CALCULATION OF COMPOSITE SPECTRA FOR BOOLEAN FUNCTIONS IN HAAR DOMAIN

Bogdan J. Falkowski
School of Electrical and Electronic Engineering
Nanyang Technological University
Block S1, 50 Nanyang Avenue
Singapore 639798
efalkowski@ntu.edu.sg

Shixing Yan
HP Labs China
HP Building
112 Jian Guo Road
Beijing, China 100022
shixing.yan@hp.com

ABSTRACT

A new method for the efficient evaluation of the Haar spectra of the logic product, logic sum, EXOR sum and negation of Boolean functions is proposed in this article. The different properties and equations are presented for the new approach. The proposed method shows the advantages in calculation of composite Haar spectra for Boolean functions over traditional method both in computation and time complexities.

1. INTRODUCTION

In many applications of computer engineering and science, where logic functions need to be analyzed or synthesized, it is useful to transform such functions to the corresponding spectral domain that provides various new insights into solving some important problems. Spectral techniques have been applied to Boolean logic classification, disjoint decomposition, parallel and serial linear decomposition, spectral translation synthesis, linearization of decision diagrams, multiplexer synthesis, evaluation of logic complexity, state assignment and testing [4, 6, 7].

The Haar transform which is based on Haar functions [2] is one of the earliest compact, dyadic, orthonormal wavelet transform [1]. The Haar transform has been used widely in many applications as for example, image coding, edge extraction, pattern recognition, and binary logic design [1, 4, 5, 6, 7]. For applications of the Haar transform in logic design, there are many efficient methods to calculate the Haar spectrum, such as fast algorithms [5, 7], calculation from disjoint cubes, and different types of decision diagrams [3, 6]. The Haar spectra of the logic product, logic sum, EXOR sum and negation of Boolean functions are frequently computed in the applications of the Haar transform in logic design. In the current method, the transformations between spectral domain and Boolean domain for the composite Haar spectra from the known Haar spectra of the Boolean functions are needed. In this article, a new method for the calculations of the composite Haar spectra directly from the Haar spectra of the Boolean functions is proposed. In the new method, there are only the calculations in spectral domain which improves both computation and time complexity over the former approach where the transformations between spectral and Boolean domains are necessary which takes a lot of time to calculate the composite Haar spectra.

2. BASIC DEFINITIONS

In this section, we introduce definitions and notations that will be used throughout the article.

The following symbols are used: Let \( x_{n} = \{ x_{p}, x_{n-1}, \ldots, x_{p}, \ldots, x_{2}, x_{1} \} \), and the symbol \( x_{p} \) stands for a data variable, \( p \) is an integer and \( 1 \leq p \leq n \). Let \( f(x_{n}) \) be the \( n \)-variable Boolean function and \( F = [ f(0), f(1), \ldots, f(j), \ldots, f(N-2), f(N-1)]^{T} \) be the truth vector of \( f(x_{n}) \), where \( 1 \leq j \leq N-1 \), \( N = 2^{p} \), and the superscript \( T \) is the transpose operator.

2.1 Operations on Boolean Functions

Let \( f_{1} \) and \( f_{2} \) be the \( n \)-variable Boolean functions, \( F_{1} \) and \( F_{2} \) be their truth vectors.

Definition 1 Let \( f^{12} \) be the logic product of \( f_{1} \) and \( f_{2} \), i.e., \( f^{12} = f_{1} \land f_{2} \). When the truth vector of \( f^{12} \) is represented by \( F^{12} \), then \( F^{12} \) can be obtained from \( F_{1} \) and \( F_{2} \) as follows:

\[
\begin{align*}
    f^{12}(i) &= f_{1}(i) \land f_{2}(i) \\
    \text{or} \quad f^{12}(i) &= f_{1}(i) \cdot f_{2}(i)
\end{align*}
\]

where \( 0 \leq i < N \).

Definition 2 Let \( f^{1/2} \) be the logic sum of \( f_{1} \) and \( f_{2} \), i.e., \( f^{1/2} = f_{1} \lor f_{2} \). When the truth vector of \( f^{1/2} \) is represented by \( F^{1/2} \), then \( F^{1/2} \) can be obtained from \( F_{1} \) and \( F_{2} \) as follows:

\[
\begin{align*}
    f^{1/2}(i) &= f_{1}(i) \lor f_{2}(i) \\
    \text{or} \quad f^{1/2}(i) &= f_{1}(i) + f_{2}(i) - f_{1}(i) \cdot f_{2}(i)
\end{align*}
\]

where \( 0 \leq i < N \).

Definition 3 Let \( f^{1\oplus 2} \) be the EXOR sum of \( f_{1} \) and \( f_{2} \), i.e., \( f^{1\oplus 2} = f_{1} \oplus f_{2} \). When the truth vector of \( f^{1\oplus 2} \) is represented by \( F^{1\oplus 2} \), then \( F^{1\oplus 2} \) can be obtained from \( F_{1} \) and \( F_{2} \) as follows:

\[
\begin{align*}
    f^{1\oplus 2}(i) &= f_{1}(i) \oplus f_{2}(i) \\
    \text{or} \quad f^{1\oplus 2}(i) &= f_{1}(i) + f_{2}(i) - 2f_{1}(i) \cdot f_{2}(i)
\end{align*}
\]

where \( 0 \leq i < N \).

Definition 4 Let \( \overline{f} \) be the negation of \( f \), and the truth vector of \( \overline{f} \) is represented by \( F \). Then the truth vector \( F \) can be obtained from \( F \) as follows:

\[
\overline{f}(i) = \overline{f(i)} \\
\text{or} \quad \overline{f}(i) = 1 - f(i)
\]

where \( 0 \leq i < N \).
2.2 Haar Transform and Spectra

The Haar functions form a complete system of $2^n$ continuous orthonormal functions over the interval $[0, 1]$.

**Definition 5** The Haar functions are defined by $[1, 2]$: 

$$\text{har}(0, 0, \theta) = 1,$$ 

$$\text{har}(i, j, \theta) = \begin{cases} \sqrt{2}^{-i/2} & \frac{i-1}{2} \leq \theta < \frac{i-1/2}{2} \\ -\sqrt{2}^{-i/2} & \frac{i}{2} \leq \theta < \frac{i+1/2}{2} \end{cases}$$

where $0 \leq \theta \leq 1$, $i = 0, 1, \cdots, n-1$ and $j = 1, 2, \cdots, 2^i$.

Discrete Haar functions can be defined as functions determined by sampling the Haar functions at $2^n$ points. For the applications in spectral analysis of switching functions, it is more convenient to work with the non-normalized system of Haar functions [4, 6], which takes only the values $0, 1$ according to the signs. The discrete non-normalized Haar transform can be generated from the non-normalized function. The Haar transform in the following texts is referred to as the non-normalized Haar transform.

**Definition 6** The forward Haar transform matrix of order $2^n$ is defined as $[4, 3, 5, 6, 7]$: 

$$\text{HAR}(n) = \begin{bmatrix} \text{HAR}(n-1) & [1, 1] \\ I_{2^{n-1}} & [1, -1] \end{bmatrix}$$

where $\text{HAR}(0) = 1$, $I_j$ is the identity matrix of order $q$, and the symbol $\otimes$ represents the Kronecker product.

The inverse Haar transform matrix can be expressed as:

$$\text{HAR}^{-1}(n) = \frac{1}{2^n} \begin{bmatrix} \text{HAR}^{-1}(n-1) & [1, 1] \\ I_{2^{n-1}} \otimes [2^{n-1} - 2^{n-1}] \end{bmatrix}$$

where $\text{HAR}^{-1}(0) = 1$.

**Definition 7** For $n$-variable Boolean function $f(x_n)$, its Haar spectrum $H = [h(0), h(1), \cdots, h(N-1)]^T$ is given by $[4, 6, 7]$: 

$$H = \text{HAR}(n) \cdot f$$

The elements of the Haar spectrum are also called as the Haar coefficients of the corresponding Boolean function.

The truth vector of $f(x_n)$ can be also derived from its Haar spectrum as follows:

$$f = \text{HAR}^{-1}(n) \cdot H$$

3. CALCULATION OF COMPOSITE SPECTRA OF HAAR TRANSFORM

Let function $f_1(x_n)$ and $f_2(x_n)$ be the $n$-variable Boolean functions, and $H_1$ and $H_2$ be the corresponding Haar spectra of these Boolean functions.

**Property 1** Let the logic product of $f_1(x_n)$ and $f_2(x_n)$ be represented by $f^{12}(x_n)$. The Haar spectrum $H^{12}$ of function $f^{12}(x_n)$ can be generated from the Haar spectra of $f_1(x_n)$ and $f_2(x_n)$ as follows:

$$H^{12} = \frac{1}{N} R^{(n)} \cdot H_2$$

where $R^{(n)}$ is a $N \times N$ matrix, i.e.,

$$R^{(n)} = \begin{bmatrix} R_{00}^{(n)} & R_{01}^{(n)} & \cdots & R_{0N-1}^{(n)} \\ R_{10}^{(n)} & R_{11}^{(n)} & \cdots & R_{1N-1}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N-1,0}^{(n)} & R_{N-1,1}^{(n)} & \cdots & R_{N-1,N-1}^{(n)} \end{bmatrix},$$

the element $R_{ij}^{(n)}$ of the matrix $R^{(n)}$ is defined in (6) on the next page.

From the commutative law in Boolean algebra, (5) can also be expressed as: $H^{12} = H^{21} = \frac{1}{N} R^{(n)} \cdot H_1,$ where

$$R^{(n)} = \begin{bmatrix} R_{00}^{(n)} & R_{01}^{(n)} & \cdots & R_{0N-1}^{(n)} \\ R_{10}^{(n)} & R_{11}^{(n)} & \cdots & R_{1N-1}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N-1,0}^{(n)} & R_{N-1,1}^{(n)} & \cdots & R_{N-1,N-1}^{(n)} \end{bmatrix},$$

and $R_{ij}^{(n)}$ is defined in (7) on the next page.

In (6) and (7), $i,j \in \{0, N-1\}$, and $i$ and $j$ can be represented in binary format as $(i_{i_{N-1}} \cdots i_1)_2$ and $(j_{j_{N-1}} \cdots j_1)_2$.

**Example 1** For the 2-variable Boolean function, the corresponding Haar spectra of the logic product can be generated using (6). The corresponding matrix $R^{(12)}$ in the equation is:

$$R^{(12)} = \begin{bmatrix} h_1(0) & h_1(1) & 2h_1(2) & 2h_1(3) \\ h_1(1) & h_1(0) & 2h_1(2) & -2h_1(3) \\ h_1(2) & h_1(2) & h_1(0) + h_1(1) & 0 \\ h_1(3) & -h_1(3) & 0 & h_1(0) - h_1(1) \end{bmatrix}.$$
\[
R^{(i)} = \begin{bmatrix}
    h_1(i) & h_1(1) & 2h_1(2) & 2h_1(3) & 4h_1(4) & 4h_1(5) & 4h_1(6) & 4h_1(7) \\
    h_1(1) & h_1(0) & 2h_1(2) & -2h_1(3) & 4h_1(4) & 4h_1(5) & -4h_1(6) & -4h_1(7) \\
    h_1(2) & h_1(2) & h_1(0) + h_1(1) & 0 & 4h_1(4) & -4h_1(5) & 0 & 0 \\
    h_1(3) & h_1(3) & h_1(0) - h_1(1) & 0 & 0 & 4h_1(6) & 0 & -4h_1(7) \\
    h_1(4) & h_1(4) & 2h_1(4) & 0 & h_1(0) + h_1(1) + 2h_1(2) & 0 & 0 & 0 \\
    h_1(5) & h_1(5) & -2h_1(5) & 0 & 0 & h_1(0) + h_1(1) - 2h_1(2) & 0 & 0 \\
    h_1(6) & -h_1(6) & 0 & 2h_1(6) & 0 & 0 & h_1(0) - h_1(1) + 2h_1(3) & 0 \\
    h_1(7) & -h_1(7) & 0 & -2h_1(7) & 0 & 0 & 0 & h_1(0) - h_1(1) - 2h_1(3) \\
\end{bmatrix}
\]

and \( f_2(x_0) \) as follows:
\[
H^{12} = H_1 + H_2 - H^{12} = H_1 + H_2 - \frac{1}{N} R^{(i)} H_2.
\]

**Property 4** Let the EXOR sum of \( f_1(x_0) \) and \( f_2(x_0) \) be represented by \( f^{12}(x_0) \). The Haar spectrum \( H^{12} \) of function \( f^{12}(x_0) \) can be generated from the Haar spectra of \( f_1(x_0) \) and \( f_2(x_0) \) as follows:
\[
H^{12} = H_1 + H_2 - 2H^{12} = H_1 + H_2 - \frac{2}{N} R^{(i)} H_2.
\]

**Property 5** Let the negation of \( f(x_0) \) be represented by \( \overline{f}(x_0) \). The Haar spectrum \( \overline{H} \) of function \( \overline{f}(x_0) \) can be generated from the Haar spectra of \( f(x_0) \) as follows:
\[
\overline{H} = K(n) - H
\]

where \( K(n) \) is an \( N \times 1 \) matrix and \( K(n) = [N, 0, 0, \cdots, 0]^T \).

When the Haar spectra of the Boolean functions have already been calculated, it is more convenient to use the above properties to evaluate the Haar spectrum of the Boolean formula composed by Boolean functions. Using traditional method, firstly we calculate the truth vectors of the Boolean functions from the known Haar spectra through inverse Haar transform applied to each Boolean function separately, then the truth vector of the Boolean formula can be obtained by the truth vectors of the Boolean functions. When the truth vector of the Boolean formula is evaluated, the corresponding Haar spectrum can be obtained by using (3). The properties shown in this section can be used to simplify the calculation of the composite Haar spectrum, since we do not need the transformations between Boolean domain and spectral domain which is necessary in traditional method. It can be noticed that the construction of the matrix \( R^{(i)} \) needs some multiplication operations. Generally, the multiplication operations have great effect in the computational efficiency, but the multiplication operations in \( R^{(i)} \) are the simple bit-shift operations in the applications where the numbers are represented in binary format since the multiplications in the
matrix are always powers-of-two. Hence, the computation complexity of the method using those properties is still much lower than the one using the traditional method. The block diagrams of the procedure for both traditional method and the method using the properties are shown in Fig. 1. It can be seen from the block diagrams that the procedure of traditional method includes the transformations between spectral domain and Boolean domain, so the corresponding inverse and forward Haar transform matrices are used in these calculations. The fast algorithm of Haar transform has been widely used to reduce the computation complexity of the calculation for Haar transform, but it still requires extra time when compared with our method.

The procedure of traditional method includes three steps which need longer processing time than the one required by proposed method. It can be seen from Fig. 1 that the time complexity of the proposed method is lower than the one using traditional method due to smaller number of steps. When the fast algorithm of Haar transform is used to reduce the computation complexity of the calculation in traditional method, the time complexity increases. Hence, the proposed method shows more advantages in time complexity than the traditional method where the fast algorithm of Haar transform is used in the practical applications.

An example is shown below for the comparison between the calculations using traditional and proposed methods.

**Example 3** Let \( f_3 \) and \( f_4 \) be 3-variable Boolean functions, and the corresponding truth vectors are \( F_3 \) and \( F_4 \), respectively. The corresponding Haar spectra of \( f_3 \) and \( f_4 \) are known as \( H_3 = [5,1,-1,0,1,0,1,-1]^T \) and \( H_4 = [5,-1,-2,-1,0,0,-1,0]^T \). The evaluations of the Haar spectrum of \( f_3 \oplus f_4 \) are shown below by traditional and proposed methods.

**Traditional Method:**

Step 1: The inverse Haar transform is needed to get the truth vectors of \( f_3 \) and \( f_4 \) from the corresponding Haar spectra as shown in (4). By using (2), the inverse Haar transform is used to reduce the calculation complexity of the calculation in traditional method, the time complexity increases. Hence, the proposed method shows more advantages in time complexity than the traditional method where the fast algorithm of Haar transform is used in the practical applications.

![Traditional Method Diagram](image)

**Proposed Method:**

![Proposed Method Diagram](image)

Figure 1: The block diagrams of the procedure for both traditional method and the proposed method.

Thus, the truth vectors \( F_3 \) and \( F_4 \) are evaluated as follows:

\[
F_3 = HAR^{-1}(3) \cdot H_3 = [1,0,1,1,0,0,1]^T,
\]

\[
F_4 = HAR^{-1}(3) \cdot H_4 = [0,0,1,1,0,1,1]^T.
\]

Step 2: Let the truth vector of \( f_3 \oplus f_4 \) be represented by \( F^{3\oplus4} \). The EXOR sum of Boolean functions is described in Definition 3, so the truth vector \( F^{3\oplus4} \) can be obtained:

\[
F^{3\oplus4} = [1,0,0,0,1,1,1,0,1,1,0,1]^T.
\]

Step 3: By using (1), the forward Haar transform is evaluated as:

\[
HAR(3) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},
\]

\[
HAR(3) \cdot F^{3\oplus4} = [4, -2, 1, 1, 1, 0, 0, 1]^T.
\]

The Haar spectrum \( H^{3\oplus4} \) can be obtained by using (3) as:

\[
H^{3\oplus4} = HAR(3) \cdot F^{3\oplus4} = [4, -2, 1, 1, 1, 0, 0, 1]^T.
\]

**Proposed Method:**

The evaluation of the Haar spectrum of EXOR sum of two Boolean functions can be processed as shown in Property 4.

Step 1: The general structure of the matrix is shown in (8) and the Haar spectra of \( f_3 \) and \( f_4 \) are known. As \( f_1 \) here corresponds to \( f_1 \) in (8), so we can construct the corresponding matrix of size \( n = 3 \) by the Haar spectrum of \( f_3 \) as:

\[
R^{3\oplus4} = \begin{bmatrix} 5 & 1 & -2 & 0 & 4 & 0 & 4 & -4 \end{bmatrix},
\]

\[
R^{3\oplus4} = \begin{bmatrix} 1 & 5 & -2 & 0 & 4 & 0 & 4 & -4 \end{bmatrix},
\]

\[
R^{3\oplus4} = \begin{bmatrix} 1 & 1 & 2 & 0 & 4 & 0 & 0 \end{bmatrix},
\]

\[
R^{3\oplus4} = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 0 & 0 \end{bmatrix}.
\]

Step 2: By using (10), the Haar spectrum is obtained as:

\[
H^{3\oplus4} = H_3 + H_4 - \frac{1}{4} R^{3\oplus4} \cdot H_3.
\]

Alternatively, from the commutative law the Haar spectrum can be obtained by:

\[
H^{3\oplus4} = H_3 + H_4 - \frac{1}{4} R^{4\oplus3} \cdot H_3.
\]

In this case, the computational costs required by applying \( R^{4\oplus3} \) are lower than using \( R^{3\oplus4} \) due to the fact that the matrix \( R^{3\oplus4} \) has more zero elements than \( R^{3\oplus4} \) since \( H_3 \) has more coefficients equal to zero than \( H_3 \). In practical applications, we can choose the Haar spectra which has bigger number of zero coefficients to construct the corresponding matrix with more zero elements which results in the final lower computational costs.
4. CONCLUSION

The method to obtain the composite Haar spectra including the Haar spectra of logic product, logic sum, EXOR sum and negation of Boolean functions is shown in this article. Based on the proposed properties, the calculations for the composite Haar spectra are all performed in the spectral domain. Comparatively, the transformations between spectral and Boolean domains lead to the higher computation and time complexities by the traditional method. The relationships between Haar spectral domain and Boolean domain are also shown from the proposed properties.

REFERENCES


