

ON DISCRETE-TIME ESTIMATORS OF SECOND-ORDER MOMENTS OF GENERALIZED ALMOST-CYCLOSTATIONARY PROCESSES

Antonio Napolitano

Dipartimento per le Tecnologie, Università di Napoli "Parthenope"
via Medina 40, I-80133, Napoli, Italy

phone: +390815474895, fax: +390815474777, email: antonio.napolitano@uniparthenope.it

ABSTRACT

In this paper, a discrete-time estimator is proposed for second-order moments of continuous-time generalized almost-cyclostationary (GACS) processes. GACS processes have statistical functions that are almost-periodic functions of time whose Fourier series expansions have both frequencies and coefficients that depend on the lag shifts of the processes. The class of GACS processes includes the almost-cyclostationary (ACS) processes which are obtained as a special case when the frequencies do not depend on the lag shifts. ACS processes filtered by Doppler channels and communications signals with time-varying parameters are further examples. The discrete-time process obtained by uniformly sampling a continuous-time GACS process is considered. It is shown that such discrete-time process is ACS and it is proved that its discrete-time cyclic correlogram is a mean-square consistent estimator of the cyclic autocorrelation function of the continuous-time GACS process, as the sampling period approaches zero and the data-record length approaches infinity.

1. INTRODUCTION

Almost-cyclostationary (ACS) processes, also called almost-periodically correlated processes, are an appropriate model for almost all modulated signals adopted in communications and for processes encountered in econometrics, acoustics, mechanics, climatology, hydrology, and biology [2], [3], [4], [11]. In communications applications, almost-cyclostationarity properties have been exploited to develop signal-selective detection and parameter-estimation algorithms, blind-channel identification and synchronization techniques, and so on.

For ACS processes, multivariate statistical functions are almost-periodic functions of time that can be expressed by (generalized) Fourier series expansions whose coefficients depend on the lag shifts of the processes and whose frequencies, referred to as cycle frequencies, do not depend on the lag shifts. In [5], the class of the *generalized almost-cyclostationary (GACS)* processes is introduced. It extends the class of the ACS processes to the case in which also the frequencies, referred to as lag-dependent cycle frequencies, depend on the lag shifts. GACS processes are an appropriate model to describe chirp signals and several angle-modulated and time-warped communication signals. Furthermore, in [6] and [7] it is shown that time-variant channels of interest in communications transform a transmitted ACS signal into a GACS one. In particular, it is shown that the GACS model can be appropriate to describe the output signal of Doppler channels due to relative motion between transmitter and receiver with nonzero relative radial acceleration when the input signal is ACS.

The problem of second-order moment estimation for continuous-time GACS processes is addressed in [10], where second-order properties in the wide-sense are shown to be completely described by the cyclic autocorrelation function and the cyclic correlogram is shown to be a mean-square consistent asymptotically Normal estimator of the cyclic autocorrelation function under mild mixing conditions expressed in term of summability of cumulants.

The discrete-time counterparts of the results of [10] are not straightforward since a discrete-time counterpart of GACS processes does not exist and uniformly sampling a continuous-time GACS process leads to a discrete-time ACS process. In particular, the GACS nature of the underlying continuous-time process can only be conjectured starting from the discrete-time ACS process of its samples [8]. Moreover, aliasing in the cycle frequency domain needs to be accounted for.

In this paper, the discrete-time cyclic correlogram of the discrete-time ACS process obtained by uniformly sampling a continuous-time GACS process is considered as an estimator for (samples of) the continuous-time cyclic autocorrelation function of the GACS process, and its mean and variance are evaluated for both finite and infinite data-record lengths. It is shown that for GACS processes no simple condition on the sampling frequency can be stated as for band-limited wide-sense stationary or ACS process in order to avoid or limit aliasing. However, it is shown that the discrete-time cyclic correlogram is a mean-square consistent estimator of the aliased cyclic autocorrelation function as the data-record length approaches infinity. In addition, sufficient conditions are provided to assure that the discrete-time cyclic correlogram be a mean-square consistent estimator of the cyclic autocorrelation function as the data-record length approaches infinity and the sampling period approaches zero.

2. GENERALIZED ALMOST-CYCLOSTATIONARY PROCESSES

A finite-power complex-valued continuous-time stochastic process $x(t)$, $t \in \mathbb{R}$, is said to be *second-order GACS in the wide sense* [10] if its autocorrelation function

$$\mathcal{R}_{xx^*}(t, \tau) \triangleq E \{x(t + \tau)x^*(t)\} \quad (1)$$

is an almost-periodic function of time. That is, for each fixed τ , $\mathcal{R}_{xx^*}(t, \tau)$ is the limit of a uniformly convergent sequence of trigonometric polynomials in t which can be written in the two following equivalent forms [5], [6]:

$$\mathcal{R}_{xx^*}(t, \tau) = \sum_{\alpha \in A_\tau} R_{xx^*}(\alpha, \tau) e^{j2\pi\alpha t} \quad (2a)$$

$$= \sum_{n \in \mathbb{I}} R_{xx^*}^{(n)}(\tau) e^{j2\pi\alpha_n(\tau)t}. \quad (2b)$$

In (2a), the real numbers α and the complex-valued functions $R_{xx^*}(\alpha, \tau)$, referred to as *cycle frequencies* and *cyclic autocorrelation functions*, are the frequencies and coefficients, respectively, of the (generalized) Fourier series expansion of $\mathcal{R}_{xx^*}(t, \tau)$ that is,

$$R_{xx^*}(\alpha, \tau) \triangleq \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathcal{R}_{xx^*}(t, \tau) e^{-j2\pi\alpha t} dt. \quad (3)$$

Furthermore, in (2a) and (2b),

$$A_\tau \triangleq \{\alpha \in \mathbb{R} : R_{xx^*}(\alpha, \tau) \neq 0\} \quad (4a)$$

$$= \bigcup_{n \in \mathbb{I}} \{\alpha \in \mathbb{R} : \alpha = \alpha_n(\tau)\} \quad (4b)$$

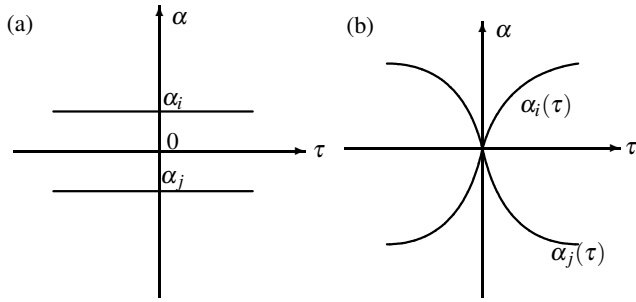


Figure 1: Support in the (α, τ) -plane of the cyclic autocorrelation function of (a) an ACS process and (b) a GACS (not ACS) process.

is a countable set, with \mathbb{I} also countable, the functions $\alpha_n(\tau)$ are referred to as *lag-dependent cycle frequencies* and the functions $R_{xx^*}^{(n)}(\tau)$, referred to as *generalized cyclic autocorrelation functions*, are defined as

$$R_{xx^*}^{(n)}(\tau) \triangleq \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathcal{R}_{xx^*}(t, \tau) e^{-j2\pi\alpha_n(\tau)t} dt. \quad (5)$$

It can be shown that, with some refinement of definition (5), it results [5], [6]

$$R_{xx^*}(\alpha, \tau) = \sum_{n \in \mathbb{I}} R_{xx^*}^{(n)}(\tau) \delta_{\alpha - \alpha_n(\tau)} \quad (6)$$

where δ_γ denotes Kronecker delta, that is, $\delta_\gamma = 1$ for $\gamma = 0$ and $\delta_\gamma = 0$ for $\gamma \neq 0$. That is, the lag-dependent cycle-frequency curves $\alpha = \alpha_n(\tau)$, $n \in \mathbb{I}$, describe the support of the cyclic autocorrelation function $R_{xx^*}(\alpha, \tau)$.

For complex processes, two second-order moments need to be considered for a complete characterization in the wide sense: The autocorrelation function (1) and the conjugate autocorrelation function $\mathcal{R}_{xx}(t, \tau) \triangleq E\{x(t+\tau)x(t)\}$. The *conjugate cyclic autocorrelation function* $R_{xx}(\alpha, \tau)$ is defined by (3) with $\mathcal{R}_{xx^*}(t, \tau)$ replaced by $\mathcal{R}_{xx}(t, \tau)$. The cyclic autocorrelation function and the conjugate cyclic autocorrelation function can be both represented by the concise notation

$$R_{xx^{(*)}}(\alpha, \tau) \triangleq \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} E\{x(t+\tau)x^{(*)}(t)\} e^{-j2\pi\alpha t} dt \quad (7)$$

where superscript $(*)$ denotes an optional complex conjugation.

The wide-sense ACS processes are obtained as a special case of GACS processes when the lag-dependent cycle frequencies are constant with respect to τ and, hence, are coincident with the cycle frequencies [5]. In such a case,

$$\mathcal{R}_{xx^{(*)}}(t, \tau) = \sum_{n \in \mathbb{I}} R_{xx^{(*)}}^{\alpha_n}(\tau) e^{j2\pi\alpha_n t}. \quad (8)$$

Moreover, $R_{xx^{(*)}}(\alpha, \tau) = R_{xx^{(*)}}^{\alpha_n}(\tau)$ for $\alpha = \alpha_n \in A$, and $R_{xx^{(*)}}(\alpha, \tau) = 0$ otherwise, with $A = \{\alpha_n\}_{n \in \mathbb{I}}$ countable. Thus, for ACS processes only one term is present in the sum in (6) and, consequently, the generalized cyclic autocorrelation functions are coincident with the cyclic autocorrelation functions.

In Figure 1, the support in the (α, τ) -plane of the cyclic autocorrelation function $R_{xx^*}(\alpha, \tau)$ is reported for (a) an ACS process and (b) a GACS (not ACS) process. For the ACS process, such a support is contained in the lines $\alpha = \alpha_n$, $n \in \mathbb{I}$, that is, lines parallel to the τ axis in correspondence of the cycle frequencies. For the GACS process, the support is constituted by the curves $\alpha = \alpha_n(\tau)$, $n \in \mathbb{I}$ (see (6)).

The GACS model turns out to be appropriate in mobile communications systems when the channel cannot be modeled as almost-periodically time-variant [6], [7]. For example, the output complex envelope $y(t)$ of the Doppler channel existing between a stationary transmitter and a moving receiver with constant relative radial acceleration is GACS when the input complex envelope $x(t)$ is ACS. In fact, the transmitted signal experiences a quadratically time-variant delay. Under the ‘‘narrow-band’’ approximation [12], the time-varying component of the delay in the complex envelope $x(\cdot)$ can be neglected obtaining the chirp-modulated signal

$$y(t) = ax(t - d_0) e^{j2\pi\nu t} e^{j\pi\gamma t^2} \quad (9)$$

where a is the complex gain, d_0 the constant delay, ν the frequency shift, and γ the chirp rate. Thus, if $x(t)$ is ACS (with autocorrelation function (8)), the autocorrelation function of $y(t)$ is given by

$$E\{y(t+\tau)y^*(t)\} = \sum_{n \in \mathbb{I}} R_{yy^*}^{(n)}(\tau) e^{j2\pi\eta_n(\tau)t} \quad (10)$$

where

$$\eta_n(\tau) = \alpha_n + \gamma\tau \quad (11)$$

$$R_{yy^*}^{(n)}(\tau) = |a|^2 R_{xx^*}^{\alpha_n}(\tau) e^{j2\pi\nu\tau} e^{j\pi\gamma\tau^2} e^{-j2\pi\alpha_n d_0} \quad (12)$$

are the lag dependent cycle frequencies and generalized cyclic autocorrelation functions, respectively. That is, $y(t)$ is GACS with lag-dependent cycle frequencies linear with slope γ [10].

Further examples of GACS processes are angle modulated signals and communications signals with time-varying parameters such as baud rate and carrier frequency [5].

3. DISCRETE-TIME ESTIMATION OF THE CYCLIC AUTOCORRELATION FUNCTION

Let us make the following assumptions on the continuous-time process $x(t)$ and the data-tapering window.

Assumptions

1a) The stochastic process $x(t)$ is (second-order) GACS in the wide sense, that is, for any choice of z_1 and z_2 in $\{x, x^*\}$,

$$E\{z_1(t+\tau)z_2(t)\} = \sum_n R_{z_1 z_2}^{(n)}(\tau) e^{j2\pi\alpha_{z_1 z_2}^{(n)}(\tau)t}. \quad (13)$$

1b) For any choice of z_1 and z_2 in $\{x, x^*\}$, the fourth-order cumulant $\text{cum}\{x(t+\tau_1), x^*(t+\tau_2), z_1(t+\tau_3), z_2(t)\}$ can be expressed as

$$\begin{aligned} & \text{cum}\{x(t+\tau_1), x^*(t+\tau_2), z_1(t+\tau_3), z_2(t)\} \\ &= \sum_n C_{xx^* z_1 z_2}^{(n)}(\tau_1, \tau_2, \tau_3) e^{j2\pi\beta_n(\tau_1, \tau_2, \tau_3)t} \end{aligned} \quad (14)$$

where $\beta_n = \beta_{xx^* z_1 z_2}^{(n)}$ for notation simplicity.

- 2) For any choice of z_1 and z_2 in $\{x, x^*\}$, $\sum_n \|R_{z_1 z_2}^{(n)}\|_\infty < \infty$.
- 3) For any choice of z_1 and z_2 in $\{x, x^*\}$, $\sum_n \|C_{xx^* z_1 z_2}^{(n)}\|_\infty < \infty$.
- 4) The process $x(t)$ has uniformly bounded fourth-order absolute moment.
- 5) The data-tapering window $w_T(t)$ is nonzero in $(-T/2, T/2)$ and can be expressed as $w_T(t) = a(t/T)/T$, with $a(t) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, continuous a.e., and with unit area.
- 6) For any choice of z_1 and z_2 in $\{x, x^*\}$,

$$\sum_n \int_{\mathbb{R}} |R_{z_1 z_2}^{(n)}(s)| ds < \infty. \quad (15)$$

- 7) For any choice of z_1 and z_2 in $\{x, x^*\}$ and $\forall \tau_1, \tau_2 \in \mathbb{R}$,

$$\sum_n \int_{\mathbb{R}} |C_{xx^* z_1 z_2}^{(n)}(s+\tau_1, s, \tau_2)| ds < \infty. \quad (16)$$

- 8) Let $z_i(t) \triangleq [x(t + \tau_i) x^{(*)}(t)]^{[*]i}$, $i = 1, \dots, k$, with $[*]_i$ optional complex conjugation. For every integer k , the magnitude of the k th-order cumulant function $\text{cum}\{z_k(t), z_i(t + s_i), i = 1, \dots, k-1\}$ is bounded by a positive summable function of s_1, \dots, s_{k-1} .
- 9) For every integer k , the processes $z_i(t)$ have bounded absolute k th-order cross-moments.

Assumptions 1a and 1b are on the almost-periodic structure of the second- and fourth-order cumulants of the process $x(t)$ and Assumptions 2–4 are on the regularity of the (generalized) Fourier series expansions of such almost-periodic functions; Assumption 5 is on the regularity properties of the data-tapering window; Assumptions 6–7 are on the finite or practically finite memory of the process, which is expressed in terms of summability of its second- and fourth-order cumulants and Assumptions 8–9 are on regularity of higher-order statistics of $x(t)$.

In [10], it is shown that under Assumptions 1–7 the continuous-time cyclic correlogram is a mean-square consistent estimator of the cyclic autocorrelation function. Moreover, its asymptotic Normality is proved under Assumptions 1–9.

Let

$$x_d(n) \triangleq x(t)|_{t=nT_s} \quad (17)$$

be the discrete-time sequences obtained by uniformly sampling with period $T_s = 1/f_s$ the continuous-time GACS processes $x(t)$. The (conjugate) cyclic autocorrelation function of the discrete-time sequence $x_d(n)$ at cycle frequency $\tilde{\alpha} \in [-1/2, 1/2)$ is defined as

$$\tilde{R}_{x_d x_d^{(*)}}(\tilde{\alpha}, m) \triangleq \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N \text{E} \left\{ x_d(n+m) x_d^{(*)}(n) \right\} e^{-j2\pi \tilde{\alpha} n}. \quad (18)$$

The (conjugate) cyclic autocorrelation function of the sampled process $x_d(n)$ is linked to the (conjugate) cyclic autocorrelation function of the continuous-time process $x(t)$ by the aliasing formula [8]

$$\tilde{R}_{x_d x_d^{(*)}}(\tilde{\alpha}, m) = \sum_{p=-\infty}^{+\infty} R_{xx^{(*)}}\left((\tilde{\alpha} + p)f_s, mT_s\right) \quad (19a)$$

$$= \sum_{p=-\infty}^{+\infty} \sum_{k \in \mathbb{I}} R_{xx^{(*)}}^{(k)}(mT_s) \delta_{\tilde{\alpha} f_s - \alpha_k(mT_s) + p f_s} \quad (19b)$$

From (19a) it follows that, in general, $\tilde{R}_{x_d x_d^{(*)}}(\tilde{\alpha}, m) \neq R_{xx^{(*)}}(\tilde{\alpha} f_s, mT_s)$ due to the presence of aliasing in the cycle frequency domain. However, if the GACS continuous-time signal $x(t)$ is such that at lag $\tau = mT_s$ there is no lag-dependent cycle frequency $\alpha_k(\tau)$, $k \in \mathbb{I}$, such that $(\tilde{\alpha} + p)f_s = \alpha_k(mT_s)$ for $p \neq 0$ then, by comparing (19b) with (6), it follows that $\tilde{R}_{x_d x_d^{(*)}}(\tilde{\alpha}, m) = R_{xx^{(*)}}(\tilde{\alpha} f_s, mT_s)$. Such an equality could be difficult to be realized in the whole domain $(\tilde{\alpha}, m) \in [-1/2, 1/2) \times \mathbb{Z}$, as a consequence of the fact that GACS signals have the power spread over an infinite bandwidth [5]. Furthermore, in [8] it is shown that the discrete-time signal obtained by uniformly sampling a continuous-time GACS signal is a discrete-time ACS signal. Thus, discrete-time ACS signals can arise from uniformly sampling ACS and non-ACS continuous-time GACS signals. Moreover, in [8] it is shown that, starting from the sampled signal, the possible ACS or non-ACS nature of the continuous-time GACS signal can only be conjectured, provided that analysis parameters such as sampling period, padding factor, and data-record length are properly chosen. Thus, the results for discrete-time processes cannot be obtained straightforwardly from those of the continuous-time case as is made in the stationary case, e.g., in [1]. In fact, unlike the case of wide-sense stationary and ACS processes, continuous-time GACS processes do not have a discrete-time counterpart. That is, discrete-time GACS processes do not exist.

From the above facts it follows that the sampling frequency f_s cannot be easily chosen in order to avoid or limit aliasing, as it happens for bandlimited wide-sense stationary and ACS signals. However, in some cases, as for the chirp-modulated signal (9), analytical results can be obtained (see Section 4).

Let $x_d(n)$ be the discrete-time processes defined in (17). Its cyclic correlogram at cycle frequency $\tilde{\alpha}$ is defined as

$$\tilde{R}_{x_d x_d^{(*)}}(\tilde{\alpha}, m; n_0, N) \triangleq \sum_{n=-N}^N v_N(n-n_0) x_d(n+m) x_d^{(*)}(n) e^{-j2\pi \tilde{\alpha} n} \quad (20)$$

where $v_N(n) \triangleq a(n/(2N+1))/(2N+1)$ is a data-tapering window with $a(t)$ as in Assumption 5.

By using (2b) with $t = nT_s$ and $\tau = mT_s$ into (20), we obtain the following result, where the made assumptions allow to interchange the order of sum and expectation operators and the order of double-index sum operations.

Theorem 1 Under Assumptions 1a, 2, and 5 on the continuous-time process $x(t)$, the expected value of the discrete-time cyclic correlogram (20) is given by

$$\begin{aligned} & \text{E} \left\{ \tilde{R}_{x_d x_d^{(*)}}(\tilde{\alpha}, m; n_0, N) \right\} \\ &= \sum_{k \in \mathbb{I}} R_{xx^{(*)}}^{(k)}(mT_s) V_{\frac{1}{N}} \left(\tilde{\alpha} - \alpha_k(mT_s)T_s \right) e^{-j2\pi [\tilde{\alpha} - \alpha_k(mT_s)T_s] n_0} \end{aligned} \quad (21)$$

where $V_{\frac{1}{N}}(v)$ is the discrete Fourier transform of the data-tapering window $v_N(n)$. \square

The function $V_{1/N}(v)$ has bandwidth of the order of $1/N$. Consequently, from (21) it follows that the expected value of the discrete-time cyclic correlogram is significantly different from zero within strips of width $1/N$ around the points with $\tilde{\alpha} = \alpha_k(mT_s)T_s$, $k \in \mathbb{I}$, in the $(\tilde{\alpha}, m)$ -plane. Such points correspond to scaled samples of the lag-dependent cycle frequencies curves $\alpha = \alpha_k(\tau)$, that is, the support curves of the continuous-time (conjugate) cyclic autocorrelation function. Since the function $V_{1/N}(v)$ is periodic in v with period 1, aliasing occurs, as in (19b), at cycle frequencies $\tilde{\alpha} \in [-1/2, 1/2)$ such that $|(\tilde{\alpha} + p) - \alpha_k(mT_s)T_s| < 1/N$ for some integer $p \neq 0$ and $k \in \mathbb{I}$. Such a phenomenon is not present for the continuous-time estimator. Moreover, a leakage phenomenon among lag-dependent cycle-frequency curves which are sufficiently close each other occurs similarly to the cyclic leakage occurring in the estimate of cyclic statistics of ACS processes [4].

By expressing the covariance of the lag-product $x_d(n+m) x_d^{(*)}(n)$ in terms of second-order moments and a fourth-order cumulant, the following result can be proved, where the made assumptions allow to interchange the order of sum and expectation operators and the order of multiple-index sum operations.

Theorem 2 Under Assumptions 1-3, and 5 on the continuous-time process $x(t)$, the covariance of the discrete-time cyclic correlogram (20) is given by

$$\begin{aligned} & \text{cov} \left\{ \tilde{R}_{x_d x_d^{(*)}}(\tilde{\alpha}_1, m_1; n_{01}, N), \tilde{R}_{x_d x_d^{(*)}}(\tilde{\alpha}_2, m_2; n_{02}, N) \right\} \\ &= \tilde{\mathcal{F}}_1 + \tilde{\mathcal{F}}_2 + \tilde{\mathcal{F}}_3 \end{aligned} \quad (22)$$

where the terms $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$, whose expressions are not reported here for sake of brevity, depend on the generalized cyclic auto- and cross-correlation functions defined in (13) and the term $\tilde{\mathcal{F}}_3$ depends on the fourth-order cumulant defined in (14). \square

Theorem 3 Under Assumptions 1a, 2, and 5 on the continuous-time process $x(t)$, the asymptotic ($N \rightarrow \infty$) expected value of the discrete-time cyclic correlogram (20) is given by

$$\lim_{N \rightarrow \infty} \text{E} \left\{ \tilde{R}_{x_d x_d^{(*)}}(\tilde{\alpha}, m; n_0, N) \right\} = \tilde{R}_{x_d x_d^{(*)}}(\tilde{\alpha}, m) \quad (23)$$

Theorem 4 Under Assumptions 1–3, 5–8 on the continuous-time process $x(t)$, the asymptotic ($N \rightarrow \infty$) covariance of the discrete-time cyclic correlogram (20) is given by

$$\lim_{N \rightarrow \infty} (2N+1) \text{cov} \left\{ \tilde{R}_{x_d x_d^{(*)}}(\tilde{\alpha}_1, m_1; n_{01}, N), \tilde{R}_{x_d x_d^{(*)}}(\tilde{\alpha}_2, m_2; n_{02}, N) \right\} = \tilde{\mathcal{F}}_1 + \tilde{\mathcal{F}}_2 + \tilde{\mathcal{F}}_3 \quad (24)$$

where

$$\begin{aligned} \tilde{\mathcal{F}}_1 \triangleq & \mathcal{E}_a \sum_{k'} \sum_{k''} \sum_{\ell=-\infty}^{+\infty} R_{xx^*}^{(k')}((m_1 - m_2 + \ell)T_s) R_{x^{(*)} x^{(*)}}^{(k'')}(\ell T_s) \\ & e^{j2\pi \alpha'_{k'}((m_1 - m_2 + \ell)T_s) m_2 T_s} e^{-j2\pi \tilde{\alpha}_1 \ell} \\ & \delta_{[\tilde{\alpha}_1 - \tilde{\alpha}_2 - \alpha'_{k'}((m_1 - m_2 + \ell)T_s) - \alpha''_{k''}(\ell T_s)] \bmod 1} \end{aligned} \quad (25)$$

$$\begin{aligned} \tilde{\mathcal{F}}_2 \triangleq & \mathcal{E}_a \sum_{k''} \sum_{k'} \sum_{\ell=-\infty}^{+\infty} R_{xx^*}^{(k'')}((m_1 + \ell)T_s) R_{x^{(*)} x^{(*)}}^{(k')}((\ell - m_2)T_s) \\ & e^{j2\pi \alpha''_{k''}((\ell - m_2)T_s) m_2 T_s} e^{-j2\pi \tilde{\alpha}_1 \ell} \\ & \delta_{[\tilde{\alpha}_1 - \tilde{\alpha}_2 - \alpha''_{k''}((m_1 + \ell)T_s) - \alpha'_{k'}((\ell - m_2)T_s)] \bmod 1} \end{aligned} \quad (26)$$

$$\begin{aligned} \tilde{\mathcal{F}}_3 \triangleq & \mathcal{E}_a \sum_k \sum_{\ell=-\infty}^{+\infty} C_{x^{(*)} x^{(*)}}^{(k)}((m_1 + \ell)T_s, \ell T_s, m_2 T_s) e^{-j2\pi \tilde{\alpha}_1 \ell} \\ & \delta_{[\tilde{\alpha}_1 - \tilde{\alpha}_2 - \beta_k((m_1 + \ell)T_s, \ell T_s, m_2 T_s)] \bmod 1} \end{aligned} \quad (27)$$

are bounded for the made assumptions. In (25)–(27), $\delta_{\tilde{\alpha} \bmod 1} = 1$ if $\tilde{\alpha} \in \mathbb{Z}$ and $\delta_{\tilde{\alpha} \bmod 1} = 0$ otherwise, and $\mathcal{E}_a \triangleq \int_{-1/2}^{1/2} |a(t)|^2 dt$.

In addition, for notation simplicity, $\alpha'_{k'}(\cdot) \equiv \alpha_{xx^*}^{(k')}(\cdot)$, $\alpha''_{k''}(\cdot) \equiv \alpha_{x^{(*)} x^{(*)}}^{(k'')}(\cdot)$, $\alpha'''_{k'''}(\cdot) \equiv \alpha_{x^{(*)} x^{(*)}}^{(k''')}(\cdot)$, and $\alpha^V_{k^V}(\cdot) \equiv \alpha_{x^{(*)} x^{(*)}}^{(k^V)}(\cdot)$. \square

From Theorem 3 it follows that the discrete-time cyclic correlogram (20) is an asymptotically ($N \rightarrow \infty$), unbiased estimator of the discrete-time cyclic autocorrelation function (18). Moreover, from Theorem 4 it follows that the variance of the discrete-time cyclic correlogram asymptotically vanishes. Consequently, the discrete-time cyclic correlogram is a mean-square consistent estimator of the discrete-time cyclic autocorrelation function, that is, of an aliased version of the continuous-time cyclic autocorrelation function (see (19a)).

In order to obtain a discrete-time estimate of (samples of) the continuous-time cyclic autocorrelation function, a further assumption is needed in order to control the amount of aliasing in (19a) and (19b) when the sampling period T_s approaches zero.

Assumption 10 For every $\tilde{\alpha}$ and m there exists a sequence $\{M_p\}_{p \in \mathbb{Z}}$ of positive numbers such that

1. $\left| R_{xx^{(*)}}(\tilde{\alpha} + p)f_s, mT_s \right| \leq M_p$
2. $\sum_{p=-\infty}^{+\infty} M_p < \infty$

Every GACS process with a finite number of lag-dependent cycle frequencies satisfies Assumption 10 since only a finite number of nonzero (bounded) terms is present in the sum over p (see, e.g., the chirp-modulated PAM signal with Nyquist pulse in Section 4).

Lemma 1 Under Assumption 10, pointwise it results

$$\lim_{T_s \rightarrow 0} \sum_{\substack{p=-\infty \\ p \neq 0}}^{+\infty} R_{xx^{(*)}}(\tilde{\alpha} + p)f_s, mT_s = 0. \quad (28)$$

From Lemma 1 it follows that the aliasing terms ($p \neq 0$) in (19a) and (19b) can be made arbitrarily small by taking the sampling period T_s sufficiently small. Furthermore, from Theorems 3 and 4 it follows that, for any fixed T_s , the discrete-time cyclic correlogram approaches in the mean-square sense the aliased cyclic autocorrelation function (19a)–(19b) as the data-record length $T = (2N+1)T_s$ approaches infinity. Consequently, for T_s sufficiently small and T sufficiently large, the discrete-time cyclic correlogram $\tilde{R}_{x_d x_d^{(*)}}(\tilde{\alpha}, m; n_0, N)$ can be made arbitrarily close to $R_{xx^{(*)}}(\alpha, \tau)|_{\alpha=\tilde{\alpha}f_s, \tau=mT_s}$ in the mean-square sense. Specifically, we have the following result.

Theorem 5 Under Assumptions 1–3, 5–7, and 10 it results that

$$\lim_{T_s \rightarrow 0} \lim_{N \rightarrow \infty} \mathbb{E} \left\{ \left| \tilde{R}_{x_d x_d^{(*)}}(\tilde{\alpha}, m; n_0, N) - R_{xx^{(*)}}(\alpha, \tau)|_{\alpha=\tilde{\alpha}f_s, \tau=mT_s} \right|^2 \right\} = 0 \quad (29)$$

where the order of the two limits cannot be interchanged. \square

4. NUMERICAL RESULTS

In this section, simulation experiments are carried out, aimed at corroborating the theoretical results of the previous sections.

In the simulation experiment, a GACS signal is obtained as the output $y(t)$ of the Doppler channel with input/output relationship (9) when the input signal $x(t)$ is a cyclostationary PAM signal. In this case, the effect of the chirp modulation is to rotate by an angle θ , where $\tan \theta = \gamma$, the support lines of the cyclic autocorrelation function of $x(t)$ (see (11)). Consequently, denoted by τ_{corr} the maximum value of τ such that $R_{xx^*}(0, \tau)$ (and, hence, $R_{xx^*}(\alpha, \tau) \forall \alpha$) is significantly different from zero, we have that the maximum cycle frequency exhibited by the GACS signal $y(t)$ is $2B + \tau_{\text{corr}} \sin \theta$. Thus, from (19a) we have that the condition $f_s \geq 2(2B + \tau_{\text{corr}} \sin \theta)$ is sufficient to prevent aliasing in the discrete-time cyclic autocorrelation function of the samples of $y(t)$. The input PAM signal has Nyquist-shaped pulse with excess bandwidth $\eta = 0.85$, stationary white binary modulating sequence, and symbol period $T_p = 10T_s$, where T_s is the sampling period. The considered Doppler channel produces a delay $d_0 = 20T_s$, a frequency shift $\nu = 0.02/T_s$, and a chirp rate $\gamma = 1.5 \cdot 10^{-3}/T_s^2$.

The sample mean and the sample standard deviation, computed by 400 Monte Carlo runs, of the cyclic correlogram of the samples of $y(t)$ are evaluated for different data-record lengths $T = NT_s$ and using a rectangular data-tapering window. In Figures 2 ($N = 2^9$) and 3 ($N = 2^{12}$), (a) magnitude of the sample mean and (b) sample standard deviation of the discrete-time cyclic correlogram are reported as functions of αT_s and τ/T_s . The adopted data-record lengths $N = 2^9$ and $N = 2^{12}$, correspond to about 52 and 411 symbols of the PAM signal. A discrete set \mathcal{A} of values of $\tilde{\alpha}$ has been considered by taking 200 cycle-frequency values in the cycle-frequency interval $[-1/8, 1/8]$.

The numerical results corroborate the theoretical results. In fact, as the data-record is increased from $N = 2^9$ to $N = 2^{12}$, both bias and standard deviation decrease. Specifically, as regards the sample mean of the cyclic correlogram, the blurred region outside the support of the cyclic autocorrelation function exhibits reduced oscillations as N is increased (see Figs. 2a and 3a) and the sample mean approaches the true cyclic autocorrelation function. Moreover, the sample mean is significantly different from zero within thin strips around the true lag-dependent cycle frequencies and the strip width becomes narrower as the data-record is increased. Furthermore, the sample standard deviation decreases as N increases (see Figs. 2b and 3b).

5. CONCLUSIONS

Continuous-time GACS processes are an appropriate model to describe the output signal of some Doppler channels excited by ACS signals. The discrete-time cyclic correlogram of the ACS sequence

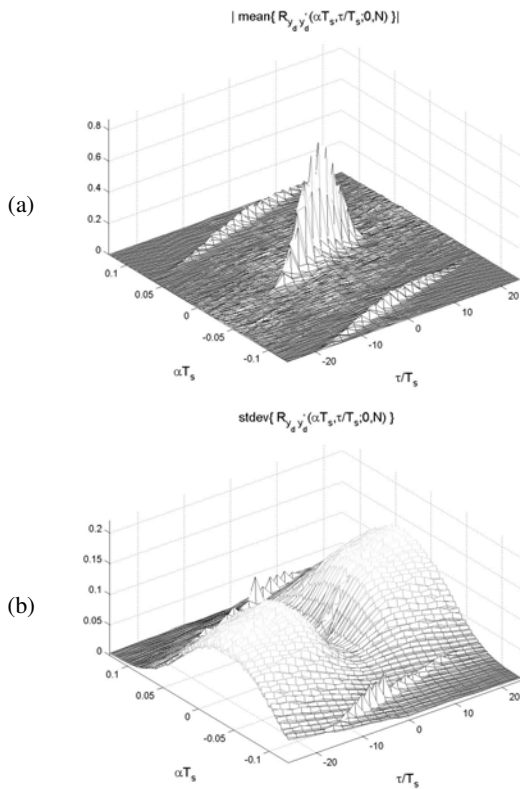


Figure 2: (a) Magnitude of the sample mean and (b) sample standard deviation of the discrete-time cyclic correlogram of the (GACS) chirp-modulated signal $y(t)$ (9), as a function of αT_s and τ/T_s , computed by a data-record length $T = NT_s$ with $N = 2^9$.

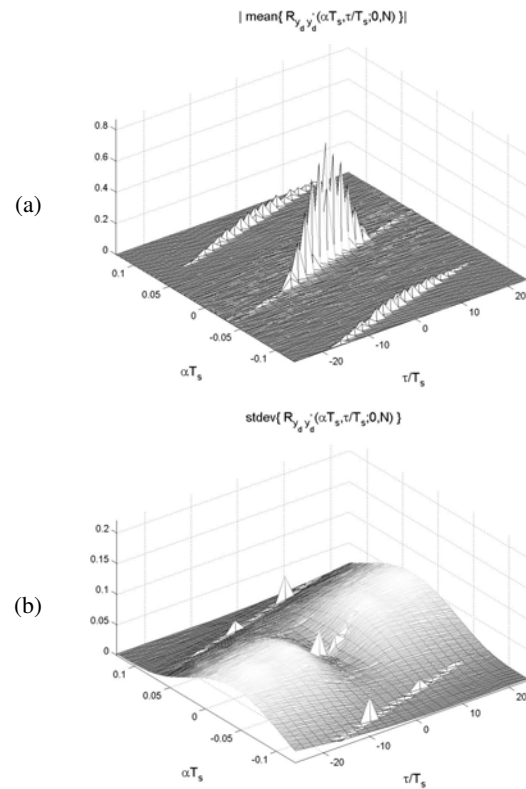


Figure 3: (a) Magnitude of the sample mean and (b) sample standard deviation of the discrete-time cyclic correlogram of the (GACS) chirp-modulated signal $y(t)$ (9), as a function of αT_s and τ/T_s , computed by a data-record length $T = NT_s$ with $N = 2^{12}$.

obtained by uniformly sampling a continuous-time GACS process has been proposed as an estimator of the continuous-time cyclic autocorrelation function. Its mean value and covariance have been evaluated for finite data-record length. Moreover, asymptotic results have been provided as the data-record length approaches infinity. Specifically, the discrete-time cyclic correlogram has been shown to be a mean-square consistent estimator of the aliased continuous-time cyclic autocorrelation function. Moreover, sufficient conditions have been provided to assure that the discrete-time cyclic correlogram approaches the continuous-time cyclic autocorrelation function in the mean-square sense, as the data-record length approaches infinity and the sampling period approaches zero.

REFERENCES

- [1] D. R. Brillinger and M. Rosenblatt, "Asymptotic theory of estimates of k th-order spectra," in *Spectral Analysis of Time Series*, B. Harris, Ed. pp. 153-188, New York: Wiley, 1967
- [2] A.V. Dandawaté and G.B. Giannakis, "Nonparametric polyspectral estimators for k th-order almost cyclostationary processes," *IEEE Transactions on Information Theory*, vol. 40, pp. 67-84, January 1994.
- [3] D. Dehay and H.L. Hurd, "Representation and estimation for periodically and almost periodically random processes," in *Cyclostationarity in Communications and Signal Processing*, (W.A. Gardner, Ed.). IEEE Press, New York, 1994.
- [4] W.A. Gardner, *Statistical Spectral Analysis: A Nonprobabilistic Approach*. Prentice Hall, Englewood Cliffs, NJ, 1988.
- [5] L. Izzo and A. Napolitano, "The higher-order theory of generalized almost-cyclostationary time-series," *IEEE Trans. Signal Processing*, vol. 46, pp. 2975-2989, November 1998.
- [6] L. Izzo and A. Napolitano, "Linear time-variant transformations of generalized almost-cyclostationary signals, Part I: Theory and method", *IEEE Transactions on Signal Processing*, vol. 50, pp. 2947-2961, December 2002.
- [7] L. Izzo and A. Napolitano, "Linear time-variant transformations of generalized almost-cyclostationary signals, Part II: Development and applications", *IEEE Transactions on Signal Processing*, vol. 50, pp. 2962-2975, December 2002.
- [8] L. Izzo and A. Napolitano, "Sampling of generalized almost-cyclostationary signals", *IEEE Transactions on Signal Processing*, vol. 51, pp. 1546-1556, June 2003.
- [9] M. Loève, *Probability Theory*. Van Nostrand, Princeton, NJ, 1963.
- [10] A. Napolitano, "Estimation of second-order cross-moments of generalized almost-cyclostationary processes," *IEEE Transactions on Information Theory*, vol. 53, n. 6, 2007.
- [11] B.M. Sadler and A.V. Dandawaté, "Nonparametric estimation of the cyclic cross spectrum," *IEEE Transactions on Information Theory*, vol. 44, pp. 351-358, January 1998.
- [12] H.L. Van Trees, *Detection, Estimation, and Modulation Theory, Part III*. John Wiley & Sons, Inc., New York, 1971.