IMPROVEMENT OF THE ZIV-ZAKAI LOWER BOUND FOR TIME DELAY ESTIMATION

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ABSTRACT

In time-delay parameter estimation theory, predicting the performance of the Maximum Likelihood Estimator (MLE) is of great interest. The Cramèr-Rao Bound (CRB) is widely used to estimate the variance of the MLE. But, in the case of low signal to noise ratio, the CRB becomes inefficient. The Ziv-Zakai Bound (ZZB) was therefore developed to estimate this variance at low SNR by taking into account the a priori parameter density. We propose in this paper an improvement of the ZZB which predicts a more accurate MLE variance. Moreover, we derive a closed-form expression of this improved ZZB using the saddlepoint approximation method.

1. INTRODUCTION

Defining the lower bounds for the Mean Square Error (MSE) of the Maximum Likelihood Estimator of a signal parameter θ from noisy observations always remains a problem of interest when the exact MSE is difficult to estimate. The most popular bounds include the well known Cramèr-Rao Bound (CRB) and the Ziv-Zakai Bound (ZZB). For high Signal to Noise Ratio (SNR), the CRB accurately approaches the MSE. In the case of low SNR, the CRB, which is only evaluated around the principal mode of the likelihood function, does not take into account the presence of undesirable peaks and hence strongly under-estimates the MSE. The ZZB, based on a prior distribution of the parameter to estimate, takes into account these undesirable peaks and better approaches the MSE at low SNR.

The ZZB [1, 2] is derived by lower-bounding the MSE with a probability of error (also called risk) of an optimal statistical test, namely the LRT (Likelihood Ratio Test). In spite of the great improvement with respect to the CRB in the case of low SNR, the BZZ can be improved to better approach the MSE. The goal of this paper is to propose a new risk, built from the Generalized Likelihood Ratio Test (GLRT), which is greater than the classical one but remains lower than the MSE. This modified ZZB is shown to better bound the MSE. This paper is devoted to the particular case of time-delay estimation but can be easily extended to other cases. The organization of the paper is as follows. Section 2 presents the problem and recalls the classical bounds. Section 3 describes an improvement of the classical ZZB by defining a new statistical test. This test being difficult to be calculated, section 4 presents a powerful tool, the Saddlepoint (SP) approximation method which enables an easy derivation of the new bound. Finally, the last section gives some examples on time-delay estimation performance and compares the new bound to the others.

2. CLASSICAL CRAMÈR-RAO AND ZIV-ZAKAI BOUNDS

We consider here a vector $\mathbf{Y} = [y(t_1), y(t_2), \dots, y(t_N)]^T$ of noisy observations $y(t_k)$, $1 \leq k \leq N$:

$$\mathbf{Y} = A\, T_\theta[\mathbf{S}] + \mathbf{N} = A\,\mathbf{S}_\theta + \mathbf{N}, \tag{1}$$

where the general transformation T_θ with unknown parameter θ to estimate (doppler shift, time-shift, time-scale, ...) is applied on a vector $\mathbf{S} = [s(t_1), s(t_2), \dots, s(t_N)]^T$ built from real samples of signal $s(t)$, where the parameter A is the positive amplitude and $\mathbf{N}$ is a real zero-mean Gaussian vector with known covariance matrix $\sigma^2\mathbf{I}$. In the sequel, we focus, without loss of generality, on time-delay estimation, *i.e.*, $\mathbf{S}_\theta = [s(t_1-\theta), s(t_2-\theta), \dots, s(t_N-\theta)]^T$. The signal $\mathbf{S}$ and its transformation $\mathbf{S}_\theta$ are supposed to respect the energy conservation, i.e.:

$$\mathbf{S}^T\,\mathbf{S} = \mathbf{S}_\theta^T\,\mathbf{S}_\theta = 1\,. \tag{2}$$

Up to a constant, the log-likelihood function f is defined, for any θ, by:

$$f_{A,\theta}(\mathbf{Y}) = \frac{-1}{2\sigma^2}\,(\mathbf{Y} - A\,\mathbf{S}_\theta)^T\,(\mathbf{Y} - A\,\mathbf{S}_\theta)\,, \tag{3}$$

and the Maximum Likelihood Estimators (MLE) are given by

$$(\hat{A}(\mathbf{Y}), \hat{\theta}(y)) = \underset{A,\theta}{\operatorname{argmax}}\, f_{A,\theta}(\mathbf{Y})\,. \tag{4}$$

For a known amplitude A, the MLE $\hat{\theta}(\mathbf{Y})$ is:

$$\hat{\theta}(\mathbf{Y}) = \underset{\theta}{\operatorname{argmax}}\, \mathbf{Y}^T\mathbf{S}_\theta\,. \tag{5}$$

When A is unknown, its estimator $\hat{A}(\mathbf{Y})$, for a given θ, is obtained by maximizing $f_{A,\theta}(\mathbf{Y})$ with respect to A:

$$\hat{A}(\mathbf{Y}) = \underset{A}{\operatorname{argmax}}\, f_{A,\theta}(\mathbf{Y}) = \mathbf{Y}^T\mathbf{S}_\theta\,. \tag{6}$$

Under condition (2), replacing $\hat{A}(\mathbf{Y})$ in (3) provides the MLE $\hat{\theta}(\mathbf{Y})$:

$$\hat{\theta}(\mathbf{Y}) = \underset{\theta}{\operatorname{argmax}}\, f_{\hat{A}(\mathbf{Y}),\theta}(\mathbf{Y}) = \underset{\theta}{\operatorname{argmax}}\, \left(\mathbf{Y}^T\mathbf{S}_\theta\right)^2\,. \tag{7}$$

In both cases (A known and unknown), estimators (5) and (7) are the same.

2.1 Cramèr Rao Bound

The efficiency of the unbiased estimator $\hat{\theta}$ is generally measured by the MSE $\varepsilon_\theta^2 = E\left[(\hat{\theta}(\mathbf{Y})-\theta)^2\right]$ which has a lower bound given by the CRB. When the constraint (2) is respected, the MLE is (for amplitude A known or unknown):

$$\varepsilon_\theta^2 \geq \left(E\left[\left(\frac{\partial f_{A,\theta}}{\partial\theta}(\mathbf{Y})\right)^2\right]\right)^{-1} = \frac{\sigma^2}{A^2}\left(\frac{\partial \mathbf{S}_\theta^T}{\partial\theta}\frac{\partial \mathbf{S}_\theta}{\partial\theta}\right)^{-1}. \quad (8)$$

By denoting σ_f^2, the spectral variance of $s(t_k)$, equation (8) leads to the well-known CRB: $\varepsilon_\theta^2 \geq \frac{\sigma^2}{A^2}\frac{1}{4\pi^2\sigma_f^2}$.

The CRB is generally easy to derive but it over-estimates the performance, particularly at low SNR. The Ziv Zakai bound allows to circumvent this problem.

2.2 The Ziv-Zakai bound [1, 2]

Let us consider here a scalar random parameter θ with *a priori* uniform Probability Density Function (PDF) $p_\theta(\theta)$ defined on $[0, T_a]$. The problem of interest is to lower-bound the MSE ε_θ^2 which has the following form:

$$\varepsilon_\theta^2 = \frac{1}{2}\int_0^{+\infty} \Pr\left(|\hat{\theta}(\mathbf{Y})-\theta| > \frac{h}{2}\right) h\,dh, \quad (9)$$

where $\Pr\left(|\hat{\theta}(\mathbf{Y})-\theta| > \frac{h}{2}\right) =$

$$\int_{-\infty}^{+\infty} \left(p_\theta(\varphi)+p_\theta(\varphi+h)\right) R_0(\varphi,h)\,d\varphi, \quad (10)$$

and $R_0(\varphi,h) =$

$$\frac{p_\theta(\varphi)}{p_\theta(\varphi)+p_\theta(\varphi+h)}\Pr\left(\hat{\theta}(\mathbf{Y}) > \varphi+\frac{h}{2}\,|\theta=\varphi\right) + \frac{p_\theta(\varphi+h)}{p_\theta(\varphi)+p_\theta(\varphi+h)}\Pr\left(\hat{\theta}(\mathbf{Y}) \leq \varphi+\frac{h}{2}\,|\theta=\varphi+h\right).$$

The term in square brackets in (11) can be seen as the probability of error $R_0(\varphi,h)$ (also called the risk) of the following test ϕ_0:

$$\phi_0 \begin{cases} H_0 & : \theta=\varphi \quad \text{if} \quad \hat{\theta}(\mathbf{Y}) \leq \varphi+\frac{h}{2} \\ H_1 & : \theta=\varphi+h \quad \text{if} \quad \hat{\theta}(\mathbf{Y}) > \varphi+\frac{h}{2} \end{cases}, \quad (11)$$

with $\begin{cases} \Pr(H_0) = \dfrac{p_\theta(\varphi)}{p_\theta(\varphi)+p_\theta(\varphi+h)} \\ \Pr(H_1) = \dfrac{p_\theta(\varphi+h)}{p_\theta(\varphi)+p_\theta(\varphi+h)}. \end{cases}$

By defining the log-Likelihood Ratio:

$$\Lambda_1(\mathbf{Y}) = f_{A,\theta=\varphi}(\mathbf{Y}) - f_{A,\theta=\varphi+h}(\mathbf{Y}) = \frac{A}{\sigma^2}\left(\mathbf{S}_\varphi^T\mathbf{Y} - \mathbf{S}_{\varphi+h}^T\mathbf{Y}\right), \quad (12)$$

the risk $R_0(\varphi,h)$ can be lower-bounded by the minimum risk $R_1(\varphi,h)$ obtained from the log-likelihood ratio of the following optimum Likelihood Ratio Test (LRT) ϕ_1:

$$\phi_1 \begin{cases} H_0 & : \theta=\varphi \quad \text{if } \Lambda_1(\mathbf{Y}) \geq 0 \\ H_1 & : \theta=\varphi+h \quad \text{otherwise.} \end{cases} \quad (13)$$

For an equally likely hypothesis, the risk of the test ϕ_1 is:

$$R_1(\varphi,h) = \frac{1}{2}\Pr\left[\Lambda_1(\mathbf{Y})<0|\,\theta=\varphi\right] + \frac{1}{2}\Pr\left[\Lambda_1(\mathbf{Y})\geq 0|\,\theta=\varphi+h\right], \quad (14)$$

and the MSE finally verifies the inequality:

$$\varepsilon_\theta^2 \geq \frac{1}{2}\int_0^{\infty} h\,dh \int_{-\infty}^{+\infty} R_1(\varphi,h)\,\left(p_\theta(\varphi)+p_\theta(\varphi+h)\right)d\varphi. \quad (15)$$

Equation (14) can be reduced by symmetry to

$$R_1(\varphi,h) = \Pr\left[\mathbf{S}_\varphi^T\mathbf{Y} - \mathbf{S}_{\varphi+h}^T\mathbf{Y} < 0 \,\Big|\, \theta=\varphi\right]. \quad (16)$$

Let us denote $\rho(h) = \mathbf{S}_\varphi^T\mathbf{S}_{\varphi+h}$, the time-delay autocorrelation function ($|\rho(h)| \leq 1$ and independent of φ). By recalling that under $\theta=\varphi$ hypothesis $\mathbf{Y}$ is a Gaussian vector $\mathcal{N}(A\mathbf{S}_\varphi, \sigma^2\mathbf{I})$, the exact expression for R_1 is independent of φ and can be obtained by:

$$R_1(h) = \Phi\left(\sqrt{\frac{A^2}{2\sigma^2}(1-\rho(h))}\right), \quad (17)$$

where Φ is the complementary error function:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_x^{+\infty} e^{-u^2/2}\,du. \quad (18)$$

When p_θ is uniform on $[0, T_a]$, replacing (17) in (15) and integrating over φ lead to the classical ZZB expressed by:

$$\varepsilon_\theta^2 \geq ZZB_1 \quad (19)$$

where

$$ZBB_1 = \int_0^{T_a} h\left(1-\frac{h}{T_a}\right)\Phi\left(\sqrt{\frac{A^2}{2\sigma^2}(1-\rho(h))}\right)dh. \quad (20)$$

A very precise and simple approximation $\widetilde{ZZB}_1$ of (20) has been described in [2]:

$$\widetilde{ZZB}_1 = \frac{T_a^2}{12}\Phi\left(\sqrt{\frac{A^2}{2\sigma^2}}\right) + \frac{\sigma^2}{4\pi^2A^2\sigma_f^2}\Gamma_{3/2}\left(\frac{A^2}{4\sigma^2}\right) - \left(\frac{\sigma^2}{4\pi^2A^2\sigma_f^2}\right)^{3/2}\frac{32}{3T_a\sqrt{2\pi}}\Gamma_2\left(\frac{A^2}{4\sigma^2}\right), \quad (21)$$

where $\Gamma_a(.)$ is the incomplete gamma function. For the low SNR, it can be proved that the bound tends to the variance $T_a^2/12$ of the *a priori* uniform PDF p_θ. Therefore, the MLE is nearly uniform on $[0, T_a]$. In the opposite, for the high SNR, the CRB given by (8) can be retrieved.

3. IMPROVED ZIV-ZAKAI BOUND

We can improve the ZZB based on the LRT with A known given by (15) or (19). When the energy does not depend on θ (2), the MLE of θ which maximizes the correlation is independent of A. Therefore, the MSE remains the same considering A known or A unknown. The goal of this section is to build a GLRT in considering A unknown which gives a different risk R_2 than the LRT risk R_1. By denoting

$$\begin{aligned}\Lambda_2(\mathbf{Y}) &= \sup_A \left(f_{A,\theta}(\mathbf{Y})|\theta=\varphi\right) - \sup_A \left(f_{A,\theta}(\mathbf{Y})|\theta=\varphi+h\right) \\ &= \frac{1}{2\sigma^2}\left(\left(\mathbf{S}_\varphi^T\mathbf{Y}\right)^2 - \left(\mathbf{S}_{\varphi+h}^T\mathbf{Y}\right)^2\right), \quad (22)\end{aligned}$$

let us now consider the following test ϕ_2 based on the GLRT,

$$\phi_2(\mathbf{Y}) = \begin{cases} H_0 & : \quad \theta=\varphi \quad \text{if } \Lambda_2(\mathbf{Y}) \geq 0 \\ H_1 & : \quad \theta = \varphi+h \quad \text{otherwise.}\end{cases} \quad (23)$$

The risk R_2 associated with the test ϕ_2 becomes:

$$\begin{aligned}R_2(\varphi,h) &= \frac{1}{2}\Pr\left[\Lambda_2(\mathbf{Y})<0|\,\theta=\varphi\right] \\ &+ \frac{1}{2}\Pr\left[\Lambda_2(\mathbf{Y})\geq 0|\,\theta=\varphi+h\right] \\ &= \Pr\left[\left(\mathbf{S}_\varphi^T\mathbf{Y}\right)^2 - \left(\mathbf{S}_{\varphi+h}^T\mathbf{Y}\right)^2 < 0 \,\middle|\, \theta=\varphi\right]. \quad (24)\end{aligned}$$

We clearly have

$$\forall \varphi, h \quad R_2(\varphi,h) \geq R_1(\varphi,h), \quad (25)$$

because the test ϕ_1 given by (13) is LRT and therefore it is Uniformly Most Powerful. The invariant test-theory states that $R_0(\varphi,h) \geq R_2(\varphi,h)$ where A is considered as a nuisance parameter [3, 4]. Indeed, ϕ_0 in (11) is invariant by scaling, *i.e.*,

$$\hat{\theta}(A\mathbf{Y}) = \underset{\theta}{\operatorname{argmax}}\left(\mathbf{S}_\theta^T A\mathbf{Y}\right)^2 = \underset{\theta}{\operatorname{argmax}}\left(\mathbf{S}_\theta^T\mathbf{Y}\right)^2, \quad (26)$$

that is $\phi_0(A\mathbf{Y}) = \phi_0(\mathbf{Y})$. Secondly, the test ϕ_2 in (23) is also invariant by scaling, and, since it is based on the GLRT, it is nearly Uniformly Most Powerful Invariant, *i.e.*, it has the minimum risk among the class of invariant tests by scaling. Recalling that ϕ_0 belongs to this class, we obtain the desired result, namely: $R_0(\varphi,h) \geq R_2(\varphi,h)$ which gives, together with (25), a more accurate lower bound for the MSE:

$$\varepsilon_\theta^2 \geq ZZB_2 \geq ZZB_1 \quad (27)$$

where

$$ZZB_2 = \frac{1}{2}\int_0^\infty \int_{-\infty}^{+\infty} R_2(\varphi,h)\left(p_\theta(\varphi)+p_\theta(\varphi+h)\right)dh\,d\varphi. \quad (28)$$

Now, let us examine how to compute the risk R_2. By denoting $\mathbf{Z} = \begin{bmatrix}\mathbf{S}_\varphi^T \\ \mathbf{S}_{\varphi+h}^T\end{bmatrix}\mathbf{Y}$, the risk $R_2(\varphi,h)$ to calculate is:

$$R_2(\varphi,h) = \Pr\left[\mathbf{Z}^T\mathbf{W}\mathbf{Z} < 0 \,\middle|\, \theta=\varphi\right], \quad (29)$$

where $\mathbf{Z}$ is the 2-dimensional Gaussian random vector with mean $E[\mathbf{Z}] = A\,[1,\rho(h)]^T$ and covariance matrix $\mathbf{\Omega} = \sigma^2\begin{pmatrix}1 & \rho(h) \\ \rho(h) & 1\end{pmatrix}$ and where $\mathbf{W} = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$. It can be noted that $\mathbf{Z}^T\mathbf{W}\mathbf{Z}$ depends only on the ratio A/σ and $\rho(h)$. The risk $R_2(\varphi,h)$ being therefore independent of φ, it will be denoted $R_2(h)$.

The PDF derivation of the quadratic form $\mathbf{Z}^T\mathbf{W}\mathbf{Z}$ is not straightforward because $\mathbf{W}$ is not positive definite. A very accurate and fast method called the saddlepoint approximation allows to give an explicit and simple expression for this PDF. The next section describes the general law approximation of such quadratic forms by the saddlepoint method.

4. THE SADDLEPOINT METHOD

4.1 Saddlepoint background and its adaptation to PDF derivation of general quadratic forms

The saddlepoint method was originally developed to approximate the PDF of the mean of n i.i.d. random variables x_i [5]. If the Moment Generating Function (MGF) of the variable x_i can be computed, the classical SP method provides the PDF of the mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ in the following manner:

- Compute the MGF, $M(t) = E\left[\exp(t\bar{x})\right]$,
- Compute the logarithm $K(t)$ of the MGF $M(t)$ and its first and second derivatives $\dot{K}(t) = \frac{dK}{dt}$ and $\ddot{K}(t) = \frac{d^2K(t)}{dt^2}$,
- Compute the root t_0, called the saddlepoint, of the equation $\dot{K}(t) = \bar{x}$,
- Compute the PDF approximation of the mean by:

$$f_n(\bar{x}) \approx \sqrt{\frac{n}{2\pi\ddot{K}(t_0)}}\exp\left(n\left(K(t_0)-t_0\bar{x}\right)\right). \quad (30)$$

This SP approximation is very precise and generally provides relative errors uniformly bounded by $1/n$ even in the tails regions. Moreover, in practice, n can be very small (starting from $n=2$). Note that if the goal is to estimate the CDF, is is more efficient to use a closed-form formula [6] instead of integrating the SP approximation of the PDF.

The SP approximation can be adapted for estimating the law $\Pr(X<\eta)$ of general quadratic forms $X = \mathbf{Z}^T\mathbf{W}\mathbf{Z}$ where $\mathbf{W}$ is a $d\times d$ not necessarily positive definite matrix. To this end, we need to compute the MGF $M(t) = E\left[\exp(tX)\right] = E\left[\exp\left(t\mathbf{Z}^T\mathbf{W}\mathbf{Z}\right)\right]$ where $\mathbf{Z}$ is a multidimensional Gaussian distribution with mean $\mathbf{m} = E[\mathbf{Z}]$ and covariance matrix $\mathbf{\Omega}$:

$$M(t) = \frac{1}{(2\pi)^{d/2}\sqrt{|\mathbf{\Omega}|}}\int_{\mathbb{R}^d}\exp\left(Q(t)\right)d\mathbf{Z}, \quad (31)$$

where $Q(t) = t\,\mathbf{Z}^T\mathbf{W}\mathbf{Z} - \frac{1}{2}\left(\mathbf{Z}-\mathbf{m}\right)^T\mathbf{\Omega}^{-1}\left(\mathbf{Z}-\mathbf{m}\right)$. This previous equation can be factorized as:

$$Q(t) = -\frac{1}{2}\left[\mathbf{Z}-\alpha(t)\right]^T\mathbf{C}^{-1}(t)\left[\mathbf{Z}-\alpha(t)\right] + D(t), \quad (32)$$

where

$$\begin{cases} D(t) = t\,\alpha^T(t)\,\mathbf{W}\,\alpha(t) - \frac{1}{2}\left[\alpha(t)-\mathbf{m}\right]^T\mathbf{\Omega}^{-1}\left[\alpha(t)-\mathbf{m}\right], \\ \mathbf{C}(t) = \left(\mathbf{\Omega}^{-1} - 2t\,\mathbf{W}\right)^{-1}, \\ \alpha(t) = \mathbf{C}(t)\,\mathbf{\Omega}^{-1}\mathbf{m}. \end{cases}$$

Therefore, it can be proved that the logarithm of the MGF of $\mathbf{X}$ is simply,

$$K(t) = \log M(t) = \frac{1}{2}\log\left(\left|\mathbf{C}(t)\,\mathbf{\Omega}^{-1}\right|\right) + D(t)\,. \qquad (33)$$

The function $K(t)$ only exists if the matrix $\left(\mathbf{\Omega}^{-1} - 2t\mathbf{W}\right)$ is positive definite. By setting $\mathbf{\Omega} = \mathbf{U}\mathbf{U}^T$ and by denoting λ_i the eigenvalues of the symmetric matrix $\mathbf{U}^T\left(\mathbf{W}+\mathbf{W}^T\right)\mathbf{U}/2$, this condition is verified if and only if $t \in]t_1,t_2[$ where $]t_1,t_2[= \bigcap_i \{t\in\mathbb{R} \;\;/\;\; 1-2t\lambda_i > 0\}$.

The SP method involves the computation of $\dot{K}(t)$ and $\ddot{K}(t)$. Some matrix properties derivations allow to write:

$$\begin{cases} \dot{K}(t) = \dfrac{1}{2}\,\mathrm{tr}\left(\mathbf{C}^{-1}(t)\,\dfrac{d\mathbf{C}(t)}{dt}\right) + \dfrac{dD(t)}{dt} \\ \ddot{K}(t) = \dfrac{1}{2}\,\mathrm{tr}\left(\dfrac{d\mathbf{C}^{-1}(t)}{dt}\mathbf{C}^{-1}(t) + \mathbf{C}^{-1}(t)\,\dfrac{d^2\mathbf{C}(t)}{dt^2}\right) + \dfrac{d^2D(t)}{dt^2}\,. \end{cases}$$

Finally, let us find the SP root solution $t_0 \in]t_1,t_2[$ of the nonlinear scalar equation: $\dot{K}(t_0) = \eta$. The saddlepoint theory states that K is convex around t_0 which allows to find numerically t_0 by Newton-Raphson like algorithm. The SP approximation PDF of $X = \mathbf{Z}^T\mathbf{W}\mathbf{Z}$ finally reduces to:

$$f(x) \approx \frac{1}{\sqrt{2\pi\ddot{K}(t_0)}}\exp\left(K(t_0) - t_0 x\right). \qquad (34)$$

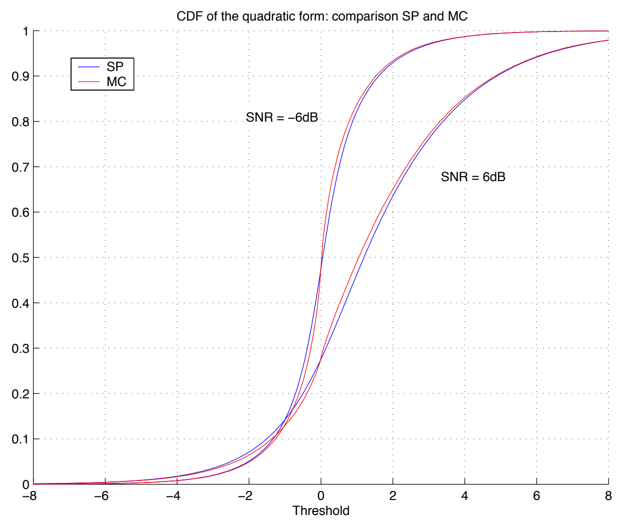

Figure 1: Comparison between SP approximation (35) and Monte-Carlo trials versus threshold η.

Instead of integrating (34) to define Cumulative Density Function (CDF), a more precise approximation was derived by [6]:

$$\Pr\left(\mathbf{Z}^T\mathbf{W}\mathbf{Z} < \eta\right) \approx 1 - \Phi(v) - \phi(v)\left(\frac{1}{u} - \frac{1}{v}\right), \qquad (35)$$

where $u = t_0\sqrt{\ddot{K}(t_0)}$, $v = \mathrm{sign}(t_0)\sqrt{2\left(t_0\,\eta - K(t_0)\right)}$ and where $\phi(v)$ is the standard Gaussian PDF.

Figure 1 presents, for two different SNR and for $\rho = 0.8$, a comparison of the SP approximation (35) and Monte-Carlo computation. This figure shows that the SP approximation is everywhere very accurate.

4.2 Closed-form Saddlepoint approximation for R_2 risk

The SP method of the previous section is applied to the approximation of the risk R_2 with $\eta = 0$ in (35) defined as in (29):

$$R_2(h) = \Pr\left(\mathbf{Z}^T\mathbf{W}\mathbf{Z} < 0\right), \qquad (36)$$

where $\mathbf{W} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and where $\mathbf{Z}$ is a 2-dimensional Gaussian vector with mean $A\,[1,\rho]^T$ and covariance matrix $\mathbf{\Omega} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. Note that ρ depends on h. The risk $R_2(h)$ depending only on A/σ, the value of σ will be chosen equal to one in the following for simplicity.

With the formulation of section 4.1, it can be shown that the function $K(t)$ determining the SP approximation have a closed-form:

$$K(t) = \frac{tA^2\left(1-\rho^2\right)(2t+1)}{1-4t^2\left(1-\rho^2\right)} - \frac{1}{2}\log\left(1-4t^2\left(1-\rho^2\right)\right). \qquad (37)$$

$K(t)$ exists for $t \in]t_1,t_2[$, with $t_2 = \dfrac{1}{2\sqrt{1-\rho^2}}$ and $t_1 = -t_2$. Solving $\dot{K}(t_0) = 0$ leads to compute the roots of a 3rd order polynomial which yields:

$$t_0 = \frac{A^2}{12} - \sqrt{\frac{A^4\left(1-\rho^2\right)+12+12A^2}{36\left(1-\rho^2\right)}}\cos\left(\frac{\pi+\Psi}{3}\right), \qquad (38)$$

where $\cos\Psi = \dfrac{A^2\left(A^4\left(1-\rho^2\right)+18A^2+72\right)\sqrt{1-\rho^2}}{\left(A^4\left(1-\rho^2\right)+12+12A^2\right)^{3/2}}$.

Finally, the ZZB_2 takes the following form:

$$ZBB_2 = \int_0^{T_a} h\left(1-\frac{h}{T_a}\right)R_2(h)\,dh\,. \qquad (39)$$

where $R_2(h)$ is computed using (35) with $\eta = 0$ and using (37) and (38). Although this paper is devoted to real signal and noise, the results remains valid for complex variables as well. The next section gives some examples of performance gain achieved between ZZB_1 and ZZB_2 in the complex case.

5. SIMULATIONS

In this section, we choose a linear frequency modulation code $s(t)$ with bandwidth B and duration T defined as:

$$s(t) = \exp\left(i\pi\frac{B}{T}t^2\right) \quad t\in[-T/2,T/2] \qquad (40)$$

whose autocorrelation function is $\rho(h) = \dfrac{\sin\pi Bh}{\pi Bh}$. We denote in the following the SNR as A^2/σ^2.

Figure 2 represents as a function of ρ the risk R_1 computed by (17) and the risk R_2 obtained by (36) for three different SNR. It can be observed that R_2 is always greater than R_1 as predicted by the theory (27).

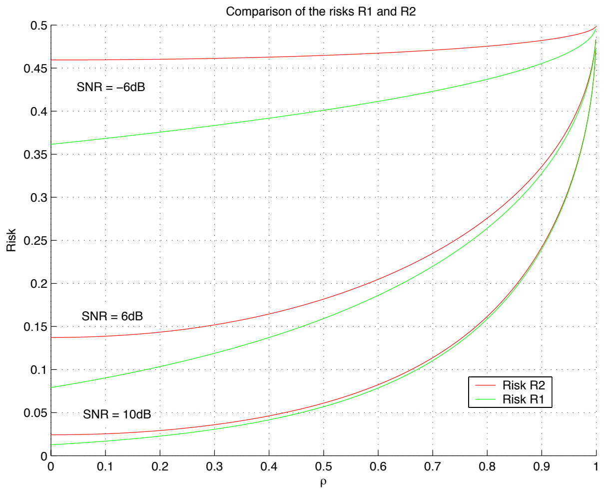

Figure 2: Comparison of the risks R_1 and R_2.

Figure 3 compares the MSE of the MLE obtained by Monte Carlo trials and the CRB with the Ziv-Zakai Bounds ZZB_1 and ZZB_2 for different SNR and for $T_a = T/2, T = 1, B = 512$. All variances have been normalized by the variance of the prior $T_a^2/12$. It can be observed that the ZZB_2 bound is greater than ZZB_1 and remains lower than the MSE of the MLE as predicted by the theory (27). The performance gain between ZZB_1 and ZZB_2 is around 3dB in the low SNR region. Thus, the ZZB_2 can be used to better predict the MLE performance. The high SNR region corresponding to the CRB is shown in Figure 4. In this figure, a comparison of ZZB for different parameters T_a of the *a priori* density p_θ is presented. It can be noted that all the ZZB bounds reach the CRB in the high SNR region.

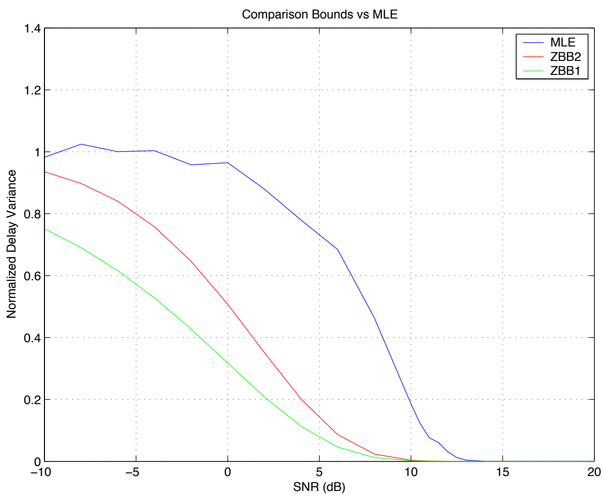

Figure 3: Comparison between the MSE, CRB and ZZBs for a given prior parameter $T_a = T/2$.

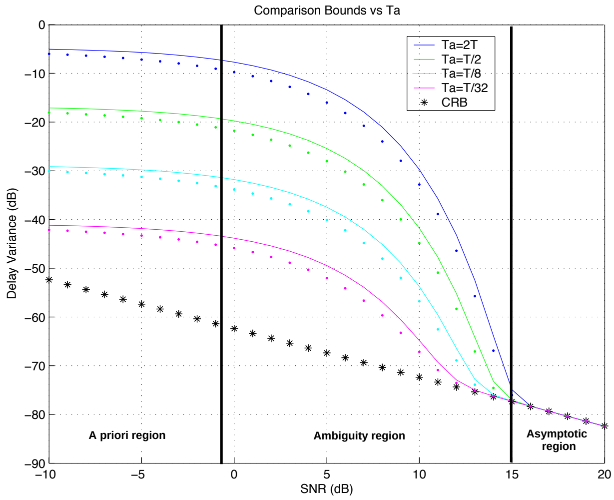

Figure 4: Comparison of the CRB (*), ZZB_1 in dotted lines and ZZB_2 in solid lines for different prior parameters T_a.

6. CONCLUSION

In this paper, we have derived an improvement of the Ziv-Zakai Bound for time-delay estimation which better approaches the Maximum Likelihood Estimator variance. The improvement is made using a GLRT test whose risk is greater than the classical one, based on the LRT. This modified bound is derived by the Saddlepoint approximation method, which gives a simple closed-form expression. Simulations show that this approximation is very accurate and prove that the time-delay MLE performance are better predicted with this proposed bound.

7. ACKNOWLEDGEMENT

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