QUATERNIONS AND BIQUATERNIONS FOR SYMMETRIC MARKOV-CHAIN SYSTEM ANALYSIS

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ABSTRACT

In this conference, we use quaternions, biquaternions, and their related algebras similar to them to model symmetric Markov systems. With these algebras, the original $N \times N$ Markov system can be reduced into an $(N/2)\times(N/2)$ or $(N/4)\times(N/4)$ system. It makes the system easier for implementing and analysis. In addition to Markov chains, the proposed idea is also helpful for simplifying the complexities of other symmetric systems whose interactions between two objects are determined by their distance.

1. INTRODUCTION

The quaternion [1][2] is a generalization of the complex algebra. Its application in signal and image processing were noticed in recent years [2][3][4]. The quaternion has been applied for filter design, feature extraction, pattern recognition, rotation analysis, differential equation analysis, and SVD decomposition, etc. In this paper, we propose another application of the quaternion and its related algebras, i.e., the Markov chain analysis.

2. QUATERNIONS AND RELATED ALGEBRAS

Quaternions [1][2] are generalizations of complex numbers. A quaternion has 4 components: real, i-, j-, and k-parts:

$$q = q_r + q_i \cdot i + q_j \cdot j + q_k \cdot k, \qquad (1)$$

and i, j, k obey the rules as below:

$$i^2 = j^2 = k^2 = -1, \quad i \cdot j = k, \quad j \cdot k = i, \quad k \cdot i = j,$$
$$j \cdot i = -k, \quad k \cdot j = -i, \quad i \cdot k = -j. \qquad (2)$$

Quaternions can be used for the color image analysis, filter design, and 3-D object analysis.

The definition of the Clifford biquaternion [6][10] is similar to that of the quaternion, but there are 8 elements:

$$q = q_0 + q_i \cdot i + q_j \cdot j + q_k \cdot k + q_I I + q_{iI} \cdot iI + q_{jI} \cdot jI + q_{kI} \cdot kI, \qquad (3)$$

where i, j, and k satisfy (2) and

$$I^2 = 1, \quad i \cdot I = I \cdot i, \quad j \cdot I = I \cdot j, \quad k \cdot I = I \cdot k. \qquad (4)$$

The Tessarine with complex coefficients [5] (we call it the complex Tessarine) is defined in the similar form.

$$q = q_r + q_i \cdot i + q_I I + q_{iI} \cdot iI. \qquad (5)$$

In this condition, $j^2 = 1$, $k^2 = -1$, $ij = ji$, $ik = ki$, and $jk = kj$. The Clifford biquaternion and the complex Tessarine have idempotent elements e_1 and e_2:

$$e_1 = (1+I)/2, \quad e_2 = (1-I)/2, \qquad (6)$$
$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1 e_2 = 0.$$

Idempotent elements are helpful for fast computation. If q is defined in (3) and p is defined as

$$p = p_0 + p_i \cdot i + p_j \cdot j + p_k \cdot k + p_I I + p_{iI} \cdot iI + p_{jI} \cdot jI + p_{kI} \cdot kI, \qquad (7)$$

then p and q can be expressed as:

$$p = p_1 e_1 + p_2 e_2, \quad q = q_1 e_1 + q_2 e_2, \qquad (8)$$

where

$$p_1 = pe_1 = (p_0 + p_i \cdot i + p_j \cdot j + p_k \cdot k + p_I + p_{iI} \cdot i + p_{jI} \cdot j + p_{kI} \cdot k)/2, \quad p_2 = pe_2. \qquad (9)$$

Similarly,

$$q_1 = qe_1 \quad \text{and} \quad q_2 = qe_2. \qquad (10)$$

Then, the product of p and q can be expressed as:

$$pq = (p_1 e_1 + p_2 e_2)(q_1 e_1 + q_2 e_2) = p_1 q_1 e_1 + p_2 q_2 e_2. \qquad (11)$$

In addition to the quaternion and the Clifford biquaternion, there are also other types of 4-D and 8-D algebras. The algebras used in this paper are:

[2-D algebras]

- Complex algebra.
- Tessarines (also called as split-complex numbers, and double complex numbers) [5][6]:

$$q = a_r + I a_I, \text{ where } I^2 = 1. \qquad (12)$$

[4-D algebras]

- Quaternions (defined in (1) and (2))
- Complex Tessarines (defined in (5)).
- Reduced biquaternion: See [3][7]. It is also called the bicomplex number (C×C). Its definition is the same as the reduced biquaternion in (5) but $I^2 = -1$.
- 4-D purely hyperbolic complex numbers:

$$q = q_r + q_I I + q_J J + q_{IJ} \cdot IJ, \quad I^2 = J^2 = 1, \ IJ = JI. \qquad (13)$$

[8-D algebras]

- Octonions (standard definition [1][9]). It has 7 imaginary parts and the multiplicative rule is shown in [6].
- Clifford biquaternions (see (3)).
- 8-D Tessarines (4-D hyperbolic complex numbers with complex coefficients)

$$q = a_r + ia_i + Ja_J + iJa_{iJ} + Ia_I + iIa_{iI} + JIa_{JI} + iJIa_{iJI},$$

where $i^2 = -1$, $I^2 = J^2 = 1$, $iI = Ii$, $iJ = Ji$, and $IJ = JI$. (14)

- Hamilton's biquaternions [1][3]: Its definition is the same as the biquaternion in (3) but $I^2 = -1$.
- Tri-complex numbers

$$q = a_r + ia_i + Ia_I + iIa_{iI} + Ja_J + iJa_{iJ} + Ka_K + iKa_{iK},$$

where $I^2 = J^2 = K^2 = -1$, $Ii = iI$, $Ji = iJ$,
$Ki = iK$, $IJ = JI$, $IK = KI$, and $JK = KJ$. (15)

3. 2×2 AND 4×4 SYMMETRIC SYSTEM

For many systems, the distance between two points will determine the mutual effect. For example, in physics, the gravitation between two objects is proportional to $1/R^2$ where R is the distance between two objects. Intensities of the electric and magnetic fields also depend on R. In social science, the interaction between two cities can also be modeled by a function of the distance between two points.

Here, we suppose that there are N points in a Markov system and the field in each point is denoted by $x_k[n]$, where $k = 1, 2, \ldots, N$. Suppose that the interaction between two dots is determined by their distance:

$$x_m[n+1] = \sum_{k=1}^{N} x_k[n] a_{m,k}, \text{ where } a_{m,k} = f\left(R_{m,k}\right), \quad (16)$$

$R_{m,k}$ is the distance between m and.

We find that in the system is symmetric, we can model the system by the complex Tessarine.

For example, for the system of Fig. 1

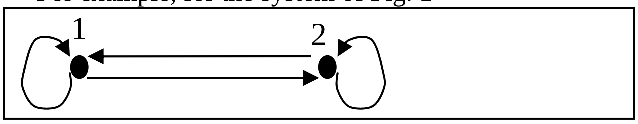

Fig. 1 A symmetric 2 × 2 system.

If the interaction is determined by the instance between two dots, it can be modeled by the following Markov chain:

$$x_1[n+1] = ax_1[n] + bx_2[n], \quad (17)$$

$$x_2[n+1] = bx_1[n] + ax_2[n]. \quad (18)$$

Then, we can define a function as

$$x[n] = x_1[n] + Ix_2[n], \quad (19)$$

where $I^2 = 1$. If we define

$$h = a + Ib, \quad (20)$$

then

$$h \cdot x[n] = ax_1[n] + bx_2[n] + I(ax_2[n] + bx_1[n]), \quad (21)$$

i.e., $x[n+1] = h \cdot x[n]$. (22)

If $x_1[n]$ and $x_2[n]$ are complex functions, then (22) is a **complex Tessarine operation**. That is, the system 1 can be modeled by a complex Tessarine operation. If $x_1[n]$ and $x_2[n]$ are quaternion functions, Fig. 1 can be modeled by a Clifford biquaternion operation.

If the effect of the system is determined by the only the distance but also the **direction**, that is,

(a) The interaction between the points p and q and that between r and s are the same $\overrightarrow{pq}$ and $\overrightarrow{rs}$ have the same distance and direction.

(b) If $\overrightarrow{pq} = -\overrightarrow{rs}$, then the interactions between p and q (denoted by $h_{p,q}$) and r and s ($h_{r,s}$)) has the relation of

$h_{p,q} = -h_{r,s}$

then

$$x_1[n+1] = ax_1[n] - bx_2[n], \quad (23)$$

$$x_2[n+1] = bx_1[n] + ax_2[n], \quad (24)$$

then, we can define a function of

$$x[n] = x_1[n] + Ix_2[n] \quad (25)$$

where $I^2 = -1$, $iI = Ii$. (26)

In this case, we can also prove that

$$x[n+1] = h \cdot x[n], \quad \text{where} \quad h = a + \iota b. \quad (27)$$

If $x_1[n]$ and $x_2[n]$ are complex functions, then the system is modeled by the following algebra:

$$q = q_0 + q_1 \cdot i + q_4 \iota + q_5 \cdot i\iota. \quad (28)$$

It is just the same as the **reduced biquaternion algebra**.

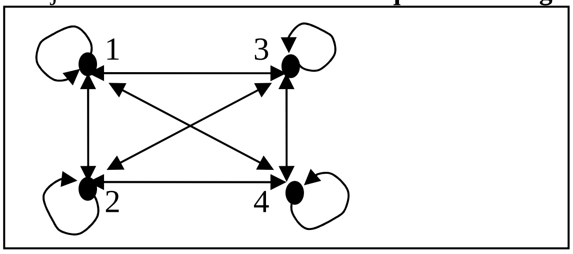

Fig. 2 A doubly symmetric 4 × 4 rectangular system.

The idea can be extended to any symmetric Markov system. For the system in Fig. 2, suppose that the values at the points 1, 2, 3, and 4 are

$x_1[n]$, $x_2[n]$, $x_3[n]$, and $x_4[n]$ (29)

and the interactions between the points m, n are denoted by $h_{m,n}$. Therefore,

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ x_3[n+1] \\ x_4[n+1] \end{bmatrix} = \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} & h_{1,4} \\ h_{2,1} & h_{2,2} & h_{2,3} & h_{2,4} \\ h_{3,1} & h_{3,2} & h_{3,3} & h_{1,4} \\ h_{4,1} & h_{4,2} & h_{4,3} & h_{4,4} \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \\ x_4[n] \end{bmatrix}. \quad (30)$$

First, note that

$h_{m,n} = h_{n,m}$. (31)

Moreover, since interactions are determined by distances,

$h_{1,1} = h_{2,2} = h_{3,3} = h_{4,4} = a$ $h_{1,2} = h_{2,1} = h_{3,4} = h_{4,3} = b$,
$h_{1,3} = h_{2,4} = h_{3,1} = h_{4,2} = c$, $h_{1,4} = h_{2,3} = h_{3,2} = h_{4,1} = d$. (32)

Then (30) can be rewritten as:

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ x_3[n+1] \\ x_4[n+1] \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \\ x_4[n] \end{bmatrix} \quad (33)$$

If we set that

$$y_1[n] = x_1[n] + Ix_2[n], \quad y_2[n] = x_3[n] + Ix_4[n], \quad (34)$$

$$h_1 = a + Ib, \qquad h_2 = c + Id.$$

Then (33) can be rewritten as

$$\begin{bmatrix} y_1[n+1] \\ y_2[n+1] \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ h_2 & h_1 \end{bmatrix} \begin{bmatrix} y_1[n] \\ y_2[n] \end{bmatrix}. \quad (35)$$

That is, after converting into the complex Tessarine form, the 4 × 4 system become a 2×2 system.

Moreover, notice that (35) is also a symmetric system. We can further reduce it into the 1 × 1 form. We can apply the 8-D Tessarines algebra. We can set that

$$y[n] = x_1[n] + Jx_2[n] + Ix_3[n] + IJx_4[n], \quad (36)$$

Then

$$y[n+1] = h\,y[n] \quad \text{where } h = a + Jb + Ic + IJd. \quad (37)$$

That is, after applying the complex Tessarines, the 4 × 4 symmetric rectangular system becomes a 2 × 2 symmetric system. After applying the 8-D Tessarines n, the 4 × 4 symmetric rectangular system becomes a one-input and one-output system.

In the case where the interactions between two dots are affected by the direction:

$$h_{m,k} = -h_{k,m} \quad \text{if } m \neq k, \quad (38)$$

then

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ x_3[n+1] \\ x_4[n+1] \end{bmatrix} = \begin{bmatrix} a & -b & -c & -d \\ b & a & -f & -c \\ c & f & a & -b \\ d & c & b & a \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \\ x_4[n] \end{bmatrix}. \quad (39)$$

Specially, if $d = f$, and $x_1[n]$, $x_2[n]$, $x_3[n]$, and $x_4[n]$ are real,

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ x_3[n+1] \\ x_4[n+1] \end{bmatrix} = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & -c \\ c & d & a & -b \\ d & c & b & a \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \\ x_4[n] \end{bmatrix} \quad (40)$$

We can compare (39) with the quaternion multiplication. If

$$z_r + iz_i + jz_j + kz_k = (x_r + ix_i + jx_j + kx_k)(y_r + iy_i + jy_j + ky_k), \quad (41)$$

then

$$\begin{bmatrix} z_r \\ z_i \\ z_j \\ z_k \end{bmatrix} = \begin{bmatrix} y_r & y_i & y_j & y_k \\ y_i & y_r & y_k & -y_j \\ y_j & -y_k & y_r & y_i \\ y_k & y_j & -y_i & y_r \end{bmatrix} \begin{bmatrix} x_r \\ x_i \\ x_j \\ x_k \end{bmatrix}. \quad (42)$$

Since

$$\begin{bmatrix} x_1[n+1] + 2cx_3[n] \\ x_2[n+1] - 2bx_1[n] \\ x_3[n+1] \\ x_4[n+1] - 2dx_1[n] \end{bmatrix} = \begin{bmatrix} a & -b & c & -d \\ -b & a & -d & -c \\ c & d & a & -b \\ -d & c & b & a \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \\ x_4[n] \end{bmatrix}, \quad (43)$$

we find that, if

$$y[n] = x_1[n] + ix_2[n] + jx_3[n] + kx_4[n], \quad (44)$$

$h = a - ib + jc - kd$, then

$$y[n+1] = y[n]h + \delta, \quad (45)$$

where $\delta = 2\left(cy_j[n] - iby_r[n] - kdy_r[n]\right)$, (46)

$y_r[n]$ and $y_j[n]$ mean the real and j- parts of $y[n]$. That is, a directional symmetric rectangular system can be represented by the **quaternion**.

Table 1 2×2 and 4×4 symmetric system simplification by quaternion and octonion.

original system	algebra	reduced system
2×2 symmetric system	**complex Tessarine**	**1-to-1 system**
2×2 directional symmetric system	reduced biquaternion	1-to-1 system
2×2 symmetric system, real input	Tessarine	1-to-1 system
2×2 symmetric system, quaternion input	Clifford biquaternion	1-to-1 system
2×2 directional symmetric system, real input	complex algebra	1-to-1 system
2×2 directional symmetric system, quaternion input	Hamilton's biquaternions	1-to-1 system
4×4 symmetric system	**8-D Tessarines**	**1-to-1 system**
4×4 directional symmetric system	complex Tessarines	2×2 system with offsets
4×4 directional symmetric system, $d = f$	octonion-V	1-to-1 system with offsets
4×4 symmetric system, real input	tri-complex numbers	1-to-1 system
4×4 directional symmetric system, real input	complex algebra	2×2 system with offsets
4×4 directional symmetric system, $d = f$, real input	quaternion	1-to-1 system with offsets

For the directional symmetric rectangular system in (40), in the case where $d \neq f$, the system can be represented by a Clifford octonion system:

$$\begin{bmatrix} y_1[n+1] \\ y_2[n+1] \end{bmatrix} = \begin{bmatrix} h_1 & h_3 \\ h_3 & h_2 \end{bmatrix} \begin{bmatrix} y_1[n] \\ y_2[n] \end{bmatrix} + \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}, \quad (47)$$

where $y_1[n] = x_1[n] + Ix_4[n]$, $y_2[n] = x_2[n] + Ix_3[n]$,

$$h_1 = a + Id, \quad h_2 = a + If, \quad h_3 = b + Ic, \quad I^2 = -1,$$

$$\delta_1 = \mathrm{Re}(h_3)\mathrm{Re}(y_2[n]), \quad \delta_2 = \mathrm{Re}(h_3)\mathrm{Im}(y_1[n]). \quad (48)$$

In (43) and (47), the inputs are real is considered. When $x_k[n]$ are complex, (44) can be rewritten as

$$y[n] = x_1[n] + Ix_2[n] + Jx_3[n] + Kx_4[n], \quad (49)$$

where the tri-complex numbers (defined in (15) is applied. In this case, (43) can still be applied and (44) is rewritten as

$$h = a - Ib + Jc - Kd,$$

$$\delta = 2\left(cy_j[n] - Iby_r[n] - Kdy_r[n]\right). \quad (50)$$

The relations between the quaternion-related algebras and the symmetric system are listed in Table 1.

4. GENERALIZE SYMMETRIC SYSTEM ANALYSIS

The results in Section 3 can be generalized for all symmetric and doubly symmetric systems.

For the example in Fig. 3, there are 6 dots and form a hexagon. We also suppose that the interactions between two points (denoted by $h_{m,n}$) are determined by their distance. Suppose that the system is symmetric respect to the dash line L_1. That is,

$$h_{1,3} = h_{2,4}, \quad h_{1,4} = h_{2,3}, \quad h_{1,5} = h_{2,6}, \quad h_{1,6} = h_{2,5},$$
$$h_{3,5} = h_{4,6}, \quad h_{3,6} = h_{4,5}, \quad h_{k,n} = h_{n,k}. \tag{51}$$

Then the system can be expressed as

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ x_3[n+1] \\ x_4[n+1] \\ x_5[n+1] \\ x_6[n+1] \end{bmatrix} = \begin{bmatrix} a & b & c & d & e & f \\ b & a & d & c & f & e \\ c & d & a & p & g & l \\ d & c & p & a & l & g \\ e & f & g & l & a & m \\ f & e & l & g & m & a \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \\ x_4[n] \\ x_5[n] \\ x_6[n] \end{bmatrix} \tag{52}$$

It is a 6 × 6 system. If we set that

$$y_1[n] = x_1[n] + Ix_2[n], \quad y_2[n] = x_3[n] + Ix_4[n], \tag{53}$$
$$y_3[n] = x_5[n] + Ix_6[n], \quad h_1 = a + Ib, \quad h_2 = c + Id,$$
$$h_3 = e + If, \quad h_4 = g + Il, \quad h_5 = a + Ip, \quad h_6 = a + Im,$$

where $I^2 = 1$ and $iI = Ii$ (i.e., the reduced biquaternion algebra is applied), then (52) can be expressed as

$$\begin{bmatrix} y_1[n+1] \\ y_2[n+1] \\ y_3[n+1] \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_2 & h_5 & h_4 \\ h_3 & h_4 & h_6 \end{bmatrix} \begin{bmatrix} y_1[n] \\ y_2[n] \\ y_3[n] \end{bmatrix}, \tag{54}$$

and the system is reduced into a 3 × 3 system by the reduced biquaternion algebra.

Furthermore, if the hexagon is also symmetric respect to L_2 (L_2 is perpendicular to L_1), i.e., in addition to (51),

$$h_{1,2} = h_{3,4}, \; h_{1,5} = h_{3,5}, \; h_{1,6} = h_{3,6}, \; h_{2,5} = h_{4,5}, \; h_{2,6} = h_{4,6}, \tag{55}$$

then the system can be expressed as the following operation:

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ x_3[n+1] \\ x_4[n+1] \\ x_5[n+1] \\ x_6[n+1] \end{bmatrix} = \begin{bmatrix} a & b & c & d & e & f \\ b & a & d & c & f & e \\ c & d & a & b & e & f \\ d & c & b & a & f & e \\ e & f & e & f & a & m \\ f & e & f & e & m & a \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \\ x_4[n] \\ x_5[n] \\ x_6[n] \end{bmatrix}. \tag{56}$$

Then, we can use the 8-D Tessarines algebra in (14) and set

$$z_1[n] = x_1[n] + Ix_2[n] + Jx_3[n] + IJx_4[n], \tag{57}$$
$$z_2[n] = x_5[n] + Ix_6[n],$$
$$s_1 = a + Ib + Jc + IJd, \qquad s_2 = e + If, \tag{58}$$
$$s_3 = e + If + Je + IJf, \qquad s_4 = a + Im,$$

and (56) can be rewritten as

$$\begin{bmatrix} z_1[n+1] \\ z_2[n+1] \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \begin{bmatrix} z_1[n] \\ z_2[n] \end{bmatrix}. \tag{59}$$

We then give another example in Fig. 4, which is an 8×8 system. Suppose that the system is symmetric respect to L_1 and the interaction between n and k is denoted by $h_{n,k}$. Then

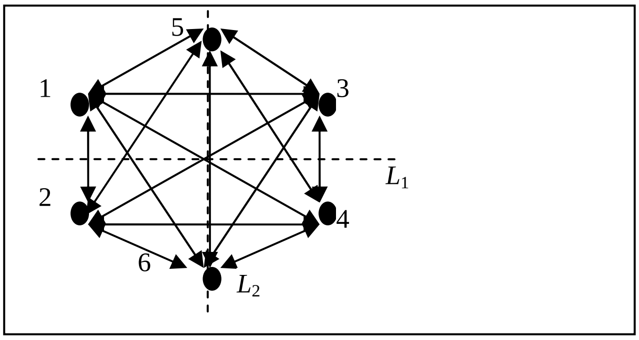

Fig. 3 A symmetric hexagon system.

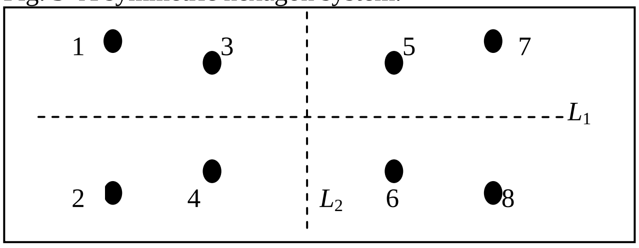

Fig. 4 A symmetric system consists of 8 points

$$h_{1,3} = h_{2,4}, \quad h_{1,4} = h_{2,3}, \quad h_{1,5} = h_{2,6}, \quad h_{1,6} = h_{2,5}, \quad h_{1,7} = h_{2,8},$$
$$h_{1,8} = h_{2,7}, \quad h_{3,5} = h_{4,6}, \quad h_{3,6} = h_{4,5}, \quad h_{3,7} = h_{4,8}, \quad h_{3,8} = h_{4,7},$$
$$h_{5,7} = h_{6,8}, \quad h_{5,8} = h_{6,7}. \tag{60}$$

Therefore, the system can be expressed as

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ x_3[n+1] \\ x_4[n+1] \\ x_5[n+1] \\ x_6[n+1] \\ x_7[n+1] \\ x_8[n+1] \end{bmatrix} = \begin{bmatrix} a & b & c & d & e & f & g & l \\ b & a & d & c & f & e & l & g \\ c & d & a & m & o & p & r & t \\ d & c & m & a & p & o & t & r \\ e & f & o & p & a & u & v & w \\ f & e & p & o & u & a & w & v \\ g & l & r & t & v & w & a & \alpha \\ l & g & t & r & w & v & \alpha & a \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \\ x_4[n] \\ x_5[n] \\ x_6[n] \\ x_7[n] \\ x_8[n] \end{bmatrix} \tag{61}$$

Then, we can apply the complex Tessarines algebra to simplify the system. We set that

$$y_1[n] = x_1[n] + Ix_2[n], \quad y_2[n] = x_3[n] + Ix_4[n], \tag{62}$$
$$y_3[n] = x_5[n] + Ix_6[n], \quad y_4[n] = x_7[n] + Ix_8[n],$$
$$h_1 = a + Ib, \quad h_2 = c + Id, \quad h_3 = e + If, \quad h_4 = g + Il,$$
$$h_5 = a + Im, \quad h_6 = o + Ip, \quad h_7 = r + It, \quad h_8 = a + Iu,$$
$$h_9 = v + Iw, \quad h_{10} = a + I\alpha, \tag{63}$$

and (64) is simplified as

$$\begin{bmatrix} y_1[n+1] \\ y_2[n+1] \\ y_3[n+1] \\ y_4[n+1] \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 \\ h_2 & h_5 & h_6 & h_7 \\ h_3 & h_6 & h_8 & h_9 \\ h_4 & h_7 & h_9 & h_{10} \end{bmatrix} \begin{bmatrix} y_1[n] \\ y_2[n] \\ y_3[n] \\ y_4[n] \end{bmatrix}, \tag{64}$$

and the original 8×8 system is reduced into the 4×4 system. Moreover, if the system in Fig. 4 is also symmetric respect to L_2, then it can be simplified by the 8-D Tessarines:

$$\begin{bmatrix} z_1[n+1] \\ z_2[n+1] \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \begin{bmatrix} z_1[n] \\ z_2[n] \end{bmatrix}, \tag{65}$$

where

$$z_1[n] = x_1[n] + Ix_2[n] + Jx_7[n] + IJx_8[n], \quad (66)$$

$$z_2[n] = x_3[n] + Ix_4[n] + Jx_5[n] + IJx_6[n],$$

$$s_1 = a + Ib + Jg + IJl \qquad s_2 = c + Id + Je + IJf , \quad (67)$$

$$s_3 = a + Im + Jo + IJp .$$

In summary, for a Markov chain system, if the interactions between two points are determined by their distance, as in (16), then we can use the quaternion or the octonion algebras to simplify the system. The rules are:

(A) If the system consists of N points and is symmetric respect to a line L_1, then we can use the **complex Tessarine algebra** to simplify the system into the

$$(N+N_1)/2 \times (N+N_1)/2 \approx N/2 \times N/2 \quad (68)$$

system, where N_1 is the number of points on L_1.

(B) If the system consists of N points and is symmetric respect to both the line L_1 and the line L_2, where L_2 is perpendicular to L_1, then we can use the **8-D Tessarine algebra** to simplify the system into the

$$(N+N_1+ N_2 +\delta)/4 \times (N+N_1+ N_2 +\delta)//4 \approx N/4 \times N/4 \quad (69)$$

system, where N_1 is the number of points on L_1 and N_2 is the number of points on L_2. If there is a point at the intersection of L_1 and L_2, then $\delta = 1$. In other conditions, $\delta = 0$.

(C) For the case where the symmetric system is directional, we can use the complex Tessarine algebra or the conventional quaternion to simplify the system (but in many conditions the offset terms should be included). The number of offset terms are near to

$$3N/4. \quad (70)$$

(D) When the inputs and the responses are real, then the symmetric system can be simplified by the complex-II algebra and the doubly symmetric system can be simplified by the 4-D purely hyperbolic complex numbers.

5. COMMUNICATION SYSTEM SIMPLIFICATION

Then, we show how to use the above concept for the signal processing application of symmetric channel communication. For the system in Fig. 5(a), the signals transmitted along channels 1 and 2. However, channels 1 and 2 interfered with each other. Then, the relations between $\{x_1(t), y_1(t)\}$ and $\{x(t), y(t)\}$ is

$$x_1(t) = \int [c(t,\tau)x(\tau) + n(t,\tau)y(\tau)]d\tau ,$$

$$y_1(t) = \int [c(t,\tau)y(\tau) + n(t,\tau)x(\tau)]d\tau , \quad (71)$$

Where $c(t, \tau)$ is the channel transmission response and $n(t, \tau)$ is the interference from another channel. Note that it is in fact a special case of (17) and (18). We can use the complex **Tessarine** algebra to simplify it into the one-channel system as Fig. 5(b), where

$$z_1(t) = \int c_1(t,\tau)z(\tau)d\tau , \quad c_1(t,\tau) = c(t,\tau) + n(t,\tau) \quad (72)$$

$$z(t) = x(t) + Iy(t) , \quad z_1(t) = x_1(t) + Iy_1(t) . \quad (73)$$

Since the system is simplified into the 1-channel case, the analysis becomes easier and many of the conventional system analysis techniques can be used.

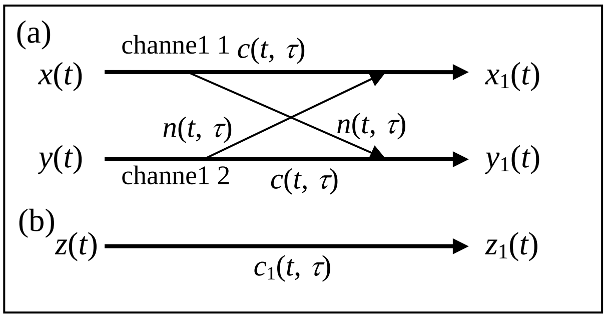

Fig. 5 The symmetric 2-channel communication system.

6. CONCLUSIONS

In this paper, we introduce the way that uses the quaternion, the biquaternion, and their related algebras to simplify the symmetric Markov chain system analysis. If an $N \times N$ system is symmetric respect to one axis or two perpendicular axes, we can use a proper algebra to reduce the system into an $(N/2) \times (N/2)$ or $(N/4) \times (N/4)$ system, respectively. With the simplification, the computation requirement is reduced, the system analysis problem can be solved in s simpler way, and many of the system properties can be analyzed by the quaternion and the related algebras.

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