( -j2\pi \hat{\psi}^{(b)}[n] \right) \right\}$$

(19)

in the complex data case. Whereas in the complex case it can be seen from (19) that it is possible to eliminate the exponential terms by premultiplying with its estimated inverse functions, this is not possible in the real case. However, simulation results indicate that the resultant frequency and phase estimates are unbiased when applying (1) for real data.

2.3 Variance and Bias, DFT-based Estimators

2.3.1 Real Data

To derive approximate expressions for bias and variance of (1) under the assumption that $a[n] \geq 0$, a Taylor series expansion approach is used [6], with the estimator being viewed as a function of the noise:

$$\hat{\psi}^{(b)}[n] = \hat{h}_n(\hat{\psi}) = h(0) + \sum_{n=0}^{N-1} \frac{\partial h_n(\hat{\psi})}{\partial \hat{\psi}} \bigg|_{\hat{\psi}=\hat{\psi}} \hat{\psi}^{(b)}[n]$$

Because (1) is given as an implicit function of $\hat{\psi}$, only, a Taylor series expansion of the squared DFT magnitude spectrum (which is equivalent to using the estimator (1))

$$J(\hat{\psi}) = \sum_{n=0}^{N-1} x[n] \exp(-j2\pi \hat{\psi}[n])$$

around the true value $\psi$, is used, yielding

$$h_{\phi}^{(b)}(\hat{\psi}) \approx \hat{\psi}^{(b)} - \frac{\partial J(\hat{\psi})}{\partial \psi} \bigg|_{\psi=\hat{\psi}} \approx \hat{\psi}^{(b)} - \frac{\partial h_n(\hat{\psi})}{\partial \hat{\psi}} \bigg|_{\hat{\psi}=\hat{\psi}} \hat{\psi}^{(b)} - \frac{\partial h_n(\hat{\psi})}{\partial \hat{\psi}} \bigg|_{\hat{\psi}=\hat{\psi}} \hat{\psi}^{(b)}$$

In matrix notation, denoting $^T$ the transpose of a matrix and $C$ the covariance matrix of the data, we obtain [6] the following expressions for the bias and the variance:

$$\text{bias} \left\{ \hat{\psi}^{(b)}[n] \right\} \approx h_{\phi}^{(b)}(0) - \hat{\psi}$$

(20)

$$\text{var} \left\{ \hat{\psi}^{(b)}[n] \right\} \approx \frac{\partial h_{\phi}^{(b)}(\hat{\psi})}{\partial \hat{\psi}} \bigg|_{\hat{\psi}=\hat{\psi}} \hat{\psi}^{(b)} - \frac{\partial h_{\phi}^{(b)}(\hat{\psi})}{\partial \hat{\psi}} \bigg|_{\hat{\psi}=\hat{\psi}} \hat{\psi}^{(b)}$$

(21)

where the last line follows from the assumption that the measurement noise is white. The derivative of $h_{\phi}^{(b)}(\hat{\psi})$ is given by

$$\frac{\partial h_{\phi}^{(b)}(\hat{\psi})}{\partial \hat{\psi}} \bigg|_{\hat{\psi}=\hat{\psi}} = \left( -\frac{1}{\hat{\xi}^{(b)}[n]} \frac{\partial \hat{\xi}^{(b)}[n]}{\partial \hat{\psi}} + \frac{\hat{\xi}^{(b)}[n]}{\hat{\psi}^{(b)}} \frac{\partial \hat{\psi}^{(b)}}{\partial \hat{\psi}} \right) \bigg|_{\hat{\psi}=\hat{\psi}} \hat{\psi}^{(b)}$$

(22)

After some further work and application of large-sample approximations under the assumption $0 < \psi < 1/2$, we obtain

$$\hat{\psi}^{(b)}[n] \approx 2\pi \sum_n a[n]n - 2\pi \sum_n a[n]n^2 \left( \sum_n a[n] \right)$$

$$\frac{\partial \hat{\psi}^{(b)}[n]}{\partial \hat{\psi}} \bigg|_{\hat{\psi}=\hat{\psi}} \approx 0$$

$$\hat{\psi}^{(b)}[n] \approx 2\pi \sum_n a[n]n - 2\pi \sum_n a[n]n^2 \left( \sum_n a[n] \right)$$

$$\frac{\partial \hat{\psi}^{(b)}[n]}{\partial \hat{\psi}} \bigg|_{\hat{\psi}=\hat{\psi}} \approx 0$$

and

$$\text{bias} \left\{ \hat{\psi}^{(b)} \right\} \approx 0$$

and

$$\text{var} \left\{ \hat{\psi}^{(b)} \right\} \approx \sigma^2 \sum_{n=0}^{N-1} \left( \frac{\sum_{n=0}^{N-1} a[n]n - \sum_{n=0}^{N-1} a[n]}{\sum_{n=0}^{N-1} a[n]} \right)^2$$

(23)

which can be used to estimate the variance of $\hat{\psi}^{(b)}$ if $\sum_{n=0}^{N-1} a[n]n$ is known.

2.3.2 Complex Data

Completely analogous to the case of real data and under the same assumptions, the following asymptotic variance expressions are obtained:

$$\text{var} \left\{ \hat{\psi}^{(b)} \right\} \approx \sigma^2 \sum_{n=0}^{N-1} \left( \frac{\sum_{n=0}^{N-1} a[n]n - \sum_{n=0}^{N-1} a[n]}{\sum_{n=0}^{N-1} a[n]} \right)^2$$

(24)

$$\text{bias} \left\{ \hat{\psi}^{(b)} \right\} \approx 0$$

$$\text{var} \left\{ \hat{\psi}^{(b)} \right\} \approx \sigma^2 \sum_{n=0}^{N-1} \left( \frac{\sum_{n=0}^{N-1} a[n]n - \sum_{n=0}^{N-1} a[n]}{\sum_{n=0}^{N-1} a[n]} \right)^2$$

(25)

$$\text{bias} \left\{ \hat{\psi}^{(b)} \right\} \approx 0$$

$$\text{var} \left\{ \hat{\psi}^{(b)} \right\} \approx \sigma^2 \sum_{n=0}^{N-1} \left( \frac{\sum_{n=0}^{N-1} a[n]n - \sum_{n=0}^{N-1} a[n]}{\sum_{n=0}^{N-1} a[n]} \right)^2$$

and

$$\text{bias} \left\{ \hat{\psi}^{(b)} \right\} \approx 0$$

$$\text{var} \left\{ \hat{\psi}^{(b)} \right\} \approx \sigma^2 \sum_{n=0}^{N-1} \left( \frac{\sum_{n=0}^{N-1} a[n]n - \sum_{n=0}^{N-1} a[n]}{\sum_{n=0}^{N-1} a[n]} \right)^2$$

(26)
2.4 Estimator, Known Modulating Function

If \( a[n] \) is known, this can be advantageously incorporated. Using the trigonometric identity

\[
\cos(\beta_1 + \beta_2) = \cos(\beta_1) \cos(\beta_2) - \sin(\beta_1) \sin(\beta_2),
\]

the model \( 3 \) can be rewritten to

\[
x[n] = x[n; \alpha; \psi_1] + w[n] = a[n] \alpha_1 \cos(2\pi \psi_1 n) + a[n] \alpha_2 \sin(2\pi \psi_1 n),
\]

with \( \alpha_1 = \cos(\phi_1) \) and \( \alpha_2 = -\sin(\phi_1) \). In matrix notation, denoting \( \alpha = [\alpha_1, \alpha_2]^T \), we have

\[
x = H_1 \psi_1 + w,
\]

with

\[
h_1 = [a[0] \gamma[0], a[1] \gamma[1], \ldots, a[N-1] \gamma[N-1]]^T
\]

\[
h_2 = [a[0] \gamma'[0], a[1] \gamma'[1], \ldots, a[N-1] \gamma'[N-1]]^T,
\]

where \( \gamma[n] = \cos(2\pi \psi_1 n) \) and \( \gamma'[n] = \sin(2\pi \psi_1 n) \). Applying the principle of separability, LS is known, this can be advantageously incorporated. Using \( \alpha_1 \) is not close to the boundaries is not necessary, nor the used large-sample approximations.

Furthermore, it is straightforward to show that

\[
\theta_1^{(c)} \approx \arctan \left( \frac{\sum_{n=0}^{N-1} a[n] x[n] \sin(2\pi \psi_1 n)}{\sum_{n=0}^{N-1} a[n] x[n] \cos(2\pi \psi_1 n)} \right).
\]  

Eq. (27) and (28) can be nicely interpreted. In principle, the estimators are very similar to the standard DFT-based estimators \( 5 \) and \( 6 \), respectively. However, here the data is weighted according to the amplitude modulating function. This makes sense because samples where \( |a[n]| \) takes on high values have higher SNR compared to samples where \( |a[n]| \) is low. The estimators \( 27 \) and \( 28 \) can be implemented computationally efficient with the FFT-algorithm, because the weighting of the data can be applied directly on the data before the estimator is applied. A further comparison of all the derived results will be given next. Note that similar results are obtained for complex data, but however, the assumption that \( \psi_1 \) is not close to the boundaries is not necessary, nor the used large-sample approximations.

2.5 Comparison, Interpretations

For brevity, we concentrate on the frequency here, but the discussion is equally valid for the phase, too. The fact that the ACRBs for known and unknown modulating functions are equal is somewhat unexpected and also the estimators \( 3 \) and \( 27 \) seem to be different at first glance. However, considering \( 19 \), we can rewrite the estimator \( 3 \) as follows:

\[
\hat{\psi}_1^{(a)} = \arg \max_{\psi_1} \sum_{n=0}^{N-1} x[n] \exp(-j\pi \psi_1 n)
\]

\[
\approx \arg \max_{\psi_1} \sum_{n=0}^{N-1} x[n] \exp(-j2\pi \psi_1 n) \exp(-j2\pi \psi_1 n)
\]

for large enough SNR so that \( \hat{\psi}_1^{(a)} \) is near \( \psi_1 \). The exponential term containing the constant phase is eliminated by the magnitude. Hence, the estimator \( 3 \) is in fact very similar to \( 27 \), but uses an estimate of the modulating function for weighting the data according to the SNR of each individual sample. This also explains the interesting phenomenon that, assuming \( a[n] \geq 0 \), for relatively low SNRs the estimator \( 3 \) often performs in fact poorer than \( 6 \) that completely neglects the modulation. For low SNRs, the estimate of the amplitude modulating function that is built into \( 3 \) performs very poorly, and hence deteriorates the estimation result. It is furthermore interesting to compare this to a result given by Stoica and Besson \([12,4]\) for a modulating function: being a stationary Gaussian random process:

\[
\var{\psi_1} \approx \frac{6}{\pi} \left[ 1 + \frac{1}{2\pi} \right]
\]

asymptotically, where the SNR \( \eta = r_0/\sigma^2 \) and

\[
r_0 = E(|a[n]|^2).
\]

The second term in brackets of (29) causes exactly the same behaviour as for the case of purely deterministic modulation. It is negligible for high SNRs, that is it makes no difference whether or not the modulating function is known. On the other hand, it significantly increases the variance for low SNRs. Hence, there is a trade-off choosing the estimator \( 3 \) or \( 5 \) depending on the available SNR, similarly to the case of multiplicative noise as described in \([1]\). The situation is completely different if the modulating function is known. Although a strict analytical proof has not been found, simulation results indicate that

\[
\var{\psi_1^{(c)}} \leq \var{\psi_1^{(b)}} \quad \text{and} \quad \var{\psi_1^{(c)}} \leq \var{\psi_1^{(a)}}
\]

regardless of \( a[n] \) and the SNR, as expected.

One strength of the used approach not to specify any particular model for the envelope when deriving the asymptotic variances and bounds is that it is quickly possible to calculate the asymptotic variances for a particular model of the envelope by simply inserting \( a[n] \) in the derived formulas.

3. SIMULATION RESULTS

For the simulations, we used the SNR definition \( 10 \log_{10} (1/\sigma^2) \), complex data, 1000 trials, and sufficient zero-padding to avoid discretization effects of the frequency axis. Fig. 1 shows the two modulating functions that have been used for simulation, where modulating functions like modulating function 1 typically occur e.g. in wide-bandwidth linear frequency modulated continuous wave radar signal processing applications. As can be seen in Fig. 2, the asymptotic estimation variances of the estimators \( 3, 5 \), and
are quite similar, but at low SNRs, it is advantageous here to use the standard DFT-based estimators. In Fig. 3 the modulating function has been chosen to be an exponential decaying function $a(n) = \exp(-0.02n)$. For high SNRs, the resulting RMSEs of the estimators that account for the unknown modulating functions achieve the asymptotic CRLB, being MLEs under the AWGN assumption. The RMSE of the standard DFT-based estimators is much higher. In any case, if the modulating function is known, the derived estimators that account therefor performs best, as expected.

4. CONCLUSION

In this paper, asymptotic performance bounds for the sinusoidal parameter estimation problem with deterministic amplitude modulation have been derived. It has been shown that the bounds are equal, regardless whether the modulating function is known or not. Furthermore, the NLS estimator in case of known modulating function has been derived, together with the asymptotic variances of standard DFT-based estimators in case of amplitude modulated data.

REFERENCES


