A NEW REASSIGNED TIME-FREQUENCY REPRESENTATION

Xiumei Li and Guoan Bi

School of EEE, Nanyang Technological University
50 Nanyang Avenue, 639798, Singapore
phone: +(65) 67904823, fax: +(65) 67933318, email: lixiumei@pmail.ntu.edu.sg, egbi@ntu.edu.sg

ABSTRACT

In this paper, the reassignment method is applied to the local polynomial periodogram to improve the readability of the time-frequency representation. Some interesting properties of the reassigned local polynomial periodogram are demonstrated. Simulations are presented to show that the reassigned local polynomial periodogram can improve the readability of the time-frequency representation, compared to the reassigned spectrogram and reassigned smoothed pseudo Wigner-Ville distribution.

1. INTRODUCTION AND REVIEW

Time-frequency representation (TFR) was proposed and widely used [1, 2] in many practical applications such as radar, sonar and communications. In the TFRs, the most popular and simplest one is the short time Fourier transform (STFT), but its resolution for the signal representations in time-frequency domain is limited by the uncertainty principle. Although the bilinear Wigner-Ville distribution (WVD) can provide optimal concentration for the chirp signal, it suffers from the cross-terms which may lead to false identification of the signal components in the TFR. The essential requirement of a good TFR is to obtain a sufficient concentration of the signal components, without significant cross-terms, so that we can interpret the signal properly.

The reassignment method is reported to improve signal concentration in the time-frequency domain, and has been generalized to all bilinear time-frequency and time-scale distributions [3]. The reassignment method of a distribution from the Cohen's class [1]

$$TFR(x;t,\omega) = \frac{1}{2\pi}\int\int \phi_{TF}(u,\Omega)WV(x;t-u,\omega-\Omega)dud\Omega, \tag{1}$$

is given by

$$RTFR(x;t',\omega') = \frac{1}{2\pi}\int\int TFR(x;t,\omega) \cdot\delta(t'-\hat{t}(x;t,\omega))\delta(\omega'-\hat{\omega}(x;t,\omega))dtd\omega, \tag{2}$$

where $WV(x;t,\omega)$ is the WVD of a given signal $x(t)$ and $\phi_{TF}(u,\Omega)$ is the distribution kernel. The reassigned time-frequency representation (RTFR) is used to improve the concentration of the signal component by reallocating its energy distribution in the time-frequency domain. The reassignment method moves the attribution point of the average operation to the gravitational center of the energy contribution $(\hat{t}(x;t,\omega),\hat{\omega}(x;t,\omega))$.The reassignment operators of the reassigned spectrogram (RSP) [3] are expressed in (3) and (4) at the top of the next page, where the subscripts Th and Dh mean that the associated STFTs use the window functions $t\cdot h(t)$ and $\frac{dh(t)}{dt}$, respectively, with

$$STFT_h = \int_{-\infty}^{+\infty} x(t+\tau)h^*(\tau)e^{-j\omega\tau}d\tau, \tag{5}$$

where $h(\tau)$ is the window function.

2. THE REASSIGNED LPP

It shows in [3] that although the reassignment method provides a higher concentration in the time-frequency domain, it cannot remove the cross-term. Therefore, a properly chosen smoothing kernel is desirable for the reassignment method to yield a high concentration of the signal components with suppressed cross-term.

The local polynomial Fourier transform (LPFT) was proposed in [4]. As the generalized form of the STFT, the kernel of the LPFT uses extra parameters to deal with the polynomial phase of the signal, resulting in a much better resolution than the STFT. Therefore the reassignment method based on the square of the modulus of the LPFT, that is the local polynomial periodogram (LPP), turns out to be very efficient for closely located signals which cannot be separated by the spectrogram (SP) and RSP. Furthermore, unlike the WVD, the LPFT is a linear transform and free of the cross-terms.

The LPFT of $x(t)$ is defined as [4]:

$$LPFT(t,\omega,\omega_1\cdots\omega_{M-1}) = \int_{-\infty}^{+\infty} x(t+\tau)h^*(\tau)e^{-j(\omega\tau+\omega_1\tau^2/2+\cdots+\omega_{M-1}\tau^M/M!)}d\tau, \tag{6}$$

where M is the highest order of the polynomial function. By setting $\omega_1 = \omega_2 = ... = \omega_{M-1} = 0$ in (6), the LPFT becomes the STFT as in (5).

Let us consider $M = 2$ for processing chirp signals in the following analysis. With the estimated parameter ω_1 of the polynomial phase obtained by polynomial time frequency transform (PTFT) [6], the LPFT can provide better resolutions than the STFT, with similar computational procedure. More details on the application of the LPFT can be found in [7].

Since the parameter ω_1 can be estimated from $|PTFT|^2$, the LPP is a bilinear TFR and the reassignment method is valid to be applied to the LPP for performance improvement [**?**]. Therefore, the reassigned LPP (RLPP) is defined as

$$RLPP(x;t',\omega') = \int\int LPP(x;t,\omega)\,\delta\left[t'-\hat{t}(x;t,\omega)\right] \cdot\delta\left[\omega'-\hat{\omega}(x;t,\omega)\right]dt\frac{d\omega}{2\pi}. \tag{7}$$

$$
\begin{aligned}
\hat{t}(x;t,\omega) &= t - \frac{\int\int u \cdot WV(h;u,\Omega)\, WV(x;t-u,\omega-\Omega)\, \frac{dud\Omega}{2\pi}}{\int\int WV(h;u,\Omega)\, WV(x;t-u,\omega-\Omega)\, \frac{dud\Omega}{2\pi}} \\
&= t - Re\left\{\frac{STFT_{Th}(x;t,\omega)\cdot STFT_h.^*(x;t,\omega)}{|STFT_h(x;t,\omega)|^2}\right\} \\
&= t - Re\left\{\frac{STFT_{Th}(x;t,\omega)}{STFT_h(x;t,\omega)}\right\} \qquad (3) \\
\hat{\omega}(x;t,\omega) &= \omega - \frac{\int\int \Omega \cdot WV(h;u,\Omega)\, WV(x;t-u,\omega-\Omega)\, \frac{dud\Omega}{2\pi}}{\int\int WV(h;u,\Omega)\, WV(x;t-u,\omega-\Omega)\, \frac{dud\Omega}{2\pi}} \\
&= \omega + Im\left\{\frac{STFT_{Dh}(x;t,\omega)\cdot STFT_h.^*(x;t,\omega)}{|STFT_h(x;t,\omega)|^2}\right\} \\
&= \omega + Im\left\{\frac{STFT_{Dh}(x;t,\omega)}{STFT_h(x;t,\omega)}\right\}. \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
\hat{t}(x;t,\omega) &= t - \frac{\int\int u \cdot WV\left(h;u,-\frac{\omega_1}{2}u+\Omega\right) WV\left(x;t-u,\omega-\frac{\omega_1}{2}u-\Omega\right) \frac{dud\Omega}{2\pi}}{\int\int WV\left(h;u,-\frac{\omega_1}{2}u+\Omega\right) WV\left(x;t-u,\omega-\frac{\omega_1}{2}u-\Omega\right) \frac{dud\Omega}{2\pi}} \\
&= t - Re\left\{\frac{LPFT_{Th}(x;t,\omega)\cdot LPFT_h^*(x;t,\omega)}{|LPFT_h(x;t,\omega)|^2}\right\} \\
&= t - Re\left\{\frac{LPFT_{Th}(x;t,\omega)}{LPFT_h(x;t,\omega)}\right\} \qquad (8) \\
\hat{\omega}(x;t,\omega) &= \omega - \frac{\int\int \left(\Omega-\frac{\omega_1}{2}u\right)\cdot WV\left(h;u,-\frac{\omega_1}{2}u+\Omega\right) WV\left(x;t-u,\omega-\frac{\omega_1}{2}u-\Omega\right) \frac{dud\Omega}{2\pi}}{\int\int WV\left(h;u,-\frac{\omega_1}{2}u+\Omega\right) WV\left(x;t-u,\omega-\frac{\omega_1}{2}u-\Omega\right) \frac{dud\Omega}{2\pi}} \\
&= \omega + Im\left\{\frac{LPFT_{Dh}(x;t,\omega)\cdot LPFT_h^*(x;t,\omega)}{|LPFT_h(x;t,\omega)|^2}\right\} \\
&= \omega + Im\left\{\frac{LPFT_{Dh}(x;t,\omega)}{LPFT_h(x;t,\omega)}\right\} \qquad (9)
\end{aligned}
$$

The expressions of the reassignment operators for the RLPP are given in (8) and (9), and their proofs can be found in [8].

It is noted that when $\omega_1 = 0$ the LPFT becomes the STFT, and the reassignment operators for the RLPP in (8) and (9) become the reassignment operators for the RSP as in (3) and (4).

3. PROPERTIES OF RLPP

Desirable properties of the RLPP are demonstrated in this Section. The details of the derivation for the properties can be referred to [8].

(a)Time and frequency shifts invariance

For a signal $y(t) = x(t-t_0)\exp^{j\omega_0 t}$, We have

$$RLPP(y;t',\omega') = RLPP(x;t'-t_0,\omega'-\omega_0).$$

It means that the shifts in time and/or frequency domain only change the location, but not the content of the *RLPP*.

(b) Time-scaling property

For a signal $y(t) = x(at)$, where a is a non-zero constant, we have

$$RLPP(y;t',\omega') = \frac{1}{|a|}RLPP(x;at',\frac{\omega'}{a}).$$

It means that the time-scaled signal with a constant $|a| > 1$ (or $|a| < 1$) has an *RLPP* that is reduced (increased) in magnitude, squeezed (expanded) in the time direction and expanded (squeezed) in the frequency direction.

(c) Energy conservation

The energy reallocation by the RLPP is consistent with the energy conservation.

$$\int\int RLPP(x;t',\omega')\frac{dt'd\omega'}{2\pi} = \int |x(t)|^2 dt,$$

when $\int\int WV\left(h;u,-\frac{\omega_1}{2}u+\Omega\right)\frac{dud\Omega}{2\pi}=1.$

This property shows that the energy of the reassigned operation of the signal in the time-frequency domain is equal to the energy of the signal in the time domain provided that the window function $h(t)$ is of unit energy.

(d) Perfect localization on chirp and impulse signals

Let us consider a chirp signal $x(t)=A\exp^{j(\omega_0 t+\alpha t^2/2)}$. When the parameter ω_1 is estimated exactly, we have $\hat{\omega}(x;t,\omega)=\omega_0+\alpha t$, which is exactly the instantaneous frequency (IF) of the chirp signal. Thus

$$RLPP(x;t',\omega') = \int\int LPP(x;t,\omega)\delta(\omega'-\omega_0-\alpha t)\delta\left[t'-\hat{t}(x;t,\omega)\right]dt\frac{d\omega}{2\pi}.$$

It shows that the RLPP of a chirp signal is totally concentrated along the instantaneous frequency of the signal, that is $\omega'=\omega_0+\alpha t$.

For an impulse signal $x(t)=A\delta(t-t_0)$, we have

$$RLPP(x;t',\omega')=\delta(t'-t_0) \cdot\int\int LPP(x;t,\omega)\,\delta\left[\omega'-\hat{\omega}(x;t,\omega)\right]dt\frac{d\omega}{2\pi},$$

which demonstrates that the RLPP of an impulse signal is totally concentrated at the time of occurrence, that is $t'=t_0$.

The localization properties show the efficiency of the presented reassigned method since the reassigned version given in (8) and (9) will always perfectly localize the chirp and impulse signals.

There are two simplified variations of the RLPP. One is the reassigned LPP along the frequency direction only which is defined as:

$$RfLPP(x;t,\omega') = \frac{1}{2\pi}\int LPP(x;t,\omega)\delta(\omega'-\hat{\omega}(x;t,\omega))d\omega.$$

The other one is the reassigned LPP along the time direction only which is defined as:

$$RtLPP(x;t',\omega) = \frac{1}{2\pi}\int LPP(x;t,\omega)\delta(t'-\hat{t}(x;t,\omega))dt.$$

The RfLPP and RtLPP share, with the RLPP, the properties of non-bilinearity, time-scaling, and energy conservation. Moreover, the RfLPP particularly has the property of frequency shift invariance and perfectly localizes the chirp components, while the RtLPP has the property of time shift invariance and perfectly localizes the impulse components. It should be noted that the parameter ω_1 affects the performance of the LPP greatly. In the RLPP, $\omega'=\omega_0+\alpha t$ with the condition that the parameter ω_1 is estimated exactly. While in the RfLPP, even when ω_1 is not exactly estimated, we can still obtain $\omega'=\omega_0+\alpha t$ with $t'=t$, which is exactly the IF of the chirp signal. Thus on the theoretical basis, for the chirp signals, the RfLPP is able to accurately represent chirp components than the RLPP with an easy implement and less computation complexity, especially for signals corrupted by the noise. This conclusion is to be confirmed in the simulations presented in the next Section. It should be noted that for the signal contains both chirp and impulse components, the RLPP can give a satisfying result for all components while the RfLPP can only concentrate the chirp components and the RtLPP can only concentrate the impulse components.

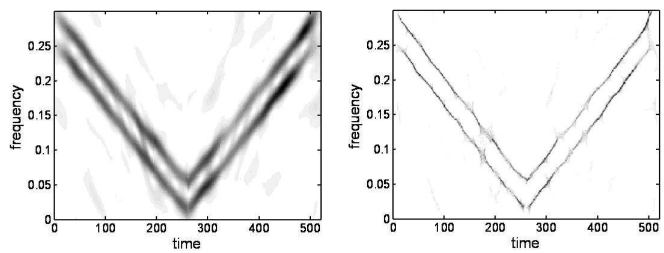
4. SIMULATIONS

This section presents simulation results to compare the signal concentration performances achieved by using different reassigned methods.

Example 1. Let us consider signal $x(t)=x_1(t)+x_2(t)$, where

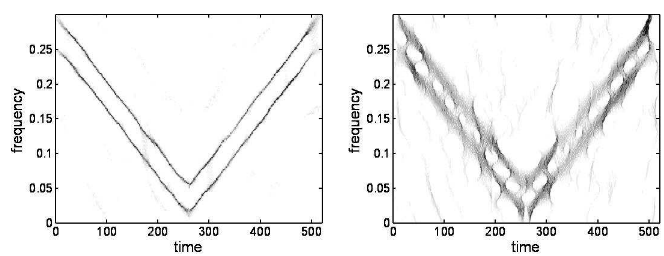
$$\begin{aligned} x_1(t) &= \exp\left[j2\pi(-0.00049t^2+0.3t)\right] \\ &\quad +\exp\left[j2\pi(-0.00049t^2+0.26t)\right], \\ &\quad 0\le t\le 256, \end{aligned}$$

$$\begin{aligned} x_2(t) &= \exp\left[j2\pi(0.00049t^2-0.2t)\right] \\ &\quad +\exp\left[j2\pi(0.00049t^2-0.24t)\right], \\ &\quad 256\le t\le 512, \end{aligned}$$



(a) LPP (b) RLPP

Figure 1: The LPP and RLPP of a two-chirp signal.

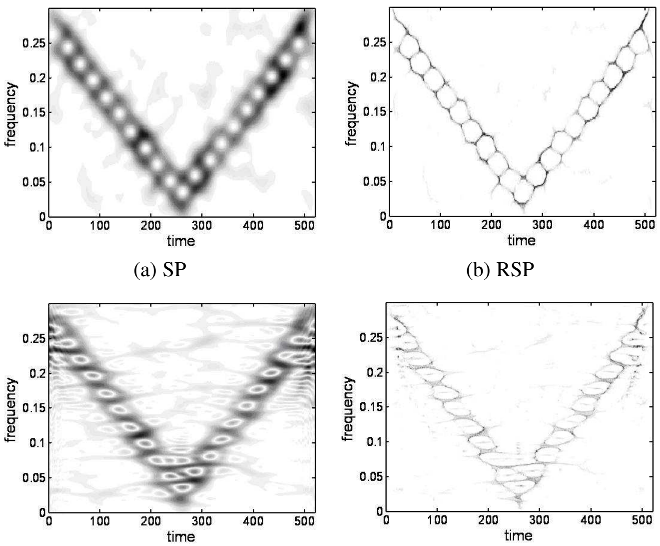


(a) RfLPP (b) RtLPP

Figure 2: Localizing chirp signals with RfLPP and RtLPP.

Figure 1 shows the LPP and RLPP for this signal with the SNR=3dB. It is observed that both the LPP and RLPP can localize the two chirp components. However, the RLPP makes a significant improvement on signal concentration.

Figure 2 presents the RfLPP and RtLPP for the same signal in Figure 1. In Figure 2 it is seen that the RfLPP can also perfectly localize the chirp signals, with even a better concentration than the RLPP, while the RtLPP becomes blurred. Similarly the RtLPP can perfectly localize the impulse signals. It means that both the RLPP and RfLPP provide an excellent property of localizing the chirp signals, and both the RLPP and the RtLPP can perfectly localize the impulse signals. For signals containing both the chirp and impulse components, the RLPP is a better choice to achieve the improved performance.



(a) SP (b) RSP (c) SPWVD (d) RSPWVD

Figure 3: Signal representations obtained from SP, RSP, SPWVD, RSPWVD.

Figure 3 shows, respectively, signal representations obtained by using the SP, RSP, SPWVD and RSPWVD for the signal in Figure 1. It clearly demonstrates that the SP and RSP cannot separate two closely located chirp components, and the SPWVD and RSPWVD suffer from the existence of cross-term interferences. The RLPP in Figure 1 and the RfLPP in Figure 2 can achieve a much better time-frequency representation than those in Figure 3 with more concentrated signal contents and without any cross terms. Furthermore, the RfLPP uses much less computation time than the RLPP because the reassignment is performed along the frequency direction only.

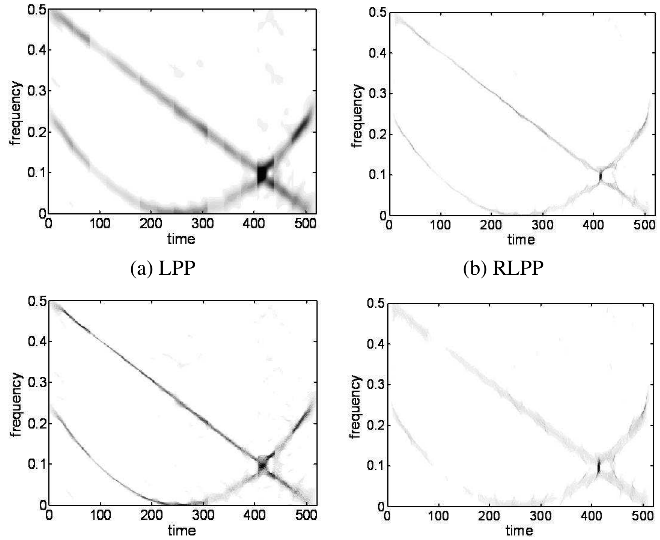
Example 2. The RLPP and RfLPP can also be extended for signal with multiple components that may contain signal other than the chirp signal. As an example, let us consider a signal containing one linear chirp component and one parabolic frequency modulated component defined as

$$\begin{aligned} x(t) &= \exp\left[j2\pi(-0.00048t^2+0.5t)\right] \\ &+ \exp\left[j2\pi(0.0000012t^3+0.00072t^2+0.25t)\right] \end{aligned}$$

with the SNR of 5 dB. The LPP, RLPP, RfLPP and RtLPP of the signal are presented in Figure 4. It is seen that the RLPP and RfLPP can still achieve a much higher concentration than the LPP. More example on multi-component signals, with its concentration measured, can be referred to [5].

5. CONCLUSION

This paper presents a new reassignment method based on the LPP, that is the RLPP, and its properties. For the chirp signals, simulation results show that the RLPP can achieve much better performance in terms of signal concentration than the RSP and RSPWVD. Moreover the RfLPP, with reduced computation complexity, can get a even better result than the RLPP for the chirp signals.



(a) LPP (b) RLPP (c) RfLPP (d) RtLPP

Figure 4: The LPP, RLPP, RfLPP and RtLPP of a signal with multiple components.

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