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GENERATION OF DUAL N-CHANNEL FILTERBANKS WITH HILBERT-TRANSFORMED MOTHER WAVELETS

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ABSTRACT

We reconsider the problem of generating a dual filterbank to a given one with N channels such that its $N-1$ mother wavelets will be related to those of the original one by the Hilbert-transform. Known solutions treat the case of unitary filterbanks [1, 2], while [3, 4] treats the biorthogonal case and [5] the overcomplete case. We present a solution which is valid for arbitrary filterbanks and obtain identical results. This fact indicates that the proposition of unitarity is not necessary at all. Moreover, the phase condition is obtained in a constructive way and is given by a closed form expression.

1. INTRODUCTION

1.1 Notation

We consider a set of N wavelets $u_\nu(t)$ given by the following equations

$$u_\nu(t) = N \cdot \sum_k h_\nu(k) \cdot u_0(Nt - kT), \quad \nu = 0...N-1, \tag{1.1}$$

with $N \in \mathbb{N}$, $N \geq 2$, being the scaling factor. For $\nu = 0$ we obtain the scaling function and for $\nu = 1...N-1$ the respective mother wavelets. The sets of coefficients $\{h_\nu(k), k \in \mathbb{Z}\}$ can be finite or infinite according to the desired application.

For the following investigations we assume, that the sets of coefficients $\{h_\nu(k)\}$ are known and that their discrete time Fourier transforms exist, i.e.

$$H_{*,\nu}(\omega) = \sum_k h_\nu(k) \cdot e^{-jk\omega T}, \ \omega \in \mathbb{R}, \nu = 0...N-1. \tag{1.2a}$$

The subscript "$*$" indicates a periodic function of the frequency ω. In the Fourier domain we can write equation (1.1) as

$$U_\nu(\omega) = H_{*,\nu}\left(\frac{\omega}{N}\right) \cdot U_0\left(\frac{\omega}{N}\right), \quad \nu = 0...N-1 \tag{1.2b}$$

or equivalently after rescaling

$$U_\nu(N\omega) = H_{*,\nu}(\omega) \cdot U_0(\omega), \quad \nu = 0...N-1. \tag{1.2c}$$

The mother wavelet spectra $U_\nu(\omega)$ result from spectral forming of the scaling function spectrum $U_0\left(\frac{\omega}{N}\right)$ with the periodic spectrum $H_{*,\nu}\left(\frac{\omega}{N}\right)$.
We explicitly note that all $H_{*,\nu}(\omega)$ are periodic with $\omega_a = \frac{2\pi}{T}$. Therefore we can write

$$H_{*,\nu}(\omega + \lambda\omega_a) = H_{*,\nu}(\omega), \quad \lambda \in \mathbb{Z}. \tag{1.2d}$$

By iterating (1.2b) we get

$$U_\nu(\omega) = U_0(0) \cdot H_{*,\nu}\left(\frac{\omega}{N}\right) \cdot H_{\infty,0}\left(\frac{\omega}{N}\right) \tag{1.3a}$$

with the non-periodic function

$$H_{\infty,0}(\omega) = \prod_{i=1}^{\infty} H_{*,0}\left(\omega \cdot N^{-i}\right). \tag{1.3b}$$

In particular the spectrum of the scaling function is

$$U_0(\omega) = U_0(0) \cdot H_{\infty,0}(\omega). \tag{1.3c}$$

All statements so far hold for arbitrary sets of wavelets as long as the functions $H_{*,\nu}(\omega)$ in equation (1.2a) exist.

1.2 The Problem

In addition to the N wavelets $u_\nu(t)$ we are now introducing a second filter bank leading to a dual set of wavelets $\{v_0(t), v_1(t), \ldots, v_{N-1}(t)\}$. For this we assume a similar set of equations as proposed in (1.1) - (1.3c).

$$v_\nu(t) = N \cdot \sum_k g_\nu(k) \cdot v_0(Nt - kT), \quad \nu = 0...N-1. \tag{1.4a}$$

As before $v_0(t)$ is a scaling function and $v_\nu(t)$ are mother wavelets for $\nu = 1...N-1$. The spectra of these wavelets are

$$V_\nu(\omega) = G_{*,\nu}\left(\frac{\omega}{N}\right) \cdot V_0\left(\frac{\omega}{N}\right) \tag{1.4b}$$

$$V_\nu(N\omega) = G_{*,\nu}(\omega) \cdot V_0(\omega) \tag{1.4c}$$

with the periodic functions $G_{*,\nu}(\omega)$,

$$G_{*,\nu}(\omega) = \sum_k g_\nu(k) \cdot e^{-jk\omega T} \tag{1.4d}$$

$$G_{*,\nu}(\omega) = G_{*,\nu}(\omega + \lambda\omega_a), \ \lambda \in \mathbb{Z}. \tag{1.4e}$$

Iterating equation (1.4b) the spectra of the wavelets become

$$V_\nu(\omega) = V_0(0) \cdot G_{*,\nu}\left(\frac{\omega}{N}\right) \cdot G_{\infty,0}\left(\frac{\omega}{N}\right) \tag{1.4f}$$

with the non-periodic function $G_{\infty,0}(\omega)$

$$G_{\infty,0}(\omega) = \prod_{i=1}^{\infty} G_{*,0}\left(\omega \cdot N^{-i}\right). \tag{1.4g}$$

These two filterbanks have to be interlinked in a way that the mother wavelets of the first filterbank $u_1, \ldots, u_{N-1}$ are Hilbert transforms to the mother wavelets $v_1, \ldots, v_{N-1}$ of the second:

$$v_\nu(t) = \mathfrak{H}\{u_\nu(t)\} = \frac{1}{\pi} \cdot \int u_\nu(t') \cdot \frac{1}{t-t'}\, dt', \quad \nu = 1...N-1. \tag{1.5a}$$

In the Fourier domain this convolution becomes

$$V_\nu(\omega) = H_H(\omega) \cdot U_\nu(\omega) \tag{1.5b}$$

with the Hilbert transformer

$$H_H(\omega) = -j \cdot \text{sign}(\omega) = e^{-j\beta_H(\omega)}, \tag{1.5c}$$

and the phase

$$\beta_H(\omega) = \frac{\pi}{2} \cdot \text{sign}(\omega). \tag{1.5d}$$

For later purposes we remark that the Hilbert transformer is practically scale invariant. With the similarity theorem of the Fourier transform,

$$\mathscr{F}^{-1}\{F(a\omega)\} = \frac{1}{|a|} \cdot f\left(\frac{t}{a}\right), \quad a \neq 0, \tag{1.5e}$$

we get

$$\mathscr{F}^{-1}\{H_H(a\omega)\} = \text{sign}(a) \cdot h_H(t). \tag{1.5f}$$

The problem is to propose a relation between $V_0(\omega)$ and $U_0(\omega)$ so that the equations (1.5a) and (1.5b) are met.

I. Selesnick gives a solution to this problem for $N = 2$ in [1], and for $N > 2$ a solution is given by C. Chaux et alea in [2]. Both approaches use the fact, that the filterbanks to be coordinated are unitary. It will turn out that this proposition can be dropped and identical results hold for arbitrary, not necessarily unitary filterbanks as well. The second point is that we prove the phase condition directly in a constructive way and obtain closed form expressions.

2. RELATIONS BETWEEN THE WAVELET SYSTEMS

2.1 Wavelet Spectra

From equation (1.5b) and (1.5c) we immediately get the spectra of the mother wavelets as

$$V_\nu(\omega) = e^{-j\beta_H(\omega)} \cdot U_\nu(\omega), \quad \nu = 1...N-1, \tag{2.1a}$$

and hence

$$|V_\nu(\omega)| = |U_\nu(\omega)|, \tag{2.1b}$$

respectively.

According to their construction the spectra of the mother wavelets differ only in the additional phase of the Hilbert transformer. From that we get an important fact for all mother wavelets:

> If the functions $u_1, \ldots, u_{N-1}$ have compact support this no longer holds for $v_1, \ldots, v_{N-1}$.

For later purposes we assume the normalization

$$U_0(0) = V_0(0). \tag{2.2}$$

2.2 Interlinking the scaling function

For reasons of presentation we do not assume a relation between $U_0(\omega)$ and $V_0(\omega)$, but choose a more indirect relation of the periodic spectra

$$G_{*,0}(\omega) = H_{*,0}(\omega) \cdot B_{*,0}(\omega) \tag{2.3a}$$

with $B_{*,0}(\omega)$ also having to be periodic with ω_a. $B_{*,0}(\omega)$ is a discrete all pass function and can be written as

$$B_{*,0}(\omega) = e^{-j\beta_{*,0}(\omega)}. \tag{2.3b}$$

Apparently $\beta_{*,0}(\omega)$ is periodic with ω_a,

$$\beta_{*,0}(\omega + \lambda\omega_a) = \beta_{*,0}(\omega), \quad \lambda \in \mathbb{Z}. \tag{2.3c}$$

This fact will be very important later.

If $H_{*,0}(\omega)$ and $G_{*,0}(\omega)$ satisfy equations (2.3a) and (2.3b), we can deduce a relation between $V_0(\omega)$ and $U_0(\omega)$ for any given phase $\beta_{*,0}(\omega)$ of the discrete allpass $B_{*,0}(\omega)$ in (2.3a). To do so we take a look at equations (1.3c) and (1.4f):

$$\begin{aligned} U_0(\omega) &= U_0(0) \cdot H_{\infty,0}(\omega), &(2.4a)\\ V_0(\omega) &= V_0(0) \cdot G_{\infty,0}(\omega). &(2.4b)\end{aligned}$$

With equations (2.2) and (2.3b) $G_{\infty,0}(\omega)$ in (2.4b) can be rewritten as

$$G_{\infty,0}(\omega) = H_{\infty,0}(\omega) \cdot e^{-j\beta_{\infty,0}(\omega)} \tag{2.5a}$$

with

$$\beta_{\infty,0}(\omega) = \sum_{i=1}^{\infty} \beta_{*,0}\left(\omega \cdot N^{-i}\right). \tag{2.5b}$$

Apparently $\beta_{\infty,0}(\omega)$ is not a periodic function. It exists if $\beta_{*,0}(\omega)$ is an odd function and is Lipschitz continuous in a vicinity of zero. In this case $\lim_{\omega\to 0}\beta_{*,0}(\omega) = 0$ and the convergence in (2.5b) is geometric.

Inserting (2.5a) and (2.5b) into (2.4b) we get with (2.4a)

$$V_0(\omega) = B_{\infty,0}(\omega) \cdot U_0(\omega), \tag{2.6a}$$

whereas

$$B_{\infty,0}(\omega) = e^{-j\beta_{\infty,0}(\omega)} \tag{2.6b}$$

generally is a non-periodic all pass function.

We emphasize, that we did not determine the periodic phase $\beta_{*,0}(\omega)$ in (2.3b). Therefore the continuous phase $\beta_{\infty,0}(\omega)$ is also not determined. In choosing anything for $\beta_{*,0}(\omega)$ we also get a unique expression for $\beta_{\infty,0}(\omega)$.

2.3 Interlinking the band pass filters

In (2.1b) and (1.5b) the relation between the mother wavelets is given by the Hilbert transform. Using the equations (2.3) for the low pass filters, relations for the band pass filters can also be obtained. We take the equations (1.3a) and (2.1a) and rewrite them as

$$\begin{aligned} U_\nu(N\omega) &= U_0(0) \cdot H_{*,\nu}(\omega) \cdot H_{\infty,0}(\omega) &(2.7a)\\ V_\nu(N\omega) &= V_0(0) \cdot G_{*,\nu}(\omega) \cdot G_{\infty,0}(\omega). &(2.7b)\end{aligned}$$

With the relations (2.5a) and (2.5b) we can write (2.7b) as

$$V_\nu(N\omega) = U_0(0) \cdot G_{*,\nu}(\omega) \cdot H_{\infty,0}(\omega) \cdot e^{-j\beta_{\infty,0}(\omega)}.$$

Writing down equation (1.5b) in this form

$$V_\nu(N\omega) = H_H(N\omega) \cdot U_\nu(N\omega), \tag{2.8a}$$

and using the relation (see equation (1.5f))

$$H_H(N\omega) = H_H(\omega) \tag{2.8b}$$

we get

$$H_{*,\nu}(\omega) \cdot H_H(\omega) \cdot U_\nu(N\omega) = G_{*,\nu}(\omega) \cdot U_\nu(N\omega) \cdot e^{-j\beta_{\infty,0}(\omega)}$$

and with that

$$H_{*,\nu}(\omega) \cdot H_H(\omega) = G_{*,\nu}(\omega) \cdot e^{-j\beta_{\infty,0}(\omega)}.$$

Finally we get

$$G_{*,\nu}(\omega) = e^{-jp_{*,\nu}(\omega)} \cdot H_{*,\nu}(\omega) \tag{2.8c}$$

with the necessarily periodic phase function $p_{*,\nu}(\omega)$

$$\left.\begin{aligned} p_{*,\nu}(\omega) &= \beta_H(\omega) - \beta_{\infty,0}(\omega)\\ &= p_{*,\nu}(\omega + \lambda\omega_a),\ \lambda \in \mathbb{Z}. \end{aligned}\right\} \tag{2.8d}$$

We remark that we do not care about possible jumps of the phase by multiples of 2π! If desired this can be done straightforwardly!

These phase functions are identical for the indices $\nu = 1...N-1$. Therefore we can write instead

$$\begin{aligned} p_*(\omega) &= \beta_H(\omega) - \beta_{\infty,0}(\omega) \\ &= p_*(\omega + \lambda\omega_a),\ \lambda \in \mathbb{Z}. \end{aligned} \tag{2.8e}$$

The task now is to determine the interlinking phase $\beta_{*,0}(\omega)$ in equation (2.3b) so that $p_*(\omega)$ in (2.8e) becomes a periodic function. This we will do in the next section using the results of [1] and [2].

3. SOLUTIONS FOR THE INTERLINKING PHASE

3.1 Solution for $N = 2$ according to I. Selesnick

According to I. Selesnick the interlinking phase $\beta_{*,0}(\omega)$ is given by

$$\beta_{*,0}(\omega) = \begin{cases} \dfrac{\omega T}{2}, & 0 \le \omega < \dfrac{\pi}{T} \\ \dfrac{\omega T}{2} - \pi, & \dfrac{\pi}{T} < \omega \le \dfrac{2\pi}{T} \end{cases} \tag{3.1a}$$

with

$$\beta_{*,0}(\omega + \lambda\omega_a) = \beta_{*,0}(\omega). \tag{3.1b}$$

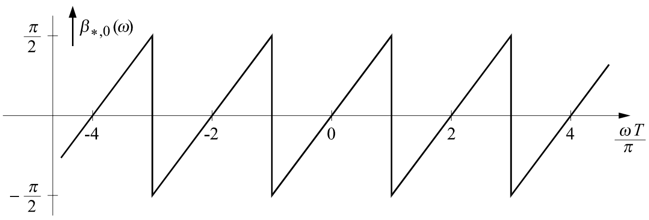

Figure 3.1: Interlinking phase $\beta_{*,0}(\omega)$ for $N = 2$

First of all we have to calculate the sum in equation (2.5b) giving the phase $\beta_{\infty,0}(\omega)$. To calculate it straightforward seems difficult, so we represent $\beta_{*,0}(\omega)$ by

$$\beta_{*,0}(\omega) = \lambda(\omega) - \sigma(\omega) \tag{3.2a}$$

with the non-periodic linear function

$$\lambda(\omega) = \frac{\omega T}{2}, \quad \omega \in \mathbb{R}, \tag{3.2b}$$

and the step function $\sigma(\omega)$, also non-periodic, defined by

$$\sigma(\omega) = \pi \cdot \sum_{m=0}^{\infty} \delta_{-1}(\omega - \omega_m), \quad \omega \ge 0, \tag{3.2c}$$

with

$$\omega_m = (2m+1) \cdot \frac{\omega_a}{2}, \quad m \in \mathbb{Z}, \tag{3.2d}$$

and

$$\sigma(-\omega) = -\sigma(\omega), \quad \omega \in \mathbb{R}. \tag{3.2e}$$

$\delta_{-1}(t)$ denotes the step function according to

$$\delta_{-1}(t) = \begin{cases} 1, & t \ge 0 \\ 0, & t < 0 \end{cases} \tag{3.2f}$$

Hence the phase $\beta_{\infty,0}(\omega)$ can be written as

$$\beta_{\infty,0}(\omega) = \ell(\omega) - s(\omega), \tag{3.3a}$$

with

$$\ell(\omega) = \sum_{i=1}^{\infty} \lambda\left(\omega \cdot 2^{-i}\right) = \frac{\omega T}{2} \tag{3.3b}$$

and

$$s(\omega) = \sum_{i=1}^{\infty} \sigma\left(\omega \cdot 2^{-i}\right). \tag{3.3c}$$

Obviously $s(\omega)$ is also an odd function

$$s(-\omega) = -s(\omega), \quad \omega \in \mathbb{R}, \tag{3.4a}$$

satisfying the functional equation

$$s(\omega) - s\left(\frac{\omega}{2}\right) = \sigma\left(\frac{\omega}{2}\right) \tag{3.4b}$$

Direct analysis of this relation for $\omega \ge 0$ leads to the assumption

$$\begin{aligned} s(\omega) &= \pi \sum_{m=1}^{\infty} \delta_{-1}(\omega - m \cdot \omega_a), \quad \omega \ge 0, \\ s(-\omega) &= -s(\omega), \qquad\qquad\qquad\qquad \omega \in \mathbb{R}. \end{aligned} \tag{3.4c}$$

Inserting the soltuion into the functional equation (3.4b) verifies its correctness.

The phase $\beta_{\infty,0}(\omega)$ now is given by

$$\beta_{\infty,0}(\omega) = \frac{\omega T}{2} - s(\omega), \quad \omega \ge 0, \tag{3.5a}$$

$$= \frac{\omega T}{2} - m\pi, \tag{3.5b}$$

for

$$\omega \in [m\omega_a, (m+1)\omega_a[,\ m \in \mathbb{N}_0 \tag{3.5c}$$

and the symmetry condition

$$\beta_{\infty,0}(-\omega) = -\beta_{\infty,0}(\omega), \quad \omega \in \mathbb{R}. \tag{3.5d}$$

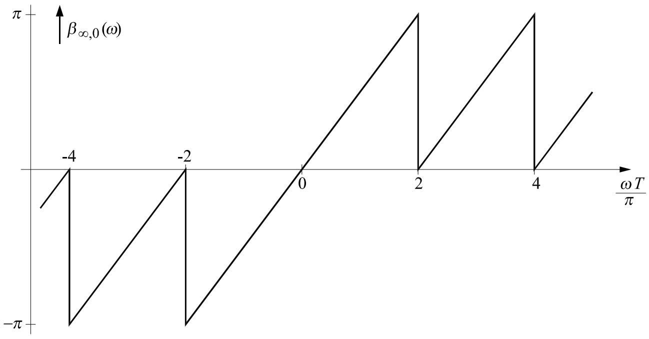

Figure 3.2: Sum phase $\beta_{\infty,0}(\omega)$ for $N = 2$

The difference phase $p_*(\omega)$ in (2.8e) now becomes

$$\begin{aligned} p_*(\omega) &= \beta_H(\omega) - \beta_{\infty,0}(\omega) \\ &= \frac{\pi}{2} \cdot \mathrm{sign}(\omega) - \frac{\omega T}{2} + m \cdot \pi, \end{aligned} \tag{3.6a}$$

with ω as in (3.5c). For $\omega \ge 0$ we can also write

$$p_*(\omega) = (2m+1)\frac{\pi}{2} - \frac{\omega T}{2} \tag{3.6b}$$

again with ω as in (3.5c).
This function $p_*(\omega)$ is periodic with ω_a and therefore

$$\left.\begin{aligned} p_*(\omega) &= \frac{\pi}{2} - \frac{\omega T}{2},\ 0 \le \omega < \omega_a \\ p_*(\omega + \lambda \cdot \omega_a) &= p_*(\omega), \quad \omega \in \mathbb{R}, \lambda \in \mathbb{Z}. \end{aligned}\right\} \tag{3.6c}$$

We notice that the solution we presented for the difference phase $p_*(\omega)$ is the very same I. Selesnick presented in [1]. The important difference is that we get the solution without postulating orthogonality!

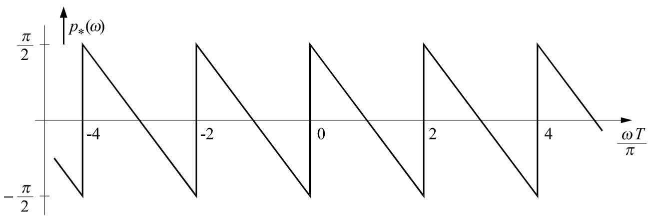

Figure 3.3: Difference phase $p_*(\omega)$ for $N=2$

3.2 Solution for $N \geq 2$

In this section we derive the result for the interlinking phase $\beta_{*,0}(\omega)$ for the case $N \geq 2$.

Defining the frequencies $\Delta_{N,m}$ or Δ_m for short as equidistant partitions of a period

$$\Delta_m = \Delta_{N,m} = m \cdot \frac{\omega_a}{N}, \quad N \in \mathbb{N}, m \in \mathbb{Z}, \tag{3.7a}$$

the interlinking phase can be written as

$$\beta_{*,0}(\omega) = \lambda(\omega) - m \cdot \pi, \tag{3.7b}$$

with

$$\Delta_m \leq \omega \leq \Delta_{m+1},\ m = 0 \ldots N-1, \tag{3.7c}$$

$$\lambda(\omega) = \frac{N-1}{2} \cdot \omega T, \quad \omega \in \mathbb{R}, \tag{3.7d}$$

$$\beta_{*,0}(\omega + \mu\omega_a) = \beta_{*,0}(\omega), \quad \omega \in \mathbb{R},\ \mu \in \mathbb{Z}. \tag{3.7e}$$

Apparently an interlinking phase of this kind is odd and continuous at zero:

$$\beta_{*,0}(0) = 0 \tag{3.7f}$$

$$\beta_{*,0}(-\omega) = -\beta_{*,0}(\omega). \tag{3.7g}$$

For $m = 1 \ldots N-1$ there are steps at Δ_m with a height of $-\pi$:

$$\lim_{\delta \to 0+} \left\{ \beta_{*,0}(\Delta_m + \delta) - \beta_{*,0}(\Delta_m - \delta) \right\} = -\pi, \tag{3.8a}$$

$$\beta_{*,0}(m \cdot \omega_a) = 0, \quad m \in \mathbb{Z}. \tag{3.8b}$$

The limits on the left and right side of these steps are

$$\begin{aligned} \beta_{*,0}(\Delta_m-) &= \frac{N-m}{N} \cdot \pi, \\ \beta_{*,0}(\Delta_m+) &= -\frac{m}{N} \cdot \pi, \quad m = 1 \ldots N-1. \end{aligned} \tag{3.8c}$$

Figure 3.4 shows the interlinking phase $\beta_{*,0}(\omega)$ for $N = 2$, 3 and 4.

Similar to the equationts (3.2) for $N=2$ we can also separate the linear terms for the cases $N \geq 2$:

$$\beta_{*,0}(\omega) = \lambda(\omega) - \sigma_1(\omega) + \sigma_N(\omega), \tag{3.9a}$$

$$\left.\begin{aligned} \sigma_1(\omega) &= \pi \cdot \sum_{\mu=1}^{\infty} \sigma_{-1}(\omega - \Delta_\mu), \quad \omega \geq 0 \\ \sigma_1(-\omega) &= -\sigma_1(\omega), \qquad \omega \in \mathbb{R}, \end{aligned}\right\} \tag{3.9b}$$

$$\left.\begin{aligned} \sigma_N(\omega) &= \pi \cdot \sum_{\mu=1}^{\infty} \sigma_{-1}(\omega - \mu\omega_a), \ \omega \geq 0 \\ \sigma_N(-\omega) &= -\sigma_N(\omega), \qquad \omega \in \mathbb{R}. \end{aligned}\right\} \tag{3.9c}$$

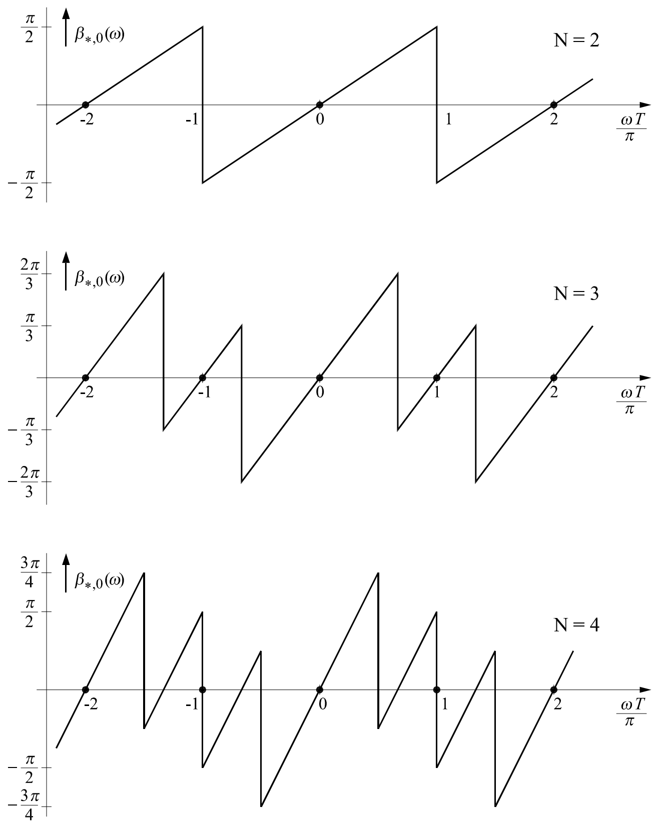

Figure 3.4: Interlinking phase $\beta_{*,0}(\omega)$ for $N = 2,3,4$

This piecewise linear function is periodic and attains the values postulated in (3.8b) and (3.8c)!
For further invertigations we also note the useful relation

$$\sigma_N(N\omega) = \sigma_1(\omega). \tag{3.9d}$$

Figure 3.5 shows the construction of the periodic phase $\beta_{*,0}(\omega)$ with non-periodic components for $N=3$.

In the next step we write the sum phase $\beta_{\infty,0}(\omega)$ as

$$\begin{aligned} \beta_{\infty,0}(\omega) &= \sum_{i=1}^{\infty} \beta_{*,0}\left(\omega \cdot N^{-i}\right) \\ &= \ell(\omega) - s_1(\omega) + s_N(\omega) \end{aligned} \tag{3.10a}$$

with

$$\ell(\omega) = \sum_{i=1}^{\infty} \lambda\left(\omega \cdot N^{-i}\right) = \frac{\omega T}{2}, \quad \omega \in \mathbb{R}, \tag{3.10b}$$

and

$$s_1(\omega) = \sum_{i=1}^{\infty} \sigma_1\left(\omega \cdot N^{-i}\right), \quad \omega \in \mathbb{R}, \tag{3.10c}$$

$$s_N(\omega) = \sum_{i=1}^{\infty} \sigma_N\left(\omega \cdot N^{-i}\right), \quad \omega \in \mathbb{R}. \tag{3.10d}$$

With (3.9d) we note the relation between these two by

$$s_N(N\omega) = s_1(\omega). \tag{3.10e}$$

Moreover the following two equations always hold

$$s_N(N\omega) = s_N(\omega) + \sigma_N(\omega) \tag{3.10f}$$

$$s_1(N\omega) = s_1(\omega) + \sigma_1(\omega). \tag{3.10g}$$

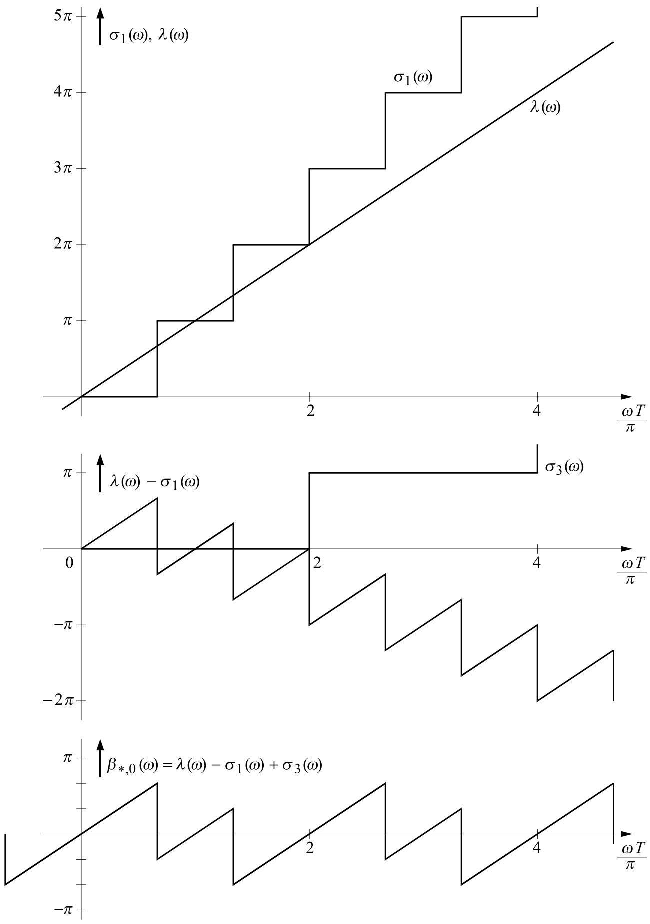

Figure 3.5: Construction of $\beta_{*,0}(\omega)$ according to (3.9) for $N = 3$

Combining (3.10e) and (3.10f) we get

$$s_1(\omega) - s_N(\omega) = \sigma_N(\omega). \tag{3.10h}$$

With that we can already explicitly write the sum phase $\beta_{\infty,0}(\omega)$ in (3.10a) as

$$\left.\begin{array}{rcll} \beta_{\infty,0}(\omega) & = & \ell(\omega) - \sigma_N(\omega) & \\ & = & \dfrac{\omega T}{2} - \sigma_N(\omega)\,, & \omega \geq 0, \\ \beta_{\infty,0}(-\omega) & = & -\beta_{\infty,0}(\omega)\,, & \omega \in \mathbb{R}. \end{array}\right\} \tag{3.11a}$$

Analyzing $\sigma_N(\omega)$ we get the relations

$$\beta_{\infty,0}(\omega) = \frac{\omega T}{2} - m \cdot \pi\,, \tag{3.11b}$$
$$m \cdot \omega_a < \omega < (m+1) \cdot \omega_a\,,\; m \in \mathbb{N}_0$$

$$\beta_{\infty,0}(\omega) = -\beta_{\infty,0}(\omega)\,, \quad \omega \in \mathbb{R}. \tag{3.11c}$$

Apparently the phase $\beta_{\infty,0}(\omega)$ is independent of N! It is identical to the phase for $N = 2$ in figure 3.2.

The phase $p_*(\omega)$ in (2.8e)

$$p_*(\omega) = \beta_H(\omega) - \beta_{\infty,0}(\omega) \tag{3.12a}$$

is also independent of N.

$$\left.\begin{array}{r} p_*(\omega) = \dfrac{\pi}{2} - \dfrac{\omega T}{2}\,,\; 0 < \omega < \omega_a, \\ p_*(\omega + \mu\omega_a) = p_*(\omega)\,, \quad \omega \in \mathbb{R},\; \mu \in \mathbb{Z} \end{array}\right\} \tag{3.12b}$$

At the discontinuities we get:

$$\begin{array}{rcl} p_*(\mu\omega_a-) & = & -\dfrac{\pi}{2}\,, \\ p_*(\mu\omega_a+) & = & \dfrac{\pi}{2}\,, \quad \mu \in \mathbb{Z}. \end{array} \tag{3.12c}$$

4. CONCLUSIONS

In this paper we have shown that it is possible to construct dual N-channel filterbank pairs with Hilbert transformed mother wavelets without postulating unitarity. Beyond that we have given a closed form to construct the interlinking phase by means of simple non-periodic functions.

REFERENCES

[1] I. W. Selesnick, “Hilbert transform pairs of wavelet bases,” *IEEE Signal Processing Letters*, vol. 8, no. 6, pp. 170–173, June 2001.

[2] C. Chaux, L. Duval, and J.-C. Pesquet, “Image analysis using a dual-tree M-band wavelet transform,” *IEEE Transactions on Image Processing*, vol. 15, no. 8, pp. 2397–2412, Aug. 2006.

[3] C. Chaux, J.-C. Pesquet, and L. Duval, “2D Dual-Tree Complex Biorthogonal M-Band Wavelet Transform,” *IEEE International Conference on Acoustics, Speech and Signal Processing, 2007. ICASSP 2007.*, vol. 3, pp. III–845–III–848, 15-20 April 2007.

[4] R. Yu and H. Ozkaramanli, “Hilbert transform pairs of biorthogonal wavelet bases,” *IEEE Trans. Signal Processing*, vol. 54, no. 6, pp. 2119–2125, June 2006.

[5] I. Selesnick, “The double-density dual-tree DWT,” *IEEE Transactions on Signal Processing*, vol. 52, no. 5, pp. 1304–1314, May 2004.

[6] ——, “The design of approximate hilbert transform pairs of wavelet bases,” *IEEE Transactions on Signal Processing*, vol. 50, no. 5, pp. 1144–1152, May 2002.

[7] I. Bayram and I. W. Selesnick, “A simple construction for the m-band dual-tree complex wavelet transform,” *Digital Signal Processing Workshop, 12th - Signal Processing Education Workshop, 4th*, pp. 596–601, Sept. 2006.

[8] P. Steffen, P. N. Heller, R. A. Gopinath, and C. S. Burrus, “Theory of regular *M*-band wavelet bases,” *IEEE Transaction on Signal Processing*, vol. 41, no. 12, pp. 3497–3511, December 1993.

[9] P. Steffen and W. Brandhuber, “Extending Vaidyanathan’s procedure to improve the performance of unitary filter banks with a fixed lowpass by using additional elementary building blocks,” in *Wavelets XII (Proceedings Volume)*, ser. Proceedings of SPIE, D. Van De Ville, V. K. Goyal, and M. Papadakis, Eds., vol. 6701, no. Part Two of Two Parts. SPIE, 20. September 2007, pp. 60 711A–1 – 60 711A–12, iSBN: 9780819468499.

[10] ——, “A Hilbert-Space Approach to the Complete Investigation of Vaidyanathan’s Procedure Applied to the Design of Unitary Filter Banks for the Generation of Orthogonal Wavelet Bases,” in *Optical and Digital Image Processing*, ser. Proceedings of SPIE, vol. 7000. SPIE, 2008, accepted for publication.

[11] C. Chaux, L. Duval, and J.-C. Pesquet, “Hilbert pairs of M-band orthonormal wavelet bases,” in *Proc. European Signal and Image Processing Conference, Vienna, Austria*, Sept. 6-10 2004, pp. 1187–1190.

[12] H. Ozkaramanli and R. Yu, “On the phase condition and its solution for Hilbert transform pairs of wavelet bases,” *IEEE Transactions on Signal Processing*, vol. 51, no. 12, pp. 3293–3294, December 2003, see also IEEE Transactions on Acoustics, Speech, and Signal Processing.