ON HYPERBOLIC COMPLEX LTI DIGITAL SYSTEMS

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ABSTRACT
An investigation of digital systems based on hyperbolic (complex) numbers is presented for the first time. We consider the algebra of hyperbolic numbers (i.e. complex numbers possessing the imaginary \( j^2 = 1 \)) as the simplest instance of a non-division algebra, which allows us to examine the impact of zero divisors in the case of DSP applications. As an example, the analysis of a general first order hyperbolic system is presented.

1. INTRODUCTION
Complex numbers \( \mathbb{C} \) have been in use for digital signal processing (DSP) purposes for a long time. They allow for compact signal representations for the first time, thereby determining the impact of zero divisors on DSP. As an example, the analysis of a general first order hyperbolic system is presented.

2. HYPERBOLIC NUMBERS
Corresponding to complex numbers \( z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}, i^2 = -1, \) a hyperbolic number \([1, 3, 7, 9]\)

\[
\begin{align*}
a &= a^r + ja^s \in \mathbb{D}, \quad a^r, a^s \in \mathbb{R}, \quad j^2 = 1, \quad j \notin \mathbb{R} \quad (1)
\end{align*}
\]

is composed of a real (\( a^r \)) and hyperbolic imaginary (\( a^s \)) part. Obviously, addition of two hyperbolic numbers is performed componentwise: \( a + b = a^r + b^r + j(a^s + b^s) \).

In contrast to complex numbers, the square of the hyperbolic imaginary unit \( j^2 = 1 \) is positive \([10]\) (such imaginaries are also applied in Clifford algebras \([2, 3]\)).

Hence, the following multiplication rule results from (1):

\[
(ab)(a^r + ja^s) = a^r b^r + a^s b^s + j(a^r b^s + a^s b^r) \quad (2)
\]

Corresponding to complex numbers, multiplication (2) is distributive over addition, associative and commutative. In conjunction with (2), hyperbolic numbers span a 2-D real vector space \( \mathbb{R}^2 \) with a specific multiplication rule. We define the hyperbolic conjugate

\[
\bar{a} = -j a^s, \quad \overline{\mathbf{a}} = \mathbf{a}, \quad \overline{a + b} = \overline{a} + \overline{b}, \quad \overline{ab} = \overline{a} \overline{b} \quad (3)
\]

with the same properties as the common complex conjugate, albeit valid only for hyperbolic numbers (cf. sec. 4.1). To separate the components of (1), the following operators are introduced applying (3): \( \mathrm{Rh} \) and \( \mathrm{Ih} \).

\[
\begin{align*}
\mathrm{Rh} \{a\} &:= \frac{a + \overline{a}}{2} = a^r, \\
\mathrm{Ih} \{a\} &:= \frac{a - \overline{a}}{2} = a^s
\end{align*}
\]

(4)

where \( \mathrm{Rh} \{a\} \) represents the real, and \( \mathrm{Ih} \{a\} \) the hyperbolic imaginary part of \( a \).

An isomorphic representation of a hyperbolic number \( a \), being completely equivalent to (1), is given by the real 2 \( \times \) 2 matrix

\[
\mathbf{M}_a = \begin{bmatrix} a^r & a^s \\ a^s & a^r \end{bmatrix} \in \mathbb{R}^{2 \times 2}, \quad a = a^r + ja^s \in \mathbb{D}. \quad (5)
\]

Any operation or property can likewise be performed using representation (1) or (5), respectively. For instance, it is readily shown that matrices of the form (5) are commutative regarding multiplication: \( \mathbf{M}_a \mathbf{M}_b = \mathbf{M}_b \mathbf{M}_a \). However, calculations applying the matrix formulation are highly redundant and are predominantly applied for analytical purposes.

The modulus \( |a| = \sqrt{a^2 + a^s} \) of a complex number \( z \in \mathbb{C} \) is equal to its Euclidean distance from the origin. For hyperbolic numbers, however, a modulus \( |a| = \sqrt{a^2 + a^s} \) does not fit to the intrinsic nature of the hyperbolic number plane. For instance, in general the square identity \([9]\) does not hold: \( |ab| \neq |a| |b| \). As a remedy, we define the quadratic form \( N(a) \) of a hyperbolic number, deviating from \( |a|^2 \), as follows:

\[
N(a) = a \overline{a} = a^2 - a^s = \det \mathbf{M}_a \in \mathbb{R}. \quad (6)
\]

It will serve as a norm, but note that (6) can be negative. Nevertheless, basic matrix algebra (\( \det \mathbf{A} \cdot \det \mathbf{B} = \det \mathbf{AB} \)) applied to (5) and (6) shows that \( N(a) \) satisfies the property \( N(ab) = N(a)b \).

In fig. 1, some constant contours of (6) are depicted in the hyperbolic number plane.
1.5 Existence of zero divisors

As a direct consequence of $j^2 = +1$, the hyperbolic numbers are not a division algebra. This means that the inverse of a hyperbolic number

$$a^{-1} = \frac{\overline{a}}{N(a)} = \frac{\overline{a} - ja}{a^2 - \overline{a}^2}$$

(7)

is not always defined. Hence, although all other requirements are met, they do not form a field as complex numbers, but a commutative ring with unity. As a result, zero divisors [4] exist: a product of two non-zero numbers can yield zero, e.g. $(1+j)(1-j) = 1 - j^2 = 0$. A zero divisor is not invertible, e.g. $(1+j)^{-1}$ (as well as $0^{-1}$ does not exist), and has a zero norm $N(\cdot)$. This is additionally confirmed with isomorphic matrix algebra (5), (6) allowing for inversion if and only if $M_d$ is non-singular (det $M_d \neq 0$). Hence, all zero divisors in $\mathbb{D}$ are determined by

$$\text{det} M_d = a^2 - \overline{a}^2 = 0 \iff |a| = |\overline{a}|,$$

(8)

forming two lines in the hyperbolic number plane (see fig. 1):

$$a_1 = a_1 (1+j), \quad a_2 = a_2 (1-j), \quad a_1, a_2 \in \mathbb{R}.$$ (9)

The product of any two numbers $a_1$ times $a_2$ results in zero.

2.2 Orthogonal representation

The existence of zero divisors allows for the orthogonal decomposition [3, 5, 9] of any hyperbolic number:

$$[\tilde{a}_1, \tilde{a}_2] = \tilde{a}_1 e_1 + \tilde{a}_2 e_2 = \tilde{a}_1 \frac{1+j}{\sqrt{2}} + \tilde{a}_2 \frac{1-j}{\sqrt{2}}, \quad \tilde{a}_1, \tilde{a}_2 \in \mathbb{R}. \quad (9)$$

This utilises an idempotent orthogonal system spanned by its base vectors $e_1, e_2 \in \mathbb{D}$, which fulfills the following properties [8]:

$$e_1 e_2 = 0, \quad e_1^2 = e_2^2 = e_1, \quad e_1 + e_2 = 1 \in \mathbb{R}. \quad (10)$$

To derive the orthogonal base vectors $e_1, e_2 \in \mathbb{D}$, we apply the algebraic rule that every idempotent element $e^2 = e$ different from $0^2 = 0$ and $1^2 = 1$ (since $a^2 - a = a(a - 1) = 0$) is a zero divisor:

$$a^2 + 2ja' + a'' - a'j = 0 \Rightarrow a^2 + a'j = \overline{a}^2 - \overline{a'}j \quad \text{and} \quad a'j = 2a' \overline{a}.$$ (11)

There exist two possible nontrivial ($a \neq 0$) solutions of (11): $a' = 0$ or $a' = \frac{1}{2}$. The former one leads to $0 = a' - a^2 \Rightarrow a' = 1 \Rightarrow e = 1$ (unit element 1), whereas the latter results in

$$a^2 = \frac{1}{4} \Rightarrow a' = \pm \frac{1}{2} \Rightarrow e_1 = \frac{1}{2}, \quad e_2 = \frac{1}{2}.$$ (12)

the two idempotent zero divisors of $\mathbb{D}$.

In the orthogonal representation, any arithmetic operation, such as addition, multiplication, division or exponentiation, is performed componentwise [3, 9]:

$$f(a) = f(\tilde{a}_1, \tilde{a}_2) = [f(\tilde{a}_1), f(\tilde{a}_2)].$$ (13)

Obviously, a calculation using the orthogonal representation saves computational complexity. For example, a hyperbolic multiplication according to (2), requires 4 real multiplications and 2 real additions, whereas it is reduced to only 2 real multiplications compliant with (13). Basically, this reflects that $\mathbb{D}$ is in fact not more than an isomorphism to the direct sum $\mathbb{R} \oplus \mathbb{R}$ (which results in the simple real vector space $\mathbb{R}^2$ without a separate multiplication rule [4]. However, just because of this isomorphism we are able to lower the computational load of multiplication, while still realising the complete structure defined by (1) and (2), respectively. To enable this alternative, an orthogonalisation

$$\tilde{a}_1 = a + a', \quad \tilde{a}_2 = a - a'$$ (14)

and a deorthogonalisation

$$a = \frac{1}{2}(\tilde{a}_1 + \tilde{a}_2), \quad a' = \frac{1}{2}(\tilde{a}_1 - \tilde{a}_2).$$ (15)

procedure, following from (9), is needed.

3. HYPERBOLIC LTI SYSTEMS

In the following, we develop an approach to LTI digital systems based on the hyperbolic number system $\mathbb{D}$. The structure of such a system is determined by (2), and consists of two distinct real subsystems $H_R(z)$ and $H_I(z)$, both existing twice: fig. 2. As in the real and complex cases, a hyperbolic LTI system is completely specified by its (hyperbolic) impulse response $h(k)$, which determines the relationship between input $x(k)$ and output $y(k)$:

$$y(k) = h(k) \ast x(k), \quad k \in \mathbb{Z}, \quad h(k), x(k), y(k) \in \mathbb{D}. \quad (16)$$

Thereby, the real-valued implementation of the convolution operator $\ast$ in (16) derives from the multiplication rule (2)

$$y(k) = h^R(k) \ast x^R(k) + h^I(k) \ast x^I(k) \ast j \left[ h_R(k) x^R(k) + h_I(k) x^I(k) \right]$$ (17)

and therefore the hyperbolic convolution adopts its properties, such as commutativity.

3.1 Hyperbolic transfer functions

For frequency domain representation, we apply the z-transform. For a hyperbolic signal $x(k)$ or impulse response $h(k)$, the z-transform is carried out componentwise using the linearity property of $\mathcal{Z} \{ \cdot \}$:

$$X(z) = \mathcal{Z} \{ x(k) \} = \sum_{k=0}^{\infty} x(k) z^{-k} + \sum_{k=\infty}^{-\infty} x^*(k) z^{-k} \in \mathbb{C} \oplus \mathbb{D}. \quad (18)$$

Due to the complex nature of the z-transform, $H(z) = \mathcal{Z} \{ h(k) \}$ according to (18) is hyperbolic with complex components, in general: $H(z) \in \mathbb{C} \oplus \mathbb{D}$ (tessarines [10], sec. 4.1). Thus, for hyperbolic systems (based on hyperbolic numbers $\mathbb{D}$ with real components) we have to deal with transfer functions which utilise a combination of two different number systems.

To separate the real and imaginary parts of a complex impulse response $h(k) \in \mathbb{C}$ in the z-domain, commonly the operator $[5, 11]$

$$\text{Re} \{ H(z) \} = \frac{H(z) + H^*(z)}{2} = \mathcal{Z} \left\{ \frac{h(k) + h^*(k)}{2} \right\}$$ (19)

is applied. Note that the two independent subsystems of a hyperbolic system are real: $H_R(z) = \text{Re} \{ H_R(z) \}$, $H_I(z) = \text{Re} \{ H_I(z) \}$. 

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Hence, similar to (19), they can be separated from the transfer function $H(z)$ of the hyperbolic system using (4):

$$H_R(z) = \text{Rh}\{H(z)\} = \frac{H(z) + \mathcal{F}(z)}{2}, \quad (20)$$

Since hyperbolic conjugation has no effect on the complex components of tessarine numbers (cf. sec. 4.1), $\mathcal{F} = z$ in (20).

Another approach to a hyperbolic system is to represent it as a real $2 \times 2$ MIMO system with vectorial input and output signals, and impulse response, respectively:

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \quad y(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix}, \quad h(k) = \begin{bmatrix} h_1(k) & h_2(k) \\ h_3(k) & h_4(k) \end{bmatrix}.$$  

In this case, the corresponding transfer matrix can readily be derived from (5):

$$H(z) = \left[ \begin{array}{c} H_R(z) \\ H_I(z) \end{array} \right] = \left[ \begin{array}{c} \text{Rh}\{H(z)\} \\ \text{Ih}\{H(z)\} \end{array} \right]. \quad (22)$$

In the special case where $\text{Rh}\{H(z)\} = H_L(z)$ and $\text{Ih}\{H(z)\} = H_H(z)$, the complex tapping equals zero.

### 3.2 Orthogonal decomposition

We do not have to implement the hyperbolic system according to (17), or (22), respectively, because the underlying algebra allows for orthogonal decomposition (as a result of the existence of zero divisors). Hence, a minimal system realisation is based on (9), and signal processing is performed according to (13), as depicted in fig. 3: the hyperbolic signal is fed into the orthogonalisier $\mathcal{F}$, paralllely processed by the (only) two orthogonal real subsystems $H_1(z)$ and $H_2(z)$, and finally recovered from the deorthogonalisator $E$. The $2 \times 2$ real MIMO representation of this processing chain is given by

$$H(z) = E \cdot \tilde{H}(z) \cdot F,$$  

which utilises the orthogonalisator

$$\tilde{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = Fx(k),$$  

following (14), and the deorthogonalisator

$$x(k) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = Ex(k) = \frac{1}{2} F\tilde{x}(k),$$  

derived from (9). The MIMO transfer matrix of the orthogonalsed system according to (24)

$$\tilde{H}(z) = \begin{bmatrix} H_1(z) & 0 \\ 0 & H_2(z) \end{bmatrix}, \quad H(z) = \left[ \begin{array}{c} \tilde{H}_1(z) \\ \tilde{H}_2(z) \end{array} \right]. \quad (27)$$
In the following, a general recursive hyperbolic LTI system of first order is analysed, resulting in the hyperbolic transfer function
\[ H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}. \tag{29} \]
Each coefficient is a hyperbolic number and can be split into two distinct real-valued parts, e.g., \( b_0 = b_0^r + j b_0^i \), \( b_1, b_i \in \mathbb{R} \). Using (20) and \( a\bar{b} + \bar{a}b = 2(\bar{a}b - a^*b^*) \), the transfer function of the first subsystem is given by:
\[ H_R(z) = \frac{1}{2} \left( \frac{b_0 + b_1 z^{-1} + \bar{b}_0 + \bar{b}_1 z^{-1}}{1 + a_1 z^{-1}} \right) \tag{30} \]
\[ = \frac{b_0 + (a_1^* - a_1) z^{-1}}{1 + a_1 z^{-1}}. \]
Applying (21) and \( a\bar{b} - ab = 2j(\bar{a}b - a^*b^*) \), we derive the second subsystem’s transfer function:
\[ H_I(z) = \frac{1}{2} \left( \frac{b_0 + b_1 z^{-1} - \bar{b}_0 - \bar{b}_1 z^{-1}}{1 + a_1 z^{-1}} \right) \tag{31} \]
\[ = \frac{b_1 - (a_1^* - a_1) z^{-1}}{1 + a_1 z^{-1}}. \]
We see that the degree of the real subsystems (30) and (31) is doubled relative to the hyperbolic system’s degree. Hence, if we want to realise a real rational transfer function of a particular degree, we may employ a hyperbolic system of half degree (although considering that a hyperbolic delay is implemented by two real delays). The same relationship also applies for complex [13] and some other hypercomplex systems [5]. The system can also be implemented according to fig. 3. For the FIR case \( a_1 \equiv 0 \), we see that the \( z^{-1} \) terms vanish in the numerators of (30) and (31), which means that the hyperbolic system (29) is reduced to a purely vectorial system (fig. 4) of only first order.

As an example, we realise the second order real system
\[ H_{real}(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}. \tag{32} \]
\[ a_1 = -0.5095, \quad a_2 = -0.1334, \quad b_0 = b_2 = 0.4320, \quad b_1 = -0.5078, \]
\[ a_1 \quad \text{is a first order hyperbolic system according to (29).} \]
\[ H_{real}(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}. \tag{32} \]
\[ a_1 = -0.5095, \quad a_2 = -0.1334, \quad b_0 = b_2 = 0.4320, \quad b_1 = -0.5078, \]
\[ a_1 \quad \text{is a first order hyperbolic system according to (29).} \]
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\[ a_1 = -0.5095, \quad a_2 = -0.1334, \quad b_0 = b_2 = 0.4320, \quad b_1 = -0.5078, \]
\[ a_1 \quad \text{is a first order hyperbolic system according to (29).} \]
A tessarine LTI system \([6, 8]\) consists of 16 (4 different) real subsystems and is, similar to the complex and hyperbolic cases (sec. 3.5), capable of doubling the degree of a hyperbolic system once more. However, it is not possible to reduce the expenditure of complex coefficient multiplication by further orthogonalisation. Hence, also for tessarine systems, the only feasible orthogonal decomposition is represented by \((14), (15), (25)\) and \((26)\).

### 4.2 \(2^N\)-dimensional hyperbolic numbers

In order to process vector signals of higher dimension \(n\), or to increase the degree multiplication (sec. 3.5), it is possible to define hyperbolic numbers with hyperbolic components. By repeating the doubling procedure \([1]\), numbers of dimension \(n = 2^N\), \(N \in \mathbb{N}\) can be obtained \([14]\). We stress that, in contrast to the step from real to hyperbolic numbers, further increase of dimension does not produce any additional issue. Every property of hyperbolic numbers is retained even for higher dimensions. Orthogonalisation can be extended to \(n\)-times decomposition. It can be performed efficiently by the (fast) Hadamard transform \([8]\).

The structure depicted in fig. 3 is in fact equal to common coupled, e.g. allpass systems. It is an open question, if it is feasible to employ higher dimensional hyperbolic LTI systems to describe coupled systems with more than two input and output ports, in order to simplify the design of such. The \(2^N\)-dimensional approach can also be combined with the use of complex components (sec. 4.1).

### 5. CONCLUSION

A first step was made to describe the properties and efficient implementation of hyperbolic digital systems in general. Since hyperbolic subalgebras emerge in many other non-division hypercomplex algebras, they represent a comprehensible example for the examination of the impact of zero divisors on DSP. However, hyperbolic systems certainly are most useful if the underlying structure is somehow related to the application aimed at. Presently, due to limitations of realisable transfer functions, they do not represent a self-contained system class. Nevertheless, further investigation has to reveal if structures, as proposed in sec. 3.5, are feasible and efficient even for general DSP purposes.

### REFERENCES


