QUATERNIONS AND BIQUATERNIONS FOR SYMMETRIC MARKOV-CHAIN SYSTEM ANALYSIS

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ABSTRACT

In this conference, we use quaternions, biquaternions, and their related algebras similar to them to model symmetric Markov systems. With these algebras, the original N×N Markov system can be reduced into an (N/2)×(N/2) or (N/4)×(N/4) system. It makes the system easier for implementing and analysis. In addition to Markov chains, the proposed idea is also helpful for simplifying the complexities of other symmetric systems whose interactions between two objects are determined by their distance.

1. INTRODUCTION

The quaternion [1][2] is a generalization of the complex algebra. Its application in signal and image processing were noticed in recent years [2][3][4]. The quaternion has been applied for filter design, feature extraction, pattern recognition, rotation analysis, differential equation analysis, and SVD decomposition, etc. In this paper, we propose another application of the quaternion and its related algebras, i.e., the Markov chain analysis.

2. QUATERNIONS AND RELATED ALGEBRAS

Quaternions [1][2] are generalizations of complex numbers. A quaternion has 4 components: real, i-, j-, and k-parts:

\[ q = q_r + q_i \cdot i + q_j \cdot j + q_k \cdot k, \]

and i, j, k obey the rules as below:

\[ i^2 = j^2 = k^2 = -1, \quad i \cdot j = k, \quad j \cdot k = i, \quad k \cdot i = j, \]

\[ j \cdot i = -k, \quad k \cdot j = -i, \quad i \cdot k = -j. \]  

(2)

Quaternions can be used for the color image analysis, filter design, and 3-D object analysis.

The definition of the Clifford biquaternion [6][10] is similar to that of the quaternion, but there are 8 elements:

\[ q = q_0 + q_1 \cdot i + q_2 \cdot j + q_3 \cdot k + \text{idempotent elements} e_1 \text{ and } e_2: \]

\[ e_1 = \frac{1+I}{2}, \quad e_2 = \frac{1-I}{2}, \]

\[ e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1 e_2 = 0. \]  

(3)

Idempotent elements are helpful for fast computation. If q is defined in (3) and p is defined as

\[ p = p_0 + p_1 \cdot i + p_2 \cdot j + p_3 \cdot k + p_4 \cdot i + p_5 \cdot j + p_6 \cdot k \]

then p and q can be expressed as:

\[ p = p_0 e_1 + p_2 e_2, \quad q = q_1 e_1 + q_2 e_2, \]

(4)

where

\[ p_0 = p e_1 = (p_0 + p_1 \cdot i + p_2 \cdot j + p_3 \cdot k), \]

\[ p_1 + p_2 \cdot i + p_3 \cdot j + p_4 \cdot k) / 2, \quad p_2 = p e_2. \]  

Similarly,

\[ q_1 = q e_1 \quad \text{and} \quad q_2 = q e_2. \]

Then, the product of p and q can be expressed as:

\[ pq = (p_0 e_1 + p_2 e_2) (q_1 e_1 + q_2 e_2) = p_0 q_1 e_1 + p_2 q_2 e_2. \]  

(5)

In this condition, \( j^2 = 1, k^2 = -1, ij = ji, ik = ki, \) and \( jk = k j. \)

The Clifford biquaternion and the complex Tessarine have idempotent elements \( e_1 \) and \( e_2: \)

\[ e_1 = (1+I)/2, \quad e_2 = (1-I)/2, \]

\[ e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1 e_2 = 0. \]  

(6)

These algebras used in this paper are:

- [2-D algebras]
  - Complex algebra.
  - Tessarines (also called as split-complex numbers, and double complex numbers) [5][6]:
    \[ q = a_0 + a_1 j, \quad \text{where} \ F = 1. \]
  - [4-D algebras]
    - Quaternions (defined in (1) and (2)).
    - Complex Tessarines (defined in (5)).
    - Reduced biquaternion: See [3][7]. It is also called the bicomplex number (C×C). Its definition is the same as the reduced biquaternion in (5) but \( F = -1. \)
    - 4-D purely hyperbolic complex numbers:
      \[ q = q_0 + q_1 I + q_2 J + q_3 I J, \quad F = F = 1, IJ = JI. \]  

(7)

- [8-D algebras]
  - Octonions (standard definition [1][9]). It has 7 imaginary parts and the multiplicative rule is shown in [6].
  - Clifford biquaternions (see (3)).
  - 8-D Tessarines (4-D hyperbolic complex numbers with complex coefficients)

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(b) If $pq = rs$, then the interactions between $p$ and $q$ (denoted by $h_{pq}$) and $r$ and $s$ ($h_{rs}$) has the relation of $h_{pq} = -h_{rs}$, then

$$x_{1}[n + 1] = ax_{1}[n] - bx_{2}[n],$$

(23)

$$x_{2}[n + 1] = bx_{1}[n] + ax_{2}[n],$$

(24)

then, we can define a function of

$$x[n] = x_{1}[n] + ix_{2}[n],$$

(25)

where $F = -1$. If $r = i$, then

(26)

In this case, we can also prove that

$$x[n + 1] = h \cdot x[n],$$

(27)

If $x_{1}[n]$ and $x_{2}[n]$ are complex functions, then the system is modeled by the following algebra:

$$q = q_{r} + q_{i} \cdot i + q_{s} \cdot i^{2}.$$

(28)

It is just the same as the reduced biquaternion algebra.

---

Fig. 2 A doubly symmetric $4 \times 4$ rectangular system.

The idea can be extended to any symmetric Markov system. For the system in Fig. 2, suppose that the values at the points 1, 2, 3, and 4 are $x_{1}[n], x_{2}[n], x_{3}[n],$ and $x_{4}[n]$ (29) and the interactions between the points $m$ and $n$ are denoted by $h_{mn}$. Therefore,

$$
x_{1}[n + 1] = \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} & h_{1,4} \\
  h_{2,1} & h_{2,2} & h_{2,3} & h_{2,4} \\
  h_{3,1} & h_{3,2} & h_{3,3} & h_{3,4} \\
  h_{4,1} & h_{4,2} & h_{4,3} & h_{4,4} \end{bmatrix} \begin{bmatrix} x_{1}[n] \\
  x_{2}[n] \\
  x_{3}[n] \\
  x_{4}[n] \end{bmatrix}
$$

(30)

First, note that

$$h_{mn} = h_{nm}.$$  

(31)

Moreover, since interactions are determined by distances,

$$h_{1,1} = h_{2,2} = h_{3,3} = h_{4,4} = a, h_{1,2} = h_{2,3} = h_{3,4} = h_{4,1} = b, h_{1,3} = h_{2,4} = h_{3,1} = h_{4,2} = c, h_{1,4} = h_{2,3} = h_{3,4} = h_{4,1} = d.$$  

(32)

Then (30) can be rewritten as:

$$
x_{1}[n + 1] = \begin{bmatrix} a & b & c & d \\
  b & a & d & c \\
  c & d & a & b \\
  d & c & b & a \end{bmatrix} \begin{bmatrix} x_{1}[n] \\
  x_{2}[n] \\
  x_{3}[n] \\
  x_{4}[n] \end{bmatrix}
$$

(33)

If we set that

$$y_{1}[n] = x_{1}[n] + ix_{2}[n], \quad y_{2}[n] = x_{3}[n] + ix_{4}[n],$$

(34)

$$h_{1} = a + ib, \quad h_{2} = c + id.$$  

Then (33) can be rewritten as:

$$
y_{1}[n + 1] = \begin{bmatrix} h_{1} & h_{2} \\
  h_{2} & h_{1} \end{bmatrix} \begin{bmatrix} y_{1}[n] \\
  y_{2}[n] \end{bmatrix},
$$

(35)
That is, after converting into the complex Tessarine form, the $4 \times 4$ system become a $2 \times 2$ system.

Moreover, notice that (35) is also a symmetric system. We can further reduce it into the $1 \times 1$ form. We can apply the 8-D Tessarines algebra. We can set that
\[
y[n] = x_1[n] + jx_2[n] + jx_3[n] + jx_4[n],
\]
Then
\[
y[n+1] = h y[n] \quad \text{where} \quad h = a + Jb + Ic + Ld.
\]
That is, after applying the complex Tessarines, the $4 \times 4$ symmetric rectangular system becomes a $2 \times 2$ symmetric system. After applying the 8-D Tessarines n, the $4 \times 4$ symmetric rectangular system becomes a one-input and one-output system.

In the case where the interactions between two dots are affected by the direction:
\[
h_{m,k} = -h_{k,m} \quad \text{if} \quad m \neq k,
\]
\[
x_1[n+1] = \begin{bmatrix} a & -b & c & -d \\ b & a & -f & -c \\ c & f & a & -b \\ d & c & b & a \end{bmatrix} x_1[n]
\]
\[
x_2[n+1] = \begin{bmatrix} a & -b & c & -d \\ b & a & -f & -c \\ c & f & a & -b \\ d & c & b & a \end{bmatrix} x_2[n]
\]
\[
x_3[n+1] = \begin{bmatrix} a & -b & c & -d \\ b & a & -f & -c \\ c & f & a & -b \\ d & c & b & a \end{bmatrix} x_3[n]
\]
\[
x_4[n+1] = \begin{bmatrix} a & -b & c & -d \\ b & a & -f & -c \\ c & f & a & -b \\ d & c & b & a \end{bmatrix} x_4[n]
\]
Specially, if $d = f$, and $x_1[n], x_2[n], x_3[n]$, and $x_4[n]$ are real,
\[
x_1[n+1] = \begin{bmatrix} a & -b & c & -d \\ b & a & -f & -c \\ c & f & a & -b \\ d & c & b & a \end{bmatrix} x_1[n]
\]
\[
x_2[n+1] = \begin{bmatrix} a & -b & c & -d \\ b & a & -f & -c \\ c & f & a & -b \\ d & c & b & a \end{bmatrix} x_2[n]
\]
\[
x_3[n+1] = \begin{bmatrix} a & -b & c & -d \\ b & a & -f & -c \\ c & f & a & -b \\ d & c & b & a \end{bmatrix} x_3[n]
\]
\[
x_4[n+1] = \begin{bmatrix} a & -b & c & -d \\ b & a & -f & -c \\ c & f & a & -b \\ d & c & b & a \end{bmatrix} x_4[n]
\]
We can compare (39) with the quaternion multiplication. If
\[
z_1 + iz_2 + jz_3 + kz_4 = (x_1 + ix_2 + jx_3 + kx_4)(y_1 + iy_2 + jy_3 + ky_4),
\]
then
\[
x_1 = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & y_1 & -y_4 & y_3 \\ y_3 & -y_4 & y_1 & y_2 \\ y_4 & y_3 & -y_2 & y_1 \end{bmatrix},
\]
Since
\[
\begin{bmatrix} x_1[n+1] + 2cx_4[n] \\ x_2[n+1] - 2bx_3[n] \\ x_3[n+1] \\ x_4[n+1] - 2dx_2[n] \end{bmatrix} = \begin{bmatrix} a & -b & c & -d \\ b & a & -f & -c \\ c & f & a & -b \\ d & c & b & a \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \\ x_4[n] \end{bmatrix},
\]
we find that, if
\[
y[n] = x_1[n] + ix_2[n] + jx_3[n] + kx_4[n],
\]
\[
h = a + ib + jc - kd, \quad \text{then}
\]
\[
y[n+1] = y[n] h + \delta,
\]
where
\[
\delta = 2 \begin{bmatrix} cy_j[n] - iby_j[n] - kdy_j[n] \end{bmatrix},
\]
y$_i[n]$ and y$_j[n]$ mean the real and j- parts of y[n]. That is, a directional symmetric rectangular system can be represented by the quaternion.

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<td>2×2 symmetric system, quaternion input</td>
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</tr>
</tbody>
</table>

For the directional symmetric rectangular system in (40), in the case where $d = f$, the system can be represented by a Clifford octonion system:
\[
\begin{bmatrix} y_1[n+1] \\ y_2[n+1] \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ h_3 & h_4 \end{bmatrix} \begin{bmatrix} y_1[n] \\ y_2[n] \end{bmatrix} + \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix},
\]
where $y_1[n] = x_1[n] + jx_2[n]$, $y_2[n] = x_2[n] + jx_3[n]$, $h_1 = a + Id$, $h_2 = a + Jf$, $h_3 = b + Ic$, $h_4 = b + Jc$, $I^2 = -1$, $J^2 = -1$, $F = -1$,
\[
\delta_1 = \text{Re}(h_1) \text{Re}(y_2[n]), \quad \delta_2 = \text{Re}(h_2) \text{Im}(y_1[n]).
\]

In (43) and (47), the inputs are real is considered. When $x_k[n]$ are complex, (44) can be rewritten as
\[
y[n] = x_1[n] + jx_2[n] + jx_3[n] + kx_4[n],
\]
where the tri-complex numbers (defined in (15)) is applied. In this case, (43) can still be applied and (44) is rewritten as
\[
h = a - Ib + Jc - kd, \quad \delta = 2 \begin{bmatrix} cy_j[n] - iby_j[n] - kdy_j[n] \end{bmatrix},
\]
The relations between the quaternion-related algebras and the symmetric system are listed in Table 1.

4. GENERALIZE SYMMETRIC SYSTEM ANALYSIS

The results in Section 3 can be generalized for all symmetric and doubly symmetric systems.
For the example in Fig. 3, there are 6 dots and form a hexagon. We also suppose that the interactions between two points (denoted by \( h_{n,n} \)) are determined by their distance. Suppose that the system is symmetric respect to the dash line \( L_1 \). That is,
\[
\begin{align*}
    h_{1,3} &= h_{2,4}, & h_{1,4} &= h_{2,3}, & h_{1,5} &= h_{2,6}, & h_{1,6} &= h_{2,5}, \\
    h_{2,5} &= h_{3,4}, & h_{3,5} &= h_{3,6}, & h_{3,6} &= h_{3,5}, & h_{3,5} &= h_{3,4}, \\
    h_{4,5} &= h_{4,6}, & h_{4,6} &= h_{4,5}, & h_{4,5} &= h_{4,6}, & h_{4,6} &= h_{4,5}.
\end{align*}
\]
Then the system can be expressed as
\[
\begin{bmatrix}
    x_1[n+1] \\
    x_2[n+1] \\
    x_3[n+1] \\
    x_4[n+1] \\
    x_5[n+1] \\
    x_6[n+1]
\end{bmatrix}
= \begin{bmatrix}
    a & b & c & d & e & f \\
    c & d & a & p & g & l \\
    d & c & p & a & l & g \\
    e & f & g & l & a & m \\
    f & e & l & g & m & a \\
\end{bmatrix}
\begin{bmatrix}
    x_1[n] \\
    x_2[n] \\
    x_3[n] \\
    x_4[n] \\
    x_5[n] \\
    x_6[n]
\end{bmatrix}
\]  
(52)

It is a 6 \times 6 system. If we set that
\[
\begin{align*}
    y_1[n] &= x_1[n] + L x_2[n], & y_2[n] &= x_3[n] + L x_4[n], \\
    y_3[n] &= x_5[n] + L x_6[n], & h_1 &= a + Ib, & h_2 &= c + Id, \\
    h_3 &= e + If, & h_4 &= g + Il, & h_5 &= a + Ip, & h_6 &= a + Im,
\end{align*}
\]
where \( I = 1 \) and \( il = jn \), then (52) can be expressed as
\[
\begin{align*}
    y_1[n+1] &= h_1 x_1[n] + h_2 x_2[n] + h_3 x_3[n], \\
    y_2[n+1] &= h_1 x_2[n] + h_2 x_3[n] + h_4 x_4[n], \\
    y_3[n+1] &= h_1 x_3[n] + h_2 x_4[n] + h_5 x_5[n], \\
\end{align*}
\]
(54)

and the system is reduced into a 3 \times 3 system by the reduced biquaternion algebra.

Furthermore, if the hexagon is also symmetric respect to \( L_2 \) (\( L_2 \) is perpendicular to \( L_1 \)), i.e., in addition to (51),
\[
\begin{align*}
    h_{1,2} &= h_{3,4}, & h_{1,5} &= h_{3,6}, & h_{1,6} &= h_{2,5}, & h_{1,6} &= h_{2,5}, \\
    h_{2,5} &= h_{3,4}, & h_{3,5} &= h_{3,6}, & h_{3,6} &= h_{3,5}, & h_{3,5} &= h_{3,4},
\end{align*}
\]
(55) then the system can be expressed as the following operation:
\[
\begin{align*}
    x_1[n+1] &= \begin{bmatrix}
        a & b & c & d & e & f \\
        c & d & a & p & g & l \\
        d & c & p & a & l & g \\
        e & f & g & l & a & m \\
        f & e & l & g & m & a \\
    \end{bmatrix}
    \begin{bmatrix}
        x_1[n] \\
        x_2[n] \\
        x_3[n] \\
        x_4[n] \\
        x_5[n] \\
    \end{bmatrix}
\end{align*}
\]
(56)

and (56) can be rewritten as
\[
\begin{align*}
    z_1[n+1] &= \begin{bmatrix}
        s_1 \\
        s_2 \\
        s_3 \\
    \end{bmatrix}
    \begin{bmatrix}
        z_1[n] \\
    \end{bmatrix},
\end{align*}
\]
(59)

We then give another example in Fig. 4, which is an 8 \times 8 system. Suppose that the system is symmetric respect to \( L_1 \) and the interaction between \( n \) and \( k \) is denoted by \( h_{n,k} \). Then
\[
\begin{align*}
    y_1[n] &= x_1[n] + L x_2[n], & y_2[n] &= x_3[n] + L x_4[n], \\
    y_3[n] &= x_5[n] + L x_6[n], & y_4[n] &= x_8[n] + L x_9[n], \\
    h_1 &= a + Ib, & h_2 &= c + Id, \\
    h_3 &= e + If, & h_4 &= g + Il, \\
    h_5 &= a + Ip, & h_6 &= a + Im,
\end{align*}
\]
(65) and (66) is simplified as
\[
\begin{align*}
    y_1[n+1] &= \begin{bmatrix}
        a & b & c & d & e & f & g & i \\
        b & a & d & c & f & e & g & i \\
        c & d & a & p & g & l & a & m \\
        d & c & p & a & l & g & m & a \\
        e & f & g & l & a & m & f & e \\
        f & e & l & g & m & a & e & f \\
    \end{bmatrix}
    \begin{bmatrix}
        x_1[n] \\
        x_2[n] \\
        x_3[n] \\
        x_4[n] \\
        x_5[n] \\
        x_6[n] \\
        x_7[n] \\
        x_8[n]
    \end{bmatrix}
\end{align*}
\]
(61)

Then, we can apply the complex Tesseractines algebra to simplify the system. We set that
\[
\begin{align*}
    y_1[n] &= x_1[n] + L x_2[n], & y_2[n] &= x_3[n] + L x_4[n], \\
    y_3[n] &= x_5[n] + L x_6[n], & y_4[n] &= x_8[n] + L x_9[n], \\
    h_1 &= a + Ib, & h_2 &= c + Id, \\
    h_3 &= e + If, & h_4 &= g + Il, \\
    h_5 &= a + Ip, & h_6 &= a + Im, \\
    h_7 &= v + Ivw, & h_8 &= a + Ix, \\
\end{align*}
\]
(63)

and (64) is simplified as
\[
\begin{align*}
    y_1[n+1] &= \begin{bmatrix}
        h_1 & h_2 & h_3 & h_4 \\
        h_2 & h_3 & h_4 & h_1 \\
        h_3 & h_4 & h_1 & h_2 \\
        h_4 & h_1 & h_2 & h_3 \\
    \end{bmatrix}
    \begin{bmatrix}
        y_1[n] \\
        y_2[n] \\
        y_3[n] \\
        y_4[n]
    \end{bmatrix}
\end{align*}
\]
(64) and the original 8 \times 8 system is reduced into the 4 \times 4 system. Moreover, if the system in Fig. 4 is also symmetric respect to \( L_2 \), then it can be simplified by the 8-D Tessarines:
\[
\begin{align*}
    z_1[n+1] &= \begin{bmatrix}
        s_1 \\
        s_2 \\
        s_3 \\
    \end{bmatrix}
    \begin{bmatrix}
        z_1[n] \\
    \end{bmatrix},
\end{align*}
\]
(65)
where 
\[ z_1[n] = x_1[n] + jx_2[n] + jx_3[n] + jLx_4[n], \]
\[ z_2[n] = x_1[n] + jx_2[n] + jx_3[n] + jLx_4[n], \]
\[ s_1 = a + ib + jg + jL, \quad s_2 = c + Id + Je + jLj, \]
\[ s_3 = a + jm + Jo + jLp. \]

In summary, for a Markov chain system, if the interactions between two points are determined by their distance, as in (16), we can use the quaternion or the octonion algebras to simplify the system. The rules are:

(A) If the system consists of \( N \) points and is symmetric respect to a line \( L_1 \), then we can use the complex Tessarine algebra to simplify the system into the \((N+N_1)/2 \times (N+N_1)/2 \approx N/2 \times N/2\) system, where \( N_1 \) is the number of points on \( L_1 \).

(B) If the system consists of \( N \) points and is symmetric respect to both the line \( L_1 \) and the line \( L_2 \), where \( L_2 \) is perpendicular to \( L_1 \), then we can use the 8-D Tessarine algebra to simplify the system into the \((N+N_1+N_2+\delta)/4 \times (N+N_1+N_2+\delta)/4 \approx N/4 \times N/4\) system, where \( N_1 \) is the number of points on \( L_1 \) and \( N_2 \) is the number of points on \( L_2 \). If there is a point at the intersection of \( L_1 \) and \( L_2 \), then \( \delta = 1 \). In other conditions, \( \delta = 0 \).

(C) For the case where the symmetric system is directional, we use the complex Tessarine algebra or the conventional quaternion to simplify the system (but in many conditions the offset terms should be included). The number of offset terms are near to \( 3N/4 \).

(D) When the inputs and the responses are real, then the symmetric system can be simplified by the complex-II algebra and the doubly symmetric system can be simplified by the 4-D pure hypercomplex numbers.

5. COMMUNICATION SYSTEM SIMPLIFICATION

Then, we show how to use the above concept for the signal processing application of symmetric channel communication. For the system in Fig. 5(a), the signals transmitted along channels 1 and 2. However, channels 1 and 2 interfere with each other. Then, the relations between \( \{x_1(t), y_1(t)\} \) and \( \{x(t), y(t)\} \) is
\[ x_1(t) = \int [c(t, \tau)x(\tau) + n(t, \tau)y(\tau)]d\tau, \]
\[ y_1(t) = \int [c(t, \tau)y(\tau) + n(t, \tau)x(\tau)]d\tau, \]
where \( c(t, \tau) \) is the channel transmission response and \( n(t, \tau) \) is the interference from another channel. Note that it is in fact a special case of (17) and (18). We can use the complex Tessarine algebra to simplify it into the one-channel system as Fig. 5(b), where
\[ z_1(t) = c(t, \tau)z(t)d\tau, \quad c(t, \tau) = c(t, \tau) + n(t, \tau) \]
\[ z(t) = x(t) + jy(t), \quad z_1(t) = x_1(t) + jy_1(t). \]

Since the system is simplified into the 1-channel case, the analysis becomes easier and many of the conventional system analysis techniques can be used.

6. CONCLUSIONS

In this paper, we introduce the way that uses the quaternion, the biquaternion, and their related algebras to simplify the symmetric Markov chain system analysis. If an \( N \times N \) system is symmetric respect to one axis or two perpendicular axes, we can use a proper algebra to reduce the system into an \((N/2) \times (N/2)\) or \((N/4) \times (N/4)\) system, respectively. With the simplification, the computation requirement is reduced, the system analysis problem can be solved in a simpler way, and many of the system properties can be analyzed by the quaternion and the related algebras.

7. REFERENCES