DEPENDENCIES BETWEEN CODING GAIN AND FILTER LENGTH
IN PARAUNITARY FILTER BANKS
DESIGNED USING QUATERNIONIC APPROACH

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ABSTRACT

In this paper, we investigate the performance limits of four-channel paraunitary filter banks designed using quaternionic approach. Our aim is to reveal how the maximum achievable coding gain depends on the filter length, linearity of the phase responses, and system one-regularity. We also try to obtain some additional insight into coefficient synthesis for paraunitary filter banks, which is often a difficult optimization problem. The results may provide useful advice for the designers who intend to employ filter banks in subband image coding as well as in other applications.

1. INTRODUCTION

It is not easy to evaluate how a filter bank performs in signal compression. Although the coding gain is mainly used for these purposes, it cannot be considered separately from other characteristics of the system.

Firstly, the coding gain can easily be calculated only if the filters are orthogonal. This is the reason for the great importance of paraunitary filter banks (PUFBs), even though the biorthogonal ones have advantages of improved design flexibility and the independene of perfect reconstruction from computational accuracy.

Secondly, other properties of the system affect the perceived quality of the signal reconstructed from quantized subband samples. These are the linearity of the phase responses and the regularity of the wavelet basis. Each of these properties also restricts the design freedom to be exploited in the maximization of the coding gain.

Finally, the coding gain depends on the filter length and number of subbands. As these parameters are directly related to the computational complexity of the filter bank, design trade-offs are unavoidable.

Obviously, such important questions have already been addressed. In spite of that, good reasons motivated us to do some research on the coding gain of PUFBs.

First of all, we have recently developed quaternionic lattice structures for 4- and 8-channel, general and linear phase (LP) PUFBs, also those with pairwise-mirror-image (PMI) symmetric structures for 4-channel PUFBs. The analysis of biorthogonal filter banks is more difficult. The coding gain is often a difficult optimization problem.

Notations: Column vectors are denoted by lower-case bold-faced characters, whereas matrices by the upper-case ones. The notation [A]mn refers to the (m, n) entry of a matrix A. I_m and J_m denote the m × m identity and reverse identity matrices, respectively. The superscript T stands for transposition.

2. CODING PERFORMANCE OF PUFBs

2.1 Coding gain

The most essential performance measure for an M-band PUFB used in data compression is the coding gain defined by [2]

$$\text{CG} = 10 \log_{10} \frac{\sum_{k=0}^{M-1} \sigma_k^2}{\left(\prod_{k=0}^{M-1} \sigma_k^2\right)^{1/2}}.$$  (1)

The subband variances \(\sigma_k^2\) correspond to the diagonal elements of the autocorrelation matrix of the transformed signal, so they can be calculated as

$$\sigma_k^2 = [\text{H} R_{xx} \text{H}^T]_{kk}. \quad (2)$$

To determine the product in the square brackets, which represents that matrix, we need the autocorrelation matrix \(R_{xx}\) of the input signal as well as the transform matrix \(\text{H}\) that describes the filter bank. The transform matrix is formed from the impulse response coefficients as follows:

$$[\text{H}]_{kn} = h_k(L - 1 - n), \quad (3)$$

where \(k = 0, \ldots, M - 1\) and \(n = 0, \ldots, L - 1\), assuming that the filters are of length \(L\).

In our experiments, the matrix \(R_{xx}\) was generated for an AR(1) input process with unit variance and the correlation coefficient of 0.95. Such a model is particularly appropriate only for natural images, and therefore other applications will require different approaches.

It should be emphasized that both (1) and (2) are valid only for PUFBs. The analysis of biorthogonal filter banks is more difficult because of correlations between their subbands.
In spite of its theoretical foundations, the coding gain is a good predictor of experimental rate-distortion performance of filter banks [3]. In image compression, however, its high value should be accompanied by additional properties of the system.

2.2 Linearity of phase responses

Linear phase responses of a filter bank are necessary to use symmetric extension to handle the boundaries of finite-length signals such as images. Unlike other approaches for obtaining nonexpansive transforms, this kind of extension does not introduce discontinuities into data and thus does not cause high-frequency artifacts [5].

2.3 Regularity and DC Leakage

It is desirable to approximate smooth signals using basis functions that have the same property. In coding, this prevents blocking artifacts [3]. That is why regular filter banks that generate smooth wavelet bases are developed.

For an M-band filter bank, regularity can be defined as the number of zeros at the mirror (aliasing) frequencies 2kπ/M, k = 1, . . . , M − 1 of the lowpass filter H0(z). To obtain K degrees of regularity, the polyphase matrix E(z) must satisfy the condition [6]

\[ \frac{d^p}{dz^p} \left( E^M(z) \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-(M-1)} \end{bmatrix} \right) \bigg|_{z = e} = c_p e, \]

where \( e = [1 \ 0 \ \cdots \ 0]^T \), and \( c_p \neq 0 \) for \( n = 0, \ldots, K - 1 \).

The one-regularity (K = 1) is of essential importance as its absence causes DC leakage, which is visible as the particularly annoying checkerboard artifact in decoded images.

3. QUATERNIONIC APPROACH TO 4-CHANNEL PUFBS

3.1 Quaternions

Quaternions discovered by Hamilton in 1843 are hypercomplex numbers of the form [7]

\[ q = q_0 + q_i j + q_j k + q_k, \quad q_0, q_i, q_j, q_k \in \mathbb{R}, \]

with one real and three distinct imaginary parts. The imaginary units: i, j, and k are related by the following equations:

\[ i^2 = j^2 = k^2 = ij = k = 1, \]

\[ ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \]

so quaternion multiplication is non-commutative. However, the conjugate

\[ \overline{q} = q_0 - q_i j - q_j k - q_k, \]

the norm (modulus)

\[ |q| = \sqrt{\overline{q}q} = \sqrt{q_0^2 + q_i^2 + q_j^2 + q_k^2}, \]

and other operations are defined similarly as in the case of ordinary complex numbers.

3.2 Quaternion multiplication matrices

Because quaternions can be identified with four-element column vectors:

\[ q \leftrightarrow q = [q_0 \ q_1 \ q_2 \ q_3]^T, \]

it is possible to represent hypercomplex arithmetic operations in vector-matrix notation. In particular, the multiplication can be written as

\[ pq \leftrightarrow \begin{bmatrix} p_1 & -p_2 & -p_3 & -p_4 \\ p_2 & p_1 & -p_4 & p_3 \\ p_3 & p_4 & p_1 & -p_2 \\ p_4 & -p_3 & -p_2 & p_1 \end{bmatrix} \times \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}, \]

using one of two multiplication matrices: the left- \( M^+ \) or right- \( M^- \) operand \( M^- \) one. Both matrices are orthogonal, or orthonormal if the quaternions have unit norm. Owing to their idiosyncratic structures, the matrices are very useful in diverse scientific applications.

3.3 Quaternion multiplier as a paraunitary building block

We have proposed to use quaternion multiplication matrices in factorizations for 4- and 8-channel PUFBS [8, 9, 10, 11]. Unlike the conventional counterparts, the lattice structures with hypercomplex transformations for 4- and 8-channel PUFBs [8, 9, 10, 11]. The design freedom is related only to the hypercomplex coefficients, although the factors in (12) can be ordered arbitrarily, and (13) can be based on both left- and right-operand quaternion multiplication matrices.

Assuming that the left-operand multiplication matrix is used in (13), the above factorization gives a one-regular filter bank iff

\[ p_0 = \pm 1, \]

where \( o = 1 + i + j + k \).

Figure 1: Graphical symbols for quaternion multipliers whose coefficient \( q \) is (a) the left- and (b) right-hand factor, respectively.
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and

\[ \Phi_{N-1} = M^{-1}(p_{N-1}) \text{diag}(J_2 \Gamma, I_2), \]  

where \( \Gamma = \text{diag}(1, -1) \), in the factorization presented in Section 3.5. For \( N = 3 \), the corresponding lattice structure is shown in Figure 2(b).

The quaternionic coefficients \( p_i \) are again restricted to be unit complex numbers. A filter bank realized using this approach is one-regular iff

\[ P_{N-1} = \pm \sqrt{2} \Phi_0^{-1} \Phi_{N-2}, \]  

where \( \alpha \) is defined as in (20).

4. METHOD OF COEFFICIENT SYNTHESIS

Our design goal is to obtain the coefficients which maximize the coding gain of the filter bank realized using a given factorization.

To ensure that hypercomplex coefficients have unit norm, optimization can be performed indirectly, by using the polar form of a quaternion

\[ \begin{align*}
q_1 &= |q| \cos \phi, \\
q_2 &= |q| \sin \phi \cos \psi, \\
q_3 &= |q| \sin \phi \sin \psi \cos \chi, \\
q_4 &= |q| \sin \phi \sin \psi \sin \chi,
\end{align*} \]

which is much more convenient to deal with than real and imaginary parts. All quaternions of a given norm can simply be produced by changing the values of \( \phi, \psi, \) and \( \chi \). The minimum ranges of the angles necessary to cover a particular subset of hypercomplex numbers are mutually dependent and can be selected in numerous ways. For example 0 \( \leq \phi < 2\pi \), 0 \( \leq \psi \leq \pi \), and 0 \( \leq \chi \leq \pi \) are sufficient to obtain an arbitrary quaternion. Complex numbers are then obtained by taking \( \psi = 0 \), which makes \( \chi \) meaningless and \( \phi \) deciding.

In this way, we can make the coding gain function of a vector of angles, each of which is one degree of design freedom. Every three elements of the vector describe one coefficient of the lattice structure for general PUFBs. In the case of (PM) LP PUFBs, there is a one-to-one correspondence between an angle and degenerated hypercomplex coefficient. The calculation of the objective function for given angle values comprises the conversion of the coefficients from their polar to rectangular form, generation of the polyphase matrix according to the appropriate factorization, assembling the transform matrix, and using it in (18).

Due to the periodicity of trigonometric functions, it is unnecessary to constrain the angles, so their values that maximize the coding gain can be searched using efficient algorithms for unconstrained optimization. We tested both \texttt{fminunc} and \texttt{fminsearch} routines provided by MATLAB to solve such problems [12]. Whereas \texttt{fminsearch} uses the simplex search method, \texttt{fminunc} uses the quasi-Newton method with numerical gradient approximation. Both are sensitive to the selection of the starting point and do not guarantee to locate the global maximum. So advanced problems require repeating the optimization for a number of different starting points, and using different methods to verify the obtained results.

5. EXPERIMENTAL RESULTS

We observed that both routines give similar results for the considered objective functions, which are highly nonlinear though continuous. There are many local maxima, but one of them is global. For a given starting point only the closest local maximum is located, so computations must usually be repeated to hit the global one. Because the convergence of \texttt{fminunc} was far better than that of \texttt{fminsearch}, which required much more computation to obtain the same result, we gave up using the latter routine after preliminary tests.

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Figure 2: Quaternionic lattice structures for 4-channel (a) general, (b) LP, and (c) PMI LP PUFBs \((N = 3)\).

3.5 Four-channel LP PUFB

The polyphase matrix of a 4-channel LP PUFB can also be factorized according to (11), but the definitions of the stages are different. Namely [6],

\[ E_0 = \frac{1}{\sqrt{2}} \Phi_0 \text{diag}(I_{M/2}, J_{M/2}) W, \]  

(15)

and

\[ G_i(z) = \frac{1}{2} \Phi_i W \text{diag}(I_{M/2}, J_{M/2}) W, \quad i = 1, \ldots, N - 1, \]  

(16)

where

\[ W = \begin{bmatrix} I_{M/2} & I_{M/2} \\ I_{M/2} & -I_{M/2} \end{bmatrix}, \]  

(17)

\[ \Phi_0 = M^{-1}(p_0) M^t(q_0), \]  

(18)

and

\[ \Phi_i = M^{-1}(p_i), \quad i = 1, \ldots, N - 1. \]  

(19)

All \( p_i \) and \( q_0 \) are unit quaternions that have the two last imaginary parts (related to \( j \) and \( k \)) set to zero. So the coefficients are unit complex numbers in fact. For \( N = 3 \), the corresponding lattice structure is shown in Figure 2(b).

A filter bank described by such a factorization is one-regular iff

\[ q_0 = \pm \frac{1}{\sqrt{2}} p_0 \cdots p_{N-1} a, \]  

(20)

where \( a = 1 + i \).

3.6 Four-channel PMI LP PUFB

To have pairwise-mirror-image symmetric magnitude responses, it is sufficient to take

\[ \Phi_i = M^{-1}(p_i), \quad i = 0, \ldots, N - 2, \]  

(21)
In main experiments, general, LP, and PMI LP PUFBs were examined, with and without the one-regularity conditions satisfied. The number of the sections in a particular factorization, \( N \), was changed from 2 to 8, which corresponds to the filter length \( L \) in the range of 8 to 32. The related changes in design freedom depend on the kind of system and are shown in Fig. 3.

For each combination of the type and length, two thousand filter banks were designed using \textit{fminunc} with different random starting points. Table 1 is the result of a simple analysis of the collected data. The most interesting quantities are also shown in Figures 4–6 as functions of filter length.

Apart from the essential statistical quantities calculated for the obtained coding gains, we are interested in the characteristics of the objective functions. These are the number of the angles to be optimized, the number of the identified local maxima, and the probability of hitting the global maximum. When calculating the latter two quantities, we combined many local maxima into one if the difference between the corresponding coding gains did not exceed 10 dB, regardless of the angle values. This seemed to be necessary because the coding gain depends only on the magnitude responses of the channel filters, not on their phase responses or order, which are not controlled during the angle optimization. Moreover, neither the coefficients of a hypercomplex lattice structure nor their polar representation are unique when the angles in (24) are unconstrained.

The in the case of LP PUFBs, the objective function has a few well distinguishable local maxima, whose number depends linearly on filter length, as Fig. 5 shows. For general PUFB, there is much more maxima and their number grows faster that filter length. Moreover, the differences between the corresponding coding gains are slight.

Regardless of the system, random selection of starting points for line search turns out to guarantee the approaching the global maximum in a moderate number of trials. For general PUFBs, the probability of hitting the maximum is inversely proportional to filter length, as Fig. 6 shows.
Thus, there are good reasons to perform both design and implementation of a filter bank using hypercomplex numbers.

The results reported in literature, concerning the maximum coding gain obtainable for a given filter length, are generally confirmed. However, our observation that in 4-channel PUFBs, the coding gain is only very slightly affected by imposing both one-regularity and the pairwise-mirror-image symmetry of magnitude responses, has the hallmarks of novelty.

The presented method of coefficient synthesis is simple and gives satisfactory results. It allowed us to characterize objective functions in terms of the number of local maxima and the probability of hitting the global maximum.

Both the reported techniques and observations should be interesting and inspiring for persons who contend with filter bank design as well as with other similar optimization problems.

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