EFFICIENT IMPLEMENTATION OF THE LOCAL CAPON SPECTRAL ESTIMATE

*Stephen R. Alty**, *Jacob Benesty*† *and Andreas Jakobsson*‡

*King's College London, Division of Engineering, London WC2R 2LS, U.K.
†INRS-EMT, Université du Québec, Montréal, Québec, H5A 1K6, Canada.
‡Dept. of Electrical Engineering, Karlstad University, SE-651 88 Karlstad, Sweden.

ABSTRACT

In this paper, we present a computationally efficient algorithm to form the well-known Capon spectral estimate. The proposed implementation, which is formed using Levinson-Durbin recursions, offers a computationally attractive alternative to existing techniques. The proposed algorithm allows the spectral estimate to be formed in $\mathcal{O}(L^2 + PL)$ operations, where P denotes the number of frequency grid points to be evaluated and L the filter length used. Numerical comparisons show that the algorithm is particularly efficient at finding high-resolution estimates, especially for limited ranges of frequencies.

1. INTRODUCTION

Spectral estimation finds applications in a wide range of fields, and has received a vast amount of interest in the literature over the last century. Due to their inherent robustness to model assumptions, there has lately been a renewed interest in *non-parametric* spectral estimators (see, e.g., [1–3]). Among the non-parametric approaches, the *data-dependent* filterbank spectral estimators have many promising properties, allowing for very accurate, computationally efficient, high-resolution estimates. In this paper, we examine the problem of forming the classical Capon spectral estimator [4, 5] in a computationally efficient manner. Given the method's wide usability, the recent literature contain several interesting and efficient ways to form the spectral estimate, see, for example, [6–14]. Herein, we propose a further refinement of these methods, suggesting an efficient implementation based on the Levinson-Durbin (LD) recursions. The proposed method offers a computationally attractive alternative to the classical Musicus' algorithm [6], especially when forming the spectral estimate of a limited number of frequencies, say P_k. Whereas Musicus' algorithm allows for an efficient evaluation of the entire frequency range using the fast Fourier transform (FFT), it is often numerically preferable to use a direct method to form the estimate over a limited range of frequencies. Here, we present an efficient reformulation of Musicus' algorithm, allowing the local estimates to be formed directly in $\mathcal{O}(L^2 + P_k L)$ operations, where L denotes the used filter length. The paper is organised as follows; in the next section, we briefly review the Capon spectral estimator and Musicus' algorithm, followed in Section 3 with the derivation of the proposed algorithm. Section 4 illustrates the complexity gain offered by the proposed method. Finally, Section 5 concludes the paper.

Please address all correspondence to steve.alty@kcl.ac.uk

2. THE CAPON SPECTRAL ESTIMATOR

Let $\{x(t); t = 0, \ldots, N-1\}$ denote the available (stationary) data sequence of which the spectrum is to be estimated, where N denotes the number of available data samples. As is well-known, the Capon spectral estimate can be formed as

$$\phi_\omega = \mathbf{h}_\omega^* \mathbf{R}_L \mathbf{h}_\omega = \frac{1}{\mathbf{f}_L^*(\omega)\mathbf{R}_L^{-1}\mathbf{f}_L(\omega)}, \tag{1}$$

where $\mathbf{h}_\omega$ denotes the frequency dependent and data-adaptive L-tap finite impulse response (FIR) filter,

$$\mathbf{h}_\omega = \begin{bmatrix} h_\omega(0) & \ldots & h_\omega(L-1) \end{bmatrix}^T, \tag{2}$$

with $(\cdot)^T$ denoting the transpose, formed as

$$\mathbf{h}_\omega = \arg\min_{\mathbf{h}_\omega} \mathbf{h}_\omega^* \mathbf{R}_L \mathbf{h}_\omega = \frac{\mathbf{R}_L^{-1}\mathbf{f}_L(\omega)}{\mathbf{f}_L^*(\omega)\mathbf{R}_L^{-1}\mathbf{f}_L(\omega)} \tag{3}$$

for a generic frequency ω. Here,

$$\mathbf{R}_L = E\{\mathbf{x}_L(t)\mathbf{x}_L^*(t)\}, \tag{4}$$

with $E\{\cdot\}$ and $(\cdot)^*$ denoting the expectation and the conjugate transpose, respectively, and

$$\mathbf{f}_L(\omega) = \begin{bmatrix} 1 & e^{j\omega} & \ldots & e^{j\omega(L-1)} \end{bmatrix}^T, \tag{5}$$

$$\mathbf{x}_L(t) = \begin{bmatrix} x(t) & \ldots & x(t+L-1) \end{bmatrix}^T, \tag{6}$$

where $M = N - L + 1$. The choice of the filter length L reflects the user's preferences, with larger filter lengths yielding higher resolution, but also higher variance, in the resulting spectral estimate (see, e.g., [1] for further details on the Capon spectral estimator). As is clear from (1), especially when noting the positive definite Toeplitz structure of $\mathbf{R}_L$, the Capon spectral estimate has a very strong inherent structure, allowing for efficient implementations. In Musicus' algorithm [1, 6], this is exploited by forming a structured estimate of $\mathbf{R}_L^{-1}$ via the LD recursion and the Gohberg-Semencul formula, as well as then evaluating the quadratic form in the denominator of (1) using the FFT. Related efficient implementations based on the Cholesky factorisation or the displacement structure of $\mathbf{R}_L^{-1}$ have also been presented in, for instance, [8, 11], with extensions to two-dimensional and time-recursive formulations in [7, 9, 10, 12].

2.1 The Musicus Algorithm

To put our proposed algorithm in context, we will here briefly review Musicus' algorithm, which essentially relies on the efficient calculation of the forward linear prediction (FLP) coefficients, say $\mathbf{a}_L$, from a given sample frame of data of length N. The popular LD algorithm is one of the most efficient and stable methods for this purpose, forming the estimated FLP coefficients as

$$a_{0,0} = 1 \quad (7)$$
$$\gamma_0 = r(0) \quad (8)$$

where $r(k)$ denotes the kth lag of the autocorrelation sequence, i.e., $r(k) = E\{x(t)x^*(t+k)\}$. Then, for $\ell = 1,\ldots,L$,

$$\kappa_\ell = -\frac{1}{\gamma_{\ell-1}}\sum_{n=0}^{\ell-1} r(n-\ell)a_{n,\ell-1}$$
$$a_{n,\ell} = \begin{cases} 1 & \text{for } n=0 \\ a_{n,\ell-1} + \kappa_\ell a^*_{\ell-n,\ell-1} & \text{for } n=1,\ldots,\ell-1 \\ \kappa_\ell & \text{for } n=\ell \end{cases}$$
$$\gamma_\ell = \gamma_{\ell-1}(1-|\kappa_\ell|^2)$$

where κ_ℓ denotes the ℓth reflection coefficient. We note that there are many other well known approaches to determine the FLP coefficients, several which may offer improved spectral estimates, such as Burg's method and the modified covariance method (see, e.g., [1]), although it should be remarked the achieved gain is usually at the expense of additional computational complexity. Without loss of generality, we will assume that the FLP coefficients have been obtained via the LD algorithm, merely noting that the evaluation of the FLP coefficients in itself offer some options for the user. Using the estimated FLP and reflection coefficients, Musicus algorithm then correlates these, i.e., for $k = 0,\ldots,L$,

$$\mu(k) = \frac{1}{\gamma_L}\sum_{n=0}^{L-k}(L+1-k-2n)a_{n,L}a^*_{n+k,L} \quad (9)$$

and as $\mu(k)$ exhibits conjugate symmetry about $k=0$,

$$\mu(k) = \mu^*(-k) \quad \text{for } k=-L,\ldots,-1. \quad (10)$$

The Capon power spectral estimate, ϕ_ω, given in (1), can then be calculated for any given relative frequency, ω, as

$$\phi_\omega = \frac{1}{\sum_{k=-L}^{L}\mu(k)e^{j\omega k}}. \quad (11)$$

When the full spectrum is required, this procedure is beneficially undertaken using a FFT of length P, where P is represents the desired number of frequency grid points. In such cases, the computational complexity of evaluating (11) is about $\mathscr{O}(P\log_2 P)$ complex operations. However, if only a *few* frequency grid points are of interest, it would be more efficient to calculate them directly using (11) alone. Such an evaluation would require $\mathscr{O}(L)$ for each frequency grid point of interest. Hence, for sufficiently short ranges and suitably short values of L, this procedure would be preferable. Notwithstanding, we will see in the next section, how by exploiting the LD recursions, one may further improve the efficiency of such an implementation.

3. THE PROPOSED IMPLEMENTATION

Reminiscent of the Musicus algorithm, we form the following set of recursions. First, we consider the vector containing the $L+1$ recent samples,

$$\mathbf{x}_{L+1}(t) = [\ x(t)\ \ \ldots\ \ x(t+L)\ ]^T, \quad (12)$$

suggesting the corresponding $(L+1)\times(L+1)$ covariance matrix

$$\mathbf{R}_{L+1} = E\{\mathbf{x}_{L+1}(t)\mathbf{x}^*_{L+1}(t)\} \quad (13)$$
$$= \begin{bmatrix} r(0) & \mathbf{r}^*_L \\ \mathbf{r}_L & \mathbf{R}_L \end{bmatrix} \quad (14)$$
$$= \begin{bmatrix} \mathbf{R}_L & \tilde{\mathbf{r}}_L \\ \tilde{\mathbf{r}}^*_L & r(0) \end{bmatrix} \quad (15)$$

where

$$\mathbf{r}_L = [\ r(1)\ \ r(2)\ \ \ldots\ \ r(L)\ ]^T \quad (16)$$
$$\tilde{\mathbf{r}}_L = [\ r(L)\ \ r(L-1)\ \ \ldots\ \ r(1)\ ]^T \quad (17)$$

Using the Schur complement, one may express $\mathbf{R}^{-1}_{L+1}$ as

$$\mathbf{R}^{-1}_{L+1} = \begin{bmatrix} \mathbf{R}^{-1}_L + \gamma^{-1}_L\mathbf{b}_L\mathbf{b}^*_L & -\gamma^{-1}_L\mathbf{b}_L \\ -\gamma^{-1}_L\mathbf{b}^*_L & \gamma^{-1}_L \end{bmatrix} \quad (18)$$

where

$$\mathbf{b}_L = \mathbf{R}^{-1}_L\tilde{\mathbf{r}}_L \quad (19)$$

is the backward predictor of length L,

$$\gamma_L = r(0) - \tilde{\mathbf{r}}^*_L\mathbf{b}_L = r(0) - \mathbf{r}^*_L\mathbf{a}_L \quad (20)$$

is the prediction error energy, and

$$\mathbf{a}_L = \mathbf{J}\mathbf{b}_L \quad (21)$$

is the forward predictor, with $\mathbf{J}$ denoting the exchange matrix. Furthermore, we note that the Fourier vector of length $L+1$ can be decomposed as

$$\mathbf{f}_{L+1}(\omega) = [\ 1\ \ e^{j\omega}\ \ \ldots\ \ e^{i\omega L}\ ]^T \quad (22)$$
$$= [\ \mathbf{f}^T_L(\omega)\ \ e^{j\omega L}\ ]^T \quad (23)$$
$$= [\ 1\ \ e^{j\omega}\mathbf{f}^T_L(\omega)\ ]^T \quad (24)$$

Pre- and post-multiplying $\mathbf{R}^{-1}_{L+1}$, using the expression in (18), with $\mathbf{f}^*_{L+1}(\omega)$ and $\mathbf{f}_{L+1}(\omega)$, respectively, yields

$$\begin{aligned}\psi_\omega &= \mathbf{f}^*_{L+1}(\omega)\mathbf{R}^{-1}_{L+1}\mathbf{f}_{L+1}(\omega) \\ &= \mathbf{f}^*_L(\omega)\mathbf{R}^{-1}_L\mathbf{f}_L(\omega) + \gamma^{-1}_L\left|\mathbf{f}^*_L(\omega)\mathbf{b}_L - e^{-j\omega L}\right| \\ &= \sum_{\ell=0}^{L}\gamma^{-1}_\ell\left|\mathbf{f}^*_\ell(\omega)\mathbf{b}_\ell - e^{-j\omega\ell}\right|, \end{aligned} \quad (25)$$

where $\mathbf{b}_0 = 0$ and $\gamma_0 = r(0)$, implying that the Capon spectral estimate, ϕ_ω, formed in (1), can be written as

$$\phi_\omega = \frac{1}{\sum_{\ell=0}^{L-1}\gamma^{-1}_\ell\left|\mathbf{f}^*_\ell(\omega)\mathbf{b}_\ell - e^{-j\omega\ell}\right|}, \quad (26)$$

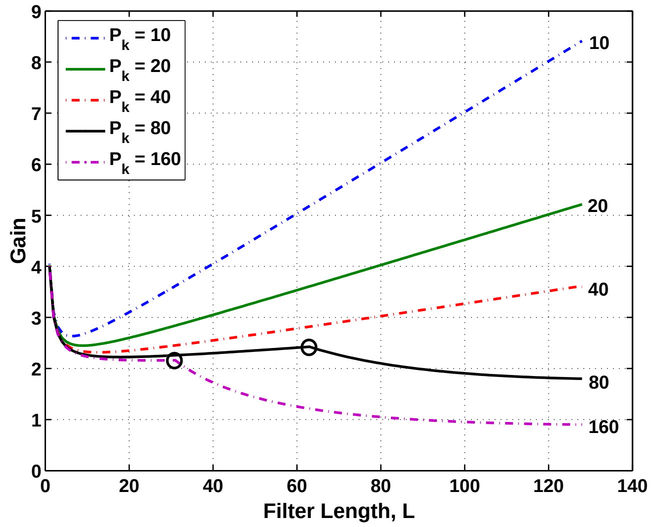

Figure 1: Numerical gain of new approach vs. Musicus' algorithm for a limited range of frequencies P_k. Here, $N = 256$, $P = 1024$ and $1 \leq L \leq 128$.

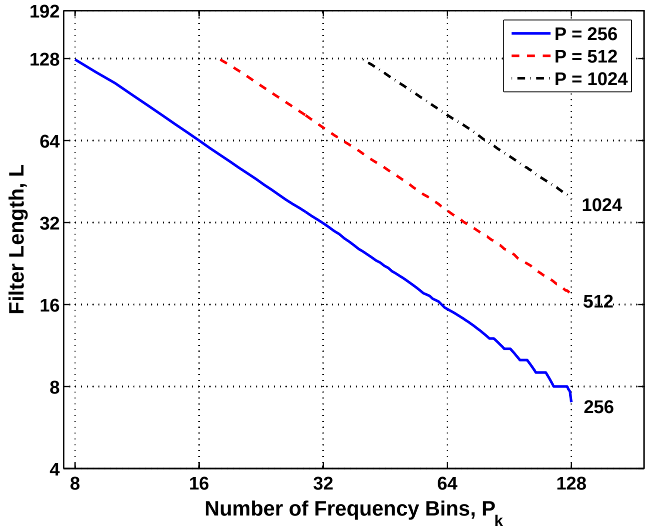

Figure 2: Trade-off point whereby FFT becomes preferred to direct approach for Musicus' algorithm for three different grid spacings, $P = \{256,\ 512\ \&\ 1024\}$.

which nicely links the autoregressive (AR) and the Capon spectral estimators in a way similar to Musicus' algorithm. As is well known, γ_ℓ and $\mathbf{b}_\ell$ can be directly computed using the LD algorithm, implying that (26) allows the Capon spectral estimate for P frequency grid points to be formed in $\mathcal{O}(PL^2)$ operations. Recalling that Musicus' algorithm only requires $\mathcal{O}(L^2 + P\log_2 P)$ operations, this seems to be an unattractive approach to implement ϕ_ω. However, the expression in (26) can be further simplified by continuing the recursions. Let

$$\nu_\ell = \mathbf{f}_\ell^*(\omega)\mathbf{b}_\ell, \tag{27}$$

and note that the backward predictor can be formed as

$$\mathbf{b}_\ell = \begin{bmatrix} 0 \\ \mathbf{b}_{\ell-1} \end{bmatrix} - \kappa_\ell \mathbf{J}_\ell \begin{bmatrix} \mathbf{b}_{\ell-1}^c \\ -1 \end{bmatrix}, \tag{28}$$

where the ℓth reflection coefficient, κ_ℓ, is formed as

$$\kappa_\ell = \gamma_{\ell-1}^{-1}\left[r(\ell) - \mathbf{b}_{\ell-1}^*\mathbf{r}_{\ell-1}\right] \tag{29}$$

and γ_ℓ is updated accordingly,

$$\gamma_\ell = \gamma_{\ell-1}(1 - |\kappa_\ell|^2). \tag{30}$$

Pre-multiplying (28) with $\mathbf{f}_\ell^*(\omega)$ yields

$$\nu_\ell = e^{-j\omega}\nu_{\ell-1} - \kappa_\ell \mathbf{f}_\ell^*(\omega)\mathbf{J}_\ell \begin{bmatrix} \mathbf{b}_{\ell-1}^c \\ -1 \end{bmatrix}. \tag{31}$$

Noting that

$$\mathbf{J}_\ell \mathbf{f}_\ell(\omega) = e^{j\omega(\ell-1)}\mathbf{f}_\ell^c(\omega), \tag{32}$$

where $(\cdot)^c$ denotes the conjugate, one can further simplify (31), obtaining

$$\nu_\ell = e^{-j\omega}\nu_{\ell-1} - \kappa_\ell\left\{e^{j\omega(\ell-1)}\nu_{\ell-1}^c - 1\right\}, \tag{33}$$

which allows ν_ℓ, for $\ell = 0, \ldots, L-1$, to be formed in $\mathcal{O}(L)$ operations, reducing the complexity of forming (26) to $\mathcal{O}(PL + L^2)$ operations for the full spectrum. The power spectrum estimate for each component, ϕ_ω, can then be determined with just, γ_ℓ and ν_ℓ, using

$$\phi_\omega = \frac{1}{\sum_{\ell=0}^{L-1} \gamma_\ell^{-1}|\nu_\ell - e^{-j\omega\ell}|^2}. \tag{34}$$

It can easily be seen that the procedure in (33) and (34) requires just $\mathcal{O}(P_k L)$ operations, giving a total computational load of just $\mathcal{O}(P_k L + L^2)$ operations for P_k frequencies of interest. It should be noted that the presented algorithm is reminiscent to the one described in [14], although the here presented approach is focused on yielding efficiency for a limited range of frequency bins.

4. COMPARISON OF COMPUTATIONAL COMPLEXITY

Herein, we compare the computational burden of the presented method to that of Musicus' algorithm, focusing on *for short frequency ranges*, i.e., where $P_k \ll P$. To clarify the main differences, it is helpful to break down the constituent parts of each method and ignore those procedures which are common to both. Hence, we can dispense with the computation required for the determination of the autocorrelation sequence $r(k)$, for $k = 0, \ldots, L-1$, as this is common to both approaches. Furthermore, we can discount the numerical load of the LD algorithm, requiring about $\mathcal{O}(L^2)$ operations, as this procedure is shared by both methods. We can then state explicitly the number of computations required for the remainder of each method to determine the value of the power spectral component at a single frequency. For Musicus' method, we require the evaluation of (9) and (11), requiring about $(L^2/2 + P_k(2L+1))$ complex operations. Whereas, for the proposed method, given the reformulation of the summation, the corresponding evaluation only

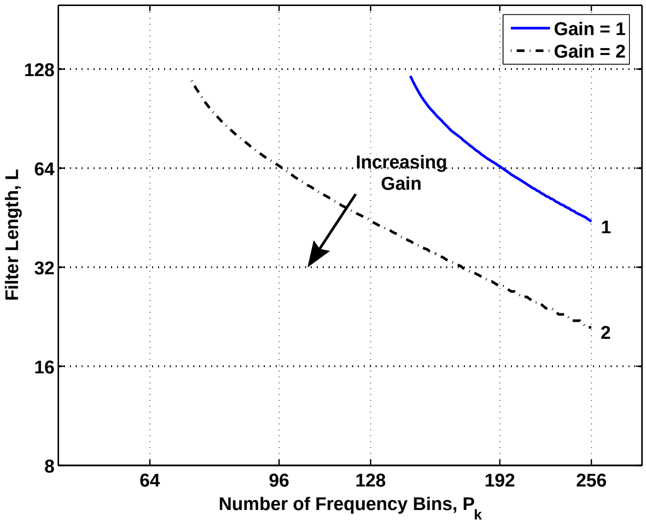

Figure 3: Gain 'isobars' for proposed vs. Musicus' method for various values of filter length, L and frequency grid points, P_k, whilst $P = 1024$.

requires about $P_k L$ complex operations. The gain, in terms of reduction of numerical burden, for the proposed method is illustrated in Figure 1. If the desired number of frequency grid points, P_k, start to approach that of the full frequency range, totalling P grid points, and depending on the desired filter length, L, then the original Musicus method using the FFT may become numerically preferable. This effect is illustrated in Figure 2. The lines represent the equilibrium or trade-off point between the direct evaluation and the full FFT for $P = \{256,\ 512\ \&\ 1024\}$. Below the lines (i.e., the bottom left corner) is the region where the direct approach is faster than applying the full length FFT. In fact, looking at the lower two traces in Figure 1, there is a distinct 'knee' point, marked with a circle in the figure, due to switching to the FFT method of evaluation when P_k is 80 and 160, for L is greater than 64 and 32, respectively. It is interesting to note, however, that the gain of the proposed method is still greater than unity even when using the FFT with Musicus for relatively large numbers of frequencies grid points, P_k, and filter lengths, L. Finally, Figure 3 demonstrates that the gain of the proposed method as compared to Musicus' approach obeys an approximate constant function of the *product* of the filter length, L, and the number of frequency bins of interest, P_k. This is most clearly shown with logarithmic axes. Also, we note from the figure that the gain increases for any given combination of L and P_k as P increases. Hence, for large filter lengths, the method is only attractive for the case when it is desired to evaluate the spectrum over a limited number of frequency grid points. Likewise, for short filter lengths, the method is preferable for even a larger number of frequency grid points (or, depending on the desired frequency resolution, perhaps the entire spectrum).

5. CONCLUSIONS

In this paper, we have shown the development of a novel computationally efficient algorithm to compute the Capon spectral estimate. The method is reminiscent to Musicus' classical algorithm, but exploit further recursions of the Levinson-Durbin algorithm, enabling an efficient evaluation of the spectrum over limited frequency regions. Numerical evaluations illustrate the achievable complexity gain as compared to Musicus' algorithm.

REFERENCES

[1] P. Stoica and R. Moses, *Spectral Analysis of Signals*. Upper Saddle River, N.J.: Prentice Hall, 2005.

[2] S. L. Marple, Jr., *Digital Spectral Analysis with Applications*. Englewood Cliffs, N.J.: Prentice-Hall, 1987.

[3] S. M. Kay, *Modern Spectral Estimation: Theory and Application*. Englewood Cliffs, N.J.: Prentice-Hall, 1988.

[4] J. Capon, "High Resolution Frequency Wave Number Spectrum Analysis," *Proc. IEEE*, vol. 57, pp. 1408–1418, 1969.

[5] R. Lacoss, "Data Adaptive Spectral Analysis Methods," *Geophysics*, vol. 36, pp. 134–148, 1971.

[6] B. Musicus, "Fast MLM Power Spectrum Estimation from Uniformly Spaced Correlations," *IEEE Trans. ASSP*, vol. 33, pp. 1333–1335, October 1985.

[7] Z.-S. Liu, H. Li, and J. Li, "Efficient Implementation of Capon and APES for Spectral Estimation," *IEEE Trans. Aero. and Elec. Syst.*, vol. 34, pp. 1314–1319, October 1998.

[8] T. Ekman, A. Jakobsson, and P. Stoica, "On Efficient Implementation of the Capon algorithm," in *European Signal Processing Conference*, (Tampere, Finland), 2000.

[9] A. Jakobsson, S. L. Marple, Jr., and P. Stoica, "Two-Dimensional Capon Spectrum Analysis," *IEEE Trans. on Signal Processing*, vol. 48, pp. 2651–2661, September 2000.

[10] E. G. Larsson and P. Stoica, "Fast Implementation of Two-Dimensional APES and Capon Spectral Estimators," *Multidimensional Systems and Signal Processing*, vol. 13, pp. 35–54, Jan. 2002.

[11] L. Wei and S. Marple, "A New Least Squares Based Minimum Variance Spectral Estimator Fast Algorithm'," in *Proc. ICASSP*, vol. IV, (Montreal), pp. 405–408, 2005.

[12] S. R. Alty, A. Jakobsson, and E. G. Larsson, "Efficient Time-Recursive Implementation of Matched Filterbank Spectral Estimators," *IEEE Trans. on Circuits and Systems I: Regular Papers*, vol. 52, pp. 516–521, March 2005.

[13] A. Jakobsson, S. R. Alty, and S. Lambotharan, "On the Implementation of the Linearly Constrained Minimum Variance Beamformer," *IEEE Trans. Circuits & Systems, Part II*, vol. 53, pp. 1059–1061, October 2006.

[14] J. Benesty, J. Chen, and Y. Huang, "Recursive and fast recursive Capon spectral estimators," *EURASIP Journal on Advances in Signal Processing*, Article ID 45194, 2007.