16th European Signal Processing Conference (EUSIPCO 2008), Lausanne, Switzerland, August 25-29, 2008, copyright by EURASIP

BLIND PARTIAL EXTRACTION OF INSTANTANEOUS MIXTURES OF SOURCES USING SECOND ORDER STATISTICS

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ABSTRACT

We introduce a criterion for blindly extracting a small subset of most interesting sources in instantaneous mixtures, based on second order statistics. The extracted sources are those for which the time varying spectral density varies the most. The gradient and an approximation to the Hessian of the criterion are derived. Based on the block diagonal form of the approximate Hessian, an algorithm of relaxation type is developed. A simulation example is provided showing the good performance of the algorithm.

1. INTRODUCTION

We consider the simplest and most widely considered blind source separation (BSS) model: $\mathbf{X}(t) = \mathbf{A}\mathbf{S}(t)$ where $\mathbf{X}(t)$ and $\mathbf{S}(t)$ are vectors of a same dimension K, representing the observations and the sources (at time t) and $\mathbf{A}$ is an invertible matrix. The goal is to extract the sources from their mixtures, without relying on any specific knowledge about them other than their independence. Several good algorithms have been proposed for this task [1, 2]. However, in many applications (biomedical for ex.), the number K of mixtures can be very large and therefore one may be interested in extracting only a small number of most interesting sources. In this work, we focus on methods based on second order statistics, which rely on the spectral diversity or time diversity or more generally time-frequency energy distribution diversity, of the sources. Thus by most interesting sources we mean those for which the spectral density (or time-frequency energy distribution) varies the most over the frequency axis (or the time-frequency plane). When there are many sources, one may expect that many of them are nearly stationary white noises: their spectral densities (or time-frequency distributions) are nearly flat. In this case, second order algorithms would have difficulties to extract the sources (in fact they fail completely if there are more than one stationary white noise sources). Extracting them just wastes computational resources and may even leads to wrong interpretation of the analysis. In fact, in BSS problem with very large number of mixtures, one routinely discards most of the extracted sources and only retains some of them.

In this paper we shall develop an algorithm for extracting only $p < K$ sources, based on second order statistics. Note that methods based on the instantaneous distribution of sources, in particular the mutual information, has been considered in [5]. Such methods exploit the non Gaussianity of the sources, while second order methods exploit their spectral and/or time diversity. The later methods basically work by jointly diagonalizing a set of (lagged) covariance or spectral matrices. Our method is based on interspectral matrices or more generally time varying interspectral matrices, at different frequencies and time points, since we use a criterion similar to the one in [4]. Note that this criterion has an interpretation as a sparsity interpretation: it extracts sources for which the time-frequency energy distribution is most concentrated, or most variable, in the time-frequency plane.

2. THE CRITERION

Since the observations are linear mixtures of sources, one would extract the sources by a linear operation, as the components $Y_i(t)$ of $\mathbf{Y}(t) = \mathbf{B}\mathbf{X}(t)$ where $\mathbf{B}$ is a $p \times K$ matrix. The process $\{Y_i(t)\}$ is random, possibly non stationary, with a time varying spectral density denoted by $f_{Y_i}(t,\omega)$. These densities can be estimated by one of the two methods described in [4]. Actually, one first estimates the time varying spectral-interspectral matrix $f_{\mathbf{X}}(t,\omega)$ of the vector process $\{\mathbf{X}(t)\}$, then deduces that of $f_{Y_i}(t,\omega)$ as the i-th diagonal element of $\mathbf{B}\hat{f}_{\mathbf{X}}(t,\omega)\mathbf{B}^{\mathrm{T}}$, $\hat{f}_{\mathbf{X}}(t,\omega)$ denoting the estimator of $f_{\mathbf{X}}(t,\omega)$ and $^{\mathrm{T}}$ the transpose. Other estimation methods can be used provided that $\hat{f}_{\mathbf{X}}(t,\omega)$ is positive definite so that $\hat{f}_{Y_i}(t,\omega) \geq 0$ for any value of $\mathbf{B}$. In practice $\hat{f}_{Y_i}(t,\omega)$ will not estimate the time varying spectral density at precisely the time-frequency point (t,ω), but only the average density over a cell centered at this point. Therefore we shall work with a finite time-frequency grid $\{t_n, \omega_m : n = 1,\ldots,N,\ m = 1,\ldots,M\}$ for which the corresponding cells are more or less disjoint. Thus our work covers two extreme cases: (i) $N = 1$ in which $f_{Y_i}(t_1,\omega_m)$ represents the spectral density of $\{Y_i(t)\}$ at the frequency ω_m (averaged over the observation period), and (ii) $M = 1$ in which $f_{Y_i}(t_n,\omega_1)$ represents the local variance of $\{Y_i(t)\}$ at time t_n.

For simplicity of notation, we denote by $\mathscr{G}$ the set of all time-frequency grid points and write $f_{\mathbf{X}}(g)$ for the time varying interspectral matrix at $g \in \mathscr{G}$ (and similarly for $\hat{f}_{\mathbf{X}}(g)$). Further, for any function h on $\mathscr{G}$, $\langle h \rangle$ denotes its average over $\mathscr{G}$. We would like to find a $p \times K$ matrix $\mathbf{B}$ such that the time varying spectral density f_{Y_i} of each component process $\{Y_i(t)\}$ of the vector process $\{\mathbf{Y}(t)\}$ varies the most with respect to $g \in \mathscr{G}$. As a measure of variability of the function f_{Y_i}, we take $\log\langle f_{Y_i}\rangle - \langle \log f_{Y_i}\rangle$, the logarithm of the ratio of arithmetic and geometric averages of the f_{Y_i}. However maximizing $\sum_{i=1}^{p}[\log\langle f_{Y_i}\rangle - \langle \log f_{Y_i}\rangle]$ would lead to extracting p times a same sources, the one which maximizes this measure of variability. Thus we need to

subtract a term to enforce the least the global correlation between the extracted sources. Since the covariance matrix of the observation vector is $\mathbf{C} = \langle f_{\mathbf{X}} \rangle$, that of the extracted source vectors is $\mathbf{BCB}^{\mathrm{T}}$. Therefore a measure of the global correlation of the extracted source can be taken as $\log\det\mathrm{diag}(\mathbf{BCB}^{\mathrm{T}}) - \log\det(\mathbf{BCB}^{\mathrm{T}})$, where diag is the operator which builds a diagonal matrix from its diagonal elements. The last expression is known, by the Hadamard inequality [6], to be non negative and can be zero if and only if the matrix $\mathbf{BCB}^{\mathrm{T}}$ is diagonal, meaning that the extracted sources are uncorrelated. Thus, noting that f_{Y_i} is the i-th diagonal element of $\mathbf{B}f_{\mathbf{X}}\mathbf{B}^{\mathrm{T}}$ hence $\langle f_{Y_i} \rangle$ is the i-th diagonal element of $\mathbf{BCB}^{\mathrm{T}}$, we are led to the criterion:

$$C(\mathbf{B}) = \langle \log\det\mathrm{diag}(\mathbf{B}f_{\mathbf{X}}\mathbf{B}^{\mathrm{T}}) \rangle - \log\det(\mathbf{BCB}^{\mathrm{T}}) \quad (1)$$

to be minimized with respect to $\mathbf{B}$. This is the theoretical criterion. In practice, the matrices $f_{\mathbf{X}}(g)$ would be replaced by their estimates $\hat{f}_{\mathbf{X}}(g)$

In this paper we focus on the case of real signals so that $\mathbf{B}$ is real (although our method can be easily extended to the complex case, in which $\mathbf{B}^{\mathrm{T}}$ should be replaced by the transpose conjugate of $\mathbf{B}$). Then $\mathrm{diag}(\mathbf{B}f_{\mathbf{X}}\mathbf{B}^{\mathrm{T}})$ is the same as $\mathrm{diag}(\mathbf{B}\Re f_{\mathbf{X}}\mathbf{B}^{\mathrm{T}})$, $\Re$ denoting the real part. Thus one may in (1) replace $f_{\mathbf{X}}$ by $\Re f_{\mathbf{X}}$. This doesn't change the matrix $\mathbf{C} = \langle f_{\mathbf{X}} \rangle$ since this matrix should be real as it represents the covariance matrix of real signals. (This is the case by choosing $\mathscr{G}$ such for any $g \in \mathscr{G}$ there is a g^* in $\mathscr{G}$ (which can be the same as g) such that $f_{\mathbf{X}}(g^*)$ is the conjugate of $f_{\mathbf{X}}(g)$).

3. GRADIENT AND HESSIAN OF THE CRITERION

Since our criterion C defined in (1) is function of matrix argument, instead of computing its gradient and Hessian directly, it is simpler to perform a Taylor expansion up to second order. Thus let $d\mathbf{B}$ be a small increment of $\mathbf{B}$, we shall compute the corresponding increment of C up to second order. The corresponding increment of $f_{\mathbf{Y}} = \mathbf{B}f_{\mathbf{X}}\mathbf{B}^{\mathrm{T}}$ is $d\mathbf{B}f_{\mathbf{XY}} + f_{\mathbf{YX}}d\mathbf{B}^{\mathrm{T}} + d\mathbf{B}f_X d\mathbf{B}^{\mathrm{T}}$ where $f_{\mathbf{XY}} = f_{\mathbf{X}}\mathbf{B}^{\mathrm{T}}$ is the time varying cross spectral density between the processes $\{\mathbf{X}(t)\}$ and $\{\mathbf{Y}(t)\}$ and similarly for $f_{\mathbf{YX}}$. The diagonal of the matrix increment of $f_{\mathbf{Y}}$ equals that of $2d\mathbf{B}f_{\mathbf{XY}} + d\mathbf{B}f_X d\mathbf{B}^{\mathrm{T}}$, since $f_{\mathbf{YX}}$ is the transpose conjugated of $f_{\mathbf{XY}}$. Thus, using the fact that $\log(a+\varepsilon) = \log a + \varepsilon/a - \frac{1}{2}(\varepsilon/a)^2 + \ldots$, the increment of $\langle \log\det\mathrm{diag}(f_{\mathbf{Y}}) \rangle$ corresponding to the increment $d\mathbf{B}$ of $\mathbf{B}$ equals

$$\Big\langle 2\mathrm{tr}[d\mathbf{B}f_{\mathbf{XY}}\mathrm{diag}(f_{\mathbf{Y}})^{-1}] + \mathrm{tr}[d\mathbf{B}f_{\mathbf{X}}d\mathbf{B}^{\mathrm{T}}\mathrm{diag}(f_{\mathbf{Y}})^{-1}] - 2\mathrm{tr}\{[\mathrm{diag}(d\mathbf{B}f_{\mathbf{XY}})\mathrm{diag}(f_{\mathbf{Y}})^{-1}]^2\}\Big\rangle + \ldots \quad (2)$$

where tr denotes the trace.

On the other hand, the increment of $\log\det(\mathbf{BCB}^{\mathrm{T}})$ corresponding to the increment $d\mathbf{B}$ of $\mathbf{B}$ is $\log\det(\mathbf{I}+\mathbf{\Delta})$ where $\mathbf{\Delta} = (d\mathbf{BCB}^{\mathrm{T}} + \mathbf{BC}d\mathbf{B}^{\mathrm{T}} + d\mathbf{BC}d\mathbf{B}^{\mathrm{T}})(\mathbf{BCB}^{\mathrm{T}})^{-1}$. Further, let $\delta_1, \delta_2, \ldots$ be the eigenvalues of $\mathbf{\Delta}$, then $\log\det(\mathbf{I}+\mathbf{\Delta}) = \sum_i \log(1+\delta_i) = \sum_i(\delta_i - \frac{1}{2}\delta_i^2 + \ldots) = \mathrm{tr}(\mathbf{\Delta}) - \frac{1}{2}\mathrm{tr}(\mathbf{\Delta}^2) + \ldots$ Therefore, the increment of $\log\det(\mathbf{BCB}^{\mathrm{T}})$ corresponding to the increment $d\mathbf{B}$ of $\mathbf{B}$ equals

$$2\mathrm{tr}[d\mathbf{BCB}^{\mathrm{T}}(\mathbf{BCB}^{\mathrm{T}})^{-1}] + \mathrm{tr}[d\mathbf{BC}d\mathbf{B}^{\mathrm{T}}(\mathbf{BCB}^{\mathrm{T}})^{-1} - d\mathbf{BCB}^{\mathrm{T}}(\mathbf{BCB}^{\mathrm{T}})^{-1}(\mathbf{BC}d\mathbf{B}^{\mathrm{T}} + d\mathbf{BCB}^{\mathrm{T}})(\mathbf{BCB}^{\mathrm{T}})^{-1}] + \ldots \quad (3)$$

since $\mathrm{tr}(\mathbf{ML}) = \mathrm{tr}(\mathbf{LM}) = \mathrm{tr}(\mathbf{M}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}})$ and $\mathbf{C}$ is symmetric

3.1 Gradient of the criterion

The gradient of the criterion is the matrix $\dot{C}(\mathbf{B})$ in the first order expansion $C(\mathbf{B}+d\mathbf{B}) = C(\mathbf{B}) + \mathrm{tr}[\dot{C}(\mathbf{B})d\mathbf{B}^{\mathrm{T}}] + \cdots$ Looking at the first order term in $d\mathbf{B}$ in (2) and (3), one gets:

$$\dot{C}(\mathbf{B}) = 2\langle \mathrm{diag}(f_{\mathbf{Y}})^{-1} f_{\mathbf{YX}} \rangle - 2(\mathbf{BCB}^{\mathrm{T}})^{-1}\mathbf{BC} \quad (4)$$

3.2 Hessian of the criterion

The formula for the Hessian, although can be obtained, is too complex and therefore we shall make some approximations. We suppose that the matrix $\mathbf{B}$ approximatively extract the sources so that there is a complementary matrix $\mathbf{B}^c$ of size $(K-p)\times K$ extracting mixtures of other sources, such that

$$\begin{bmatrix} \mathbf{B} \\ \mathbf{B}^c \end{bmatrix} f_{\mathbf{X}}(g) [\mathbf{B}^{\mathrm{T}} \;\; \mathbf{B}^{c\mathrm{T}}] \approx \begin{bmatrix} \mathrm{diag}[f_{\mathbf{Y}}(g)] & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^c f_{\mathbf{X}}(g)\mathbf{B}^{c\mathrm{T}} \end{bmatrix}$$

for all $g \in \mathscr{G}$. The matrix $\mathbf{B}^c$ can be determined uniquely up to pre-multiplication by an invertible matrix, by setting $\langle \mathbf{B}^c f_{\mathbf{X}} \mathbf{B}^{\mathrm{T}} \rangle = \mathbf{B}^c\mathbf{CB}^{\mathrm{T}}$ to exactly $\mathbf{0}$. Thus

$$f_{\mathbf{X}} \approx \begin{bmatrix} \mathbf{B} \\ \mathbf{B}^c \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{diag}(f_{\mathbf{Y}}) & 0 \\ 0 & \mathbf{B}^c f_{\mathbf{X}} \mathbf{B}^{c\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{B}^{\mathrm{T}} & \mathbf{B}^{c\mathrm{T}} \end{bmatrix}^{-1}, \quad (5)$$

the approximation being unchanged when $\mathbf{B}^c$ is pre-multiplied by an invertible matrix.

Equality $\mathbf{B}^c\mathbf{CB} = \mathbf{0}$ yields that

$$\mathbf{C} = \begin{bmatrix} \mathbf{B} \\ \mathbf{B}^c \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{BCB}^{\mathrm{T}} & 0 \\ 0 & \mathbf{B}^c\mathbf{CB}^{c\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{B}^{\mathrm{T}} & \mathbf{B}^{c\mathrm{T}} \end{bmatrix}^{-1}. \quad (6)$$

Therefore $d\mathbf{BC} = [\mathscr{E}\mathbf{BCB}^{\mathrm{T}} \;\; \mathscr{E}^c\mathbf{B}^c\mathbf{CB}^{c\mathrm{T}}][\mathbf{B}^{\mathrm{T}} \;\; \mathbf{B}^{c\mathrm{T}}]^{-1}$ where we have put

$$d\mathbf{B}\begin{bmatrix} \mathbf{B} \\ \mathbf{B}^c \end{bmatrix}^{-1} = [\mathscr{E} \;\; \mathscr{E}^c] \quad (\mathscr{E} \text{ square matrix}).$$

Hence, the second order terms in (3) can be written as

$$\mathrm{tr}[\mathscr{E}^c\mathbf{B}^c\mathbf{CB}^{c\mathrm{T}}\mathscr{E}^{\mathrm{T}}(\mathbf{BCB}^{\mathrm{T}})^{-1} - \mathscr{E}^2].$$

On the other hand, the approximation (5) for $f_{\mathbf{X}}$ yields

$$d\mathbf{B}f_{\mathbf{X}}d\mathbf{B}^{\mathrm{T}} \approx \mathscr{E}\mathrm{diag}(f_{\mathbf{Y}})\mathscr{E}^{\mathrm{T}} + \mathscr{E}^c\mathbf{B}^c f_{\mathbf{X}}\mathbf{B}^{c\mathrm{T}}\mathscr{E}^{c\mathrm{T}}$$

and $d\mathbf{B}f_{\mathbf{XY}} \approx \mathscr{E}\mathrm{diag}(f_{\mathbf{Y}})$. Further, $\mathbf{BCB}^{\mathrm{T}} = \langle f_{\mathbf{Y}} \rangle$ may be approximated by $\mathrm{diag}(\langle f_{\mathbf{Y}} \rangle)$. Finally, by (2) and (3), the second order terms in the Taylor expansion of $C(\mathbf{B}+d\mathbf{B})$ are approximately

$$\sum_{1\le i\ne j\le K}\sum \Big(\mathscr{E}_{ij}^2\Big\langle\frac{f_{Y_j}}{f_{Y_i}}\Big\rangle + \mathscr{E}_{ij}\mathscr{E}_{ji}\Big) + \sum_{i=1}^{K}\mathscr{E}_{i\cdot}^c\Big[\Big\langle\frac{\mathbf{B}^c f_X \mathbf{B}^{c\mathrm{T}}}{f_{Y_i}}\Big\rangle - \frac{\mathbf{B}^c\mathbf{CB}^{c\mathrm{T}}}{\langle f_{Y_i}\rangle}\Big]\mathscr{E}_{i\cdot}^{c\mathrm{T}}$$

where f_{Y_i} denotes i-th diagonal element of $f_{\mathbf{Y}}$ and $\mathscr{E}_{ij}$ and $\mathscr{E}_{i\cdot}^c$ denote the general element of $\mathscr{E}$ and the i-th row of $\mathscr{E}^c$.

The above results show that it is more convenient to work with the relative increments $\mathscr{E}, \mathscr{E}^c$ instead of $d\mathbf{B}$. In term of

these increments the approximate Hessian is block diagonal with 2×2 blocks

$$2\begin{bmatrix} \langle f_{Y_j}/f_{Y_i}\rangle & 1 \\ 1 & \langle f_{Y_i}/f_{Y_j}\rangle \end{bmatrix}, \qquad 1 \le i < j \le K,$$

corresponding to $\mathscr{E}_{ij}, \mathscr{E}_{ji}$, and $(K-p)\times(K-p)$ blocks

$$2(\langle \mathbf{B}^c f_X \mathbf{B}^{c\mathrm{T}}/f_{Y_i}\rangle - \mathbf{B}^c\mathbf{C}\mathbf{B}^{c\mathrm{T}}/\langle f_{Y_i}\rangle), \qquad i = 1,\dots,K,$$

corresponding to the vector $\mathscr{E}^c_{i\cdot}$. Note that the row and column of the Hessian matrix corresponding to the increment $\mathscr{E}_{ii}$ is identically zero, which is simply the consequence of the fact that the criterion is scale invariant: it is unchanged when one multiplies the i-th row of its argument by $\mathscr{E}_{ii}$.

3.3 Discussion

One may use the above formulas for the gradient and the approximate Hessian to implement a quasi Newton algorithm. However, there is a major difficulty: the approximate Hessian can be non positive definite, which breaks the algorithm. In fact we can find a counter example (not shown here for lack of space) in which the (theoretical) criterion admits a saddle point and the approximate Hessian is exact at this point and is not positive definite. Thus quasi Newton algorithm *will not be used*. However, The above formula for the Hessian has an interesting consequence: one can work with the pairs of increments $(\mathscr{E}_{ij}, \mathscr{E}_{ji})$ and the increment vectors $\mathscr{E}^c_{i\cdot}$ independently. This suggests a form of relaxation algorithm described below.

4. THE RELAXATION ALGORITHM

The idea behind the relaxation method is to minimize the criterion with respect to a subset of variables, keeping the other fixed, then repeat the procedure with another subset until all variables have been covered, then start all over again until convergence. From the result of previous section, a natural candidate of these subsets of variables are the pairs $(\mathscr{E}_{ij}, \mathscr{E}_{ji}), 1 \le i < j \le K$ and the vectors $\mathscr{E}^c_{i\cdot}, i = 1,\dots,K$. This leads to two kinds of update.

4.1 Intra-source update

This corresponds to keeping $\mathscr{E}^c = \mathbf{0}$, hence the transformation $\mathbf{B} \to \mathbf{B} + d\mathbf{B} = (\mathbf{I}+\mathscr{E})\mathbf{B}$ would simply (try to) only demix the extracted sources by adding to each of them a linear combination of the others. The matrices $f_{\mathbf{Y}}(g) = \mathbf{B} f_{\mathbf{X}}(g)\mathbf{B}^{\mathrm{T}}$ are transformed into $(\mathbf{I}+\mathscr{E}) f_{\mathbf{Y}}(g)(\mathbf{I}+\mathscr{E})^{\mathrm{T}}$, and thus the criterion, in term of $\mathscr{E}$, can be written as

$$\langle \log\det\mathrm{diag}[(\mathbf{I}+\mathscr{E}) f_{\mathbf{Y}} (\mathbf{I}+\mathscr{E})^{\mathrm{T}}]\rangle - 2\log\det|\mathbf{I}+\mathscr{E}| + \text{const.}$$

This is the same as the criterion in [3] for jointly approximately diagonalizing several matrices. Thus, one may apply the same method as in this paper, which leads to taking[1]

$$\begin{bmatrix} \mathscr{E}_{ij} \\ \mathscr{E}_{ij} \end{bmatrix} = \frac{-2}{1+\sqrt{1-4h_{ij}h_{ji}}} \begin{bmatrix} h_{ij} \\ h_{ji} \end{bmatrix}$$

where

$$\begin{bmatrix} h_{ij} \\ \bar{h}_{ji} \end{bmatrix} = \begin{bmatrix} \langle f_{Y_j}/f_{Y_i}\rangle & 1 \\ 1 & \langle f_{Y_i}/f_{Y_j}\rangle \end{bmatrix}^{-1} \begin{bmatrix} \langle f_{Y_iY_j}/f_{Y_i}\rangle \\ \langle f_{Y_jY_i}/f_{Y_j}\rangle \end{bmatrix},$$

[1] This formula is the simplified version of the one in [3] as the h_{ij} are real

$f_{Y_iY_j}$ being the ij element of $f_{\mathbf{Y}}$. Here $f_{\mathbf{Y}}$ is computed from the current value of $\mathbf{B}$ and once the $\mathscr{E}_{ij}, \mathscr{E}_{ij}$ has been obtained as above, $\mathbf{B}$ is updated by pre-multiplication with $\mathbf{I}+\mathscr{E}$ where $\mathscr{E}$ has only $\mathscr{E}_{ij}, \mathscr{E}_{ij}$ as non zero elements at the ij and ji place. The above transformations will be done for all pairs (i,j), $1 < i < j < K$, which constitute an intra-source update.

4.2 Extra-source update

In this update, each i-th row of $\mathbf{B}$ will be successively added a vector of the form $\mathscr{E}^c_{i\cdot}\mathbf{B}^c$, which corresponds to adding a linear combination of observed mixtures to the extracted sources. One may take as $\mathscr{E}^c_{i\cdot}$ the one provided by the quasi-Newton algorithm, but then we could encounter the problem of non positivity of the Hessian and this choice does not even ensure that the criterion is decreased. Our idea is to take $\mathscr{E}^c_{i\cdot}$ of the form $\alpha_i \mathbf{e}_i$ and adjust the coefficient α_i so that the criterion is (almost) maximally decreased. The choice of $\mathbf{e}_i$ will be discussed later.

Let the i-th row of $\mathbf{B}$ be added $\alpha_i \mathbf{e}_i \mathbf{B}^c = \alpha_i \mathbf{d}_i$ and the other rows are kept fixed. Then $\langle \log\det\mathrm{diag}(f_{\mathbf{Y}})\rangle$ is added

$$\Big\langle \log\Big(1 + \frac{2\alpha_i \mathbf{d}_i f_{\mathbf{X}Y_i} + \alpha_i^2 \mathbf{d}_i f_{\mathbf{X}} \mathbf{d}_i^{\mathrm{T}}}{f_{Y_i}}\Big)\Big\rangle$$

where $f_{\mathbf{X}Y_i}$ is the i-th column of $f_{\mathbf{XY}}$. On the other hand, the matrix $\mathbf{B}\mathbf{C}\mathbf{B}^{\mathrm{T}}$ has their elements unchanged except the i-th diagonal element, which is added $\alpha_i^2 \mathbf{d}_i \mathbf{C} \mathbf{d}_i^{\mathrm{T}}$ (since $\mathbf{d}_i \mathbf{C}\mathbf{B}^{\mathrm{T}} = \mathbf{e}_i \mathbf{B}^c \mathbf{C}\mathbf{B}^{\mathrm{T}} = \mathbf{0}$). Thus $\log\det(\mathbf{B}\mathbf{C}\mathbf{B}^{\mathrm{T}})$ is changed by $\log(1+\alpha_i^2 \mathbf{d}_i \mathbf{C}\mathbf{d}_i^{\mathrm{T}}/\sigma_i^2)$ where σ_i^2 denotes the inverse of the i-th diagonal element of $(\mathbf{B}\mathbf{C}\mathbf{B}^{\mathrm{T}})^{-1}$. It follows that the criterion is changed by

$$\Big\langle \log\Big(1 + \frac{2\alpha_i \mathbf{d}_i f_{\mathbf{X}Y_i} + \alpha_i^2 \mathbf{d}_i f_{\mathbf{X}} \mathbf{d}_i^{\mathrm{T}}}{f_{Y_i}}\Big)\Big\rangle - \log\Big(1 + \alpha_i^2 \frac{\mathbf{d}\mathbf{C}\mathbf{d}^{\mathrm{T}}}{\sigma_i^2}\Big)$$

By the Jensen inequality, the above expression is bounded by

$$\log\Big(\frac{1 + 2\alpha_i \mathbf{d}_i \langle f_{\mathbf{X}Y_i}/f_{Y_i}\rangle + \alpha_i^2 \mathbf{d}_i \langle f_{\mathbf{X}}/f_{Y_i}\rangle \mathbf{d}_i^{\mathrm{T}}}{1 + \alpha_i^2 \mathbf{d}_i \mathbf{C} \mathbf{d}_i^{\mathrm{T}}/\sigma_i^2}\Big). \qquad (7)$$

Equality is achieved if the function $(2\alpha_i \mathbf{d}_i f_{\mathbf{X}Y_i} + \alpha_i^2 \mathbf{d}_i f_{\mathbf{X}} \mathbf{d}_i^{\mathrm{T}})/f_{Y_i}$ is constant. Further, if $\alpha_i \mathbf{d}_i$ is small so that this function is small, then (7) is a good approximation to the actual increment of the criterion (corresponding to the increment $\alpha_i \mathbf{d}_i$ of the i-th row of $\mathbf{B}$). Therefore, instead of minimizing the increment of the criterion, which is difficult, we shall minimize (7) instead. We note that if the attained minimum is negative, then the criterion must be decreased.

The derivative of the fraction in (7) with respect to α_i can be found to be the fraction

$$2\frac{g_i + h_i\alpha_i - g_i\alpha_i^2 \mathbf{d}_i \mathbf{C}\mathbf{d}_i^{\mathrm{T}}/\sigma_i^2}{(1+\alpha_i^2 \mathbf{d}_i \mathbf{C}\mathbf{d}_i^{\mathrm{T}}/\sigma_i^2)^2},$$

where we have put

$$g_i = \mathbf{d}_i \Big\langle \frac{f_{\mathbf{X}Y_i}}{f_{Y_i}} \Big\rangle, \qquad h_i = \mathbf{d}_i \Big(\Big\langle \frac{f_{\mathbf{X}}}{f_{Y_i}} \Big\rangle - \frac{\mathbf{C}}{\sigma_i^2} \Big) \mathbf{d}_i^{\mathrm{T}}. \qquad (8)$$

The numerator of the above fraction is a quadratic polynomial in α_i, which admits two real roots:

$$\frac{h_i \pm \sqrt{h_i^2 + 4g_i^2 \mathbf{d}_i \mathbf{C}\mathbf{d}_i^{\mathrm{T}}/\sigma_i^2}}{2g_i \mathbf{d}_i \mathbf{C}\mathbf{d}_i^{\mathrm{T}}/\sigma_i^2}$$

The leading coefficient of this polynomial has the opposite sign as g_i. Thus if $g_i < 0$, this polynomial is positive before the smaller root, negative between the roots and positive again after the larger root. Since (7) has derivative of the same sign as this polynomial, it admits a global minimum at the larger root. If $g_i > 0$, this polynomial is negative before the smaller root, positive between the roots and negative again after the larger root, hence (7) admits a global minimum at the smaller root. In both case, one sees that (7) admits a global minimum at

$$\begin{aligned}\alpha_i^* &= \Big(h_i - \sqrt{h_i^2 + 4g_i^2 \mathbf{d}_i \mathbf{C}\mathbf{d}_i^{\mathrm{T}}/\sigma_i^2}\Big) \Big/ (2g_i \mathbf{d}_i \mathbf{C}\mathbf{d}_i^{\mathrm{T}}/\sigma_i^2) \\ &= -2g_i \Big/ \Big(h_i + \sqrt{h_i^2 + 4g_i^2 \mathbf{d}_i \mathbf{C}\mathbf{d}_i^{\mathrm{T}}/\sigma_i^2}\Big) \end{aligned} \quad (9)$$

In the case $g_i = 0$, (7) reduces to

$$\log\Big(\frac{1+\alpha_i^2\langle f_{\mathbf{X}}/f_{Y_i}\rangle}{1+\alpha_i^2 \mathbf{d}_i \mathbf{C}\mathbf{d}_i^{\mathrm{T}}/\sigma_i^2}\Big) = \log\Big(1 + \frac{h_i\alpha_i^2}{1+\alpha_i^2 \mathbf{d}_i \mathbf{C}\mathbf{d}_i^{\mathrm{T}}/\sigma_i^2}\Big),$$

therefore if $h_i > 0$ it attains its global minimum 0 at 0 and if $h_i < 0$ it attains its global minimum $\log(1 + h_i\sigma_i^2/\mathbf{d}_i\mathbf{C}\mathbf{d}_i^{\mathrm{T}})$ at $\pm\infty$. Thus the minimum point is still α_i^* as given in (9), using the first formula if $h_i < 0$ and the second formula if $h_i > 0$. Finally, in the case both g_i and h_i vanish, (7) is a constant and formulas (9) become indeterminate.

We now consider the choice of $\mathbf{e}_i$. The minimum of (7), that is its value at α_i^*, equals

$$\log\Big(1 + \alpha_i^* \frac{2g_i + h_i\alpha_i^*}{1+\mathbf{d}_i\mathbf{C}\mathbf{d}_i^{\mathrm{T}}/\sigma_i^2}\Big) = \log(1 + g_i\alpha_i^*)$$

since α_i^* satisfies $g_i + h_i\alpha_i^2 - g_i\alpha_i^2 \mathbf{d}\mathbf{C}\mathbf{d}^{\mathrm{T}}/\sigma_i^2 = 0$. Thus the best choice of $\mathbf{e}_i$ would be the one which maximizes $-g_i\alpha_i^*$ where g_i and α_i^* are given in (8) and (9). Unfortunately, it is not possible to obtain a closed form formula for such choice. Therefore, we shall consider two alternative choices

- *The quasi-Newton choice*: Consider the expansion of $C(\mathbf{B} + \mathbf{u}_i\mathbf{e}_i\mathbf{B}^c)$ with respect to the (small) row vector $\mathbf{e}_i$, where $\mathbf{u}_i$ is the column vector with 1 on the i-th row and 0 elsewhere. From the results of section 3, the first and second order terms of this expansion are $2\mathbf{\Gamma}_{i\cdot}\mathbf{e}_i^{\mathrm{T}}$ and $\mathbf{e}_i\mathbf{H}_i\mathbf{e}_i^{\mathrm{T}}$, where $\mathbf{\Gamma}_{i\cdot}$ is the i-th row of $\mathbf{\Gamma} = \frac{1}{2}\dot{C}(\mathbf{B})\mathbf{B}^{c\mathrm{T}}$ and hence equals $\langle f_{Y_i\mathbf{X}}/f_{Y_i}\rangle\mathbf{B}^{c\mathrm{T}}$ and $\mathbf{H}_i = \mathbf{B}^c(\langle f_{\mathbf{X}}/f_{Y_i}\rangle - \mathbf{C}/\sigma_i^2)\mathbf{B}^{c\mathrm{T}}$. Thus minimizing this expansion would yield $\mathbf{e}_i = \mathbf{\Gamma}_{i\cdot}\mathbf{H}_i^{-1}$. Note that for this choice, $g_i = h_i = \mathbf{\Gamma}_{i\cdot}\mathbf{H}_i^{-1}\mathbf{\Gamma}_{i\cdot}^{\mathrm{T}}$. *Such choice however does not guarantees the convergence of the algorithm* (see below).
- *The simple choice*: It consists in approximating $\langle f_{\mathbf{X}}/f_{y_i}\rangle$ by $\langle f_X\rangle\langle f_{Y_i}^{-1}\rangle = \mathbf{C}\langle f_{y_i}^{-1}\rangle$, which results in approximating $\mathbf{H}_i$ by a multiple of $\mathbf{B}^c\mathbf{C}\mathbf{B}^{c\mathrm{T}}$. Since a constant can be absorbed in α_i, this leads to choosing $\mathbf{e}_i = \mathbf{\Gamma}_{i\cdot}(\mathbf{B}^c\mathbf{C}\mathbf{B}^{c\mathrm{T}})^{-1}$. An advantage of this choice is that $\mathbf{d}_i = \langle f_{Y_i\mathbf{X}}^{\mathrm{T}}/f_{Y_i}\rangle\mathbf{B}^{c\mathrm{T}}(\mathbf{B}^c\mathbf{C}\mathbf{B}^{c\mathrm{T}})^{-1}\mathbf{B}^c$ can be computed without computing $\mathbf{B}^c$, since by (6)

$$\mathbf{B}^{c\mathrm{T}}(\mathbf{B}^c\mathbf{C}\mathbf{B}^{c\mathrm{T}})^{-1}\mathbf{B}^c = \mathbf{C}^{-1} - \mathbf{B}^{\mathrm{T}}(\mathbf{B}\mathbf{C}\mathbf{B}^{\mathrm{T}})^{-1}\mathbf{B}.$$

Further, $g_i = \mathbf{d}_i\mathbf{C}\mathbf{d}_i^{\mathrm{T}}$ and $h_i = \mathbf{d}_i\langle f_{\mathbf{X}}/f_{Y_i}\rangle\mathbf{d}_i^T - g_i/\sigma_i^2$.

4.3 Convergence of the algorithm

For the intra-source update, the criterion is decreased unless $h_{ij} = 0$ for all $i \neq j$, which is equivalent to that $\dot{C}(\mathbf{B})\mathbf{B}^{\mathrm{T}} = \mathbf{0}$. For the extra-source update, note that by (9):

$$-g_i\alpha_i^* = \frac{\sqrt{h_i^2 + 4g_i^2\mathbf{d}_i\mathbf{C}\mathbf{d}_i^{\mathrm{T}}/\sigma_i^2} - h_i}{2\mathbf{d}_i\mathbf{C}\mathbf{d}_i^{\mathrm{T}}/\sigma_i^2}$$

hence the criterion is decreased unless $g_i = 0$ and $h_i \geq 0$. However, unlike the intra-source update, $g_i = 0$ may not entail that $\mathbf{\Gamma}_{i\cdot} = \mathbf{0}$. This is the case for the quasi-Newton choice since then $g_i = \mathbf{\Gamma}_{i\cdot}^{\mathrm{T}}\mathbf{H}_i^{-1}\mathbf{\Gamma}_{i\cdot}$ but the matrix $\mathbf{H}_i$ may not be positive definite and thus g_i can be zero without $\mathbf{\Gamma}_{i\cdot}$ being so. For the simple choice however, $g_i = \mathbf{e}_i\mathbf{B}^c\mathbf{C}\mathbf{B}^{c\mathrm{T}}\mathbf{e}_i^{\mathrm{T}}$, thus $g_i = 0$ entails $\mathbf{e}_i = \mathbf{0}$ which entails $\mathbf{\Gamma}_{i\cdot} = \mathbf{e}_i(\mathbf{B}\mathbf{C}\mathbf{B}^{c\mathrm{T}}) = \mathbf{0}$. For this choice, the algorithm will converge to (at least) a local minimum since the criterion is decreased at each step until its gradient vanishes.

4.4 Complexity of the algorithm

We compare the complexity of our algorithm with respect to the one which extract all sources. Note that the later is a particular of the former by taking $p = K$ so that there is no extra-source update. It can be seen that for the full extraction algorithm, the computation cost is dominated by the calculation of the matrices $f_{\mathbf{Y}}(g)$ which is of the order $2K^3|\mathscr{G}|$ flops (floating point operations) where $|\mathscr{G}|$ is the cardinal of $\mathscr{G}$. In the partial extraction algorithm, this number reduces to $(pK + p^2)K|\mathscr{G}|$. For the extra-source update, one needs to compute the $\langle f_{Y_i\mathbf{X}}/f_{Y_i}\rangle$ which requires about $pK^2|\mathscr{G}|$ flops. If one adopts the simple choice, the extra cost to obtain $\mathbf{d}_i$, g_i and h_i is negligible provided that a prewhitenning has been performed to reduce $\mathbf{C}$ to the identity matrix. Thus the partial extraction algorithm is cheaper for small p.

5. A SIMULATION EXAMPLE

As a simple example, we generate 3 source sequences of length 512. They are independent white Gaussian processes (of zero mean and unit variance) modulated with some positive processes, the later being the the exponential of certain autoregressive process of second order. Figure 1 plots the modulation processes, referred to as standard deviation profiles of the sources, and the sources themselves.

We generate further 7 independent sources which are simply Gaussian white noises. The 10 sources are mixed and the problem is to recover the 3 interesting non-stationary sources.

Note that our algorithm is invariant with respect to linear transformation in the sense that applying it to the data $\mathbf{X}(t)$ starting with a separating matrix $\mathbf{B}_0$ is the same as applying it to the data $\mathbf{T}\mathbf{X}(t)$ starting with $\mathbf{B}_0\mathbf{T}^{-1}$, for any invertible matrix $\mathbf{T}$. Thus we may apply our algorithm to the unmixed sources, but starts it at the initial global matrix $\mathbf{G}_0 = \mathbf{B}_0\mathbf{A}$. We adopt the simple choice for the extra-source update. The algorithm will yield a global matrix $\mathbf{G}$ which relates the recovered sources to the original ones. As we have no idea about the mixing matrix, it is natural to chose the initial separating matrix $\mathbf{B}_0$ randomly, which entails that the initial global matrix $\mathbf{G}_0$ is random, although has a different distribution than that of $\mathbf{B}_0$. In this simulation, we simply draw the elements of $\mathbf{G}_0$ as independent standard normal variates.

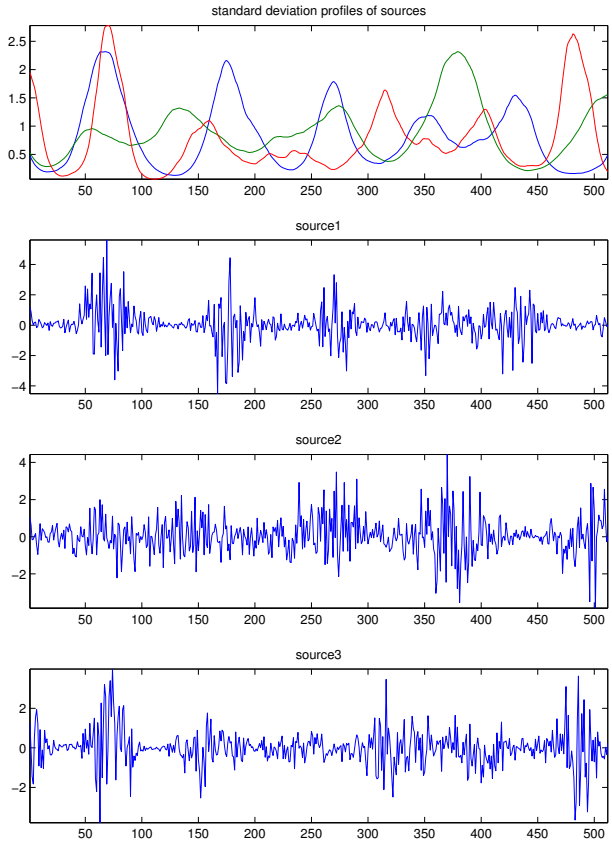

Figure 1: Standard deviation profiles of the 3 sources and these sources themselves

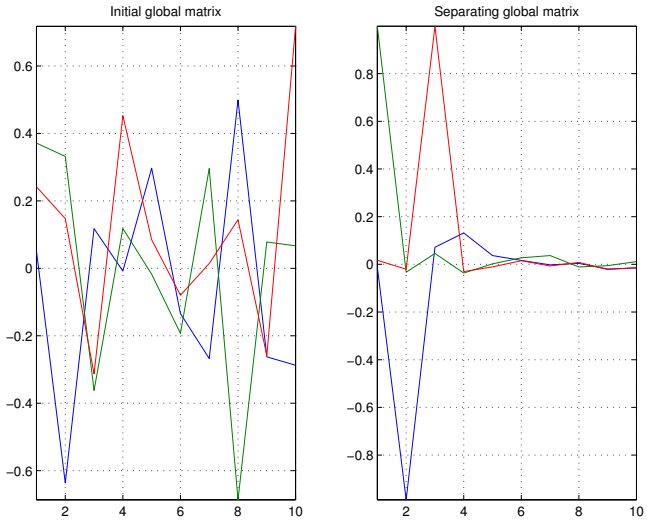

Figure 2: Extracting 3 sources: the rows of the initial global matrix (left panel) and of the final global matrix obtained by our algorithm (right panel, plotted as function of the column index

As the sources processes are uncorrelated, the algorithm will be based only on the local variance of the extracted sources ($M = 1$). The local variance is obtained by taking sample variance (without centering) over moving blocks of length 32. The blocks are overlapping half of its length, yielding a total of 31 blocks.

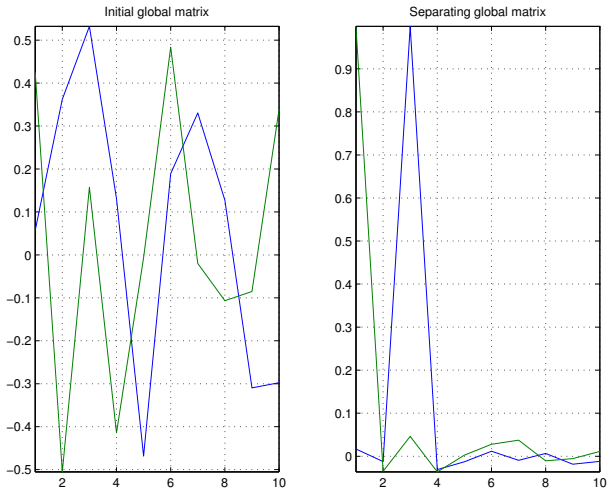

Figure 3: Extracting 2 sources: the rows of the initial global matrix (left panel) and of the final global matrix obtained by our algorithm (right panel, plotted as function of the column index

The results of the algorithm for extracting 3 sources ($p = 3$) are shown in figure 2. The left panel displays the randomly drawn initial global matrix, plotted row by row, the right panel displays the obtained final global matrix. (In the plots, the rows of the global matrices are normalized to have unit norm so that they have comparable magnitude.) The final global matrix shows that the 3 non-stationary sources are extracted correctly: each of its row contains a single large term situated at the 2nd, 1st and 3rd columns (this order however cannot be seen from the graph). The algorithm converges after 25 iterations.

We have also applied our algorithm to extract only 2 sources ($p = 2$). Source 1 and 3 are extracted correctly (see figure 3). They are the more non-stationary sources as can be seen from figure 1. The algorithm converges quicker, after 19 iterations.

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