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ON THE CONCEPT OF TIME-FREQUENCY DISTRIBUTIONS BASED ON COMPLEX-LAG MOMENTS

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ABSTRACT

The instantaneous frequency law is an important tool for the analysis of time-frequency content of a signal. Generally, the analyzed signals are composed of several time-frequency components characterized by various non-linear contents. To deal with the nonlinearity of time-frequency content, several time-frequency distributions have been proposed, mainly based on warping techniques. Recently, we introduced the time-frequency distributions based on complex-lag arguments. Its property to strongly reduce, in the case of a single component, the inner interferences due to the time-frequency non-linearity has been shown. In this paper we prove its capabilities to deal with multi-component signals. This property is achieved by an appropriate combination of complex-lag moments computed for different complex lags sets. Numerical examples prove the benefits of the concept defined in this paper.

Index Terms— *Time-frequency analysis*, *Signal Representations*

1. INTRODUCTION

The analysis of signals characterized by complex time-frequency behaviour is a very challenging topic, due to the richness of the information describing the analyzed phenomena. In a large number of applications the analysis of the time-frequency (T-F) content provides an efficient solution for the characterization of diverse physical phenomena. Wave propagation trough time-varying dispersive channels, micro-Doppler effects or mechanical signals are just three examples requiring an efficient time-frequency analysis of signals arising from these applications [1]. The signals associated to these applications are generally characterized by many non-linear time-frequency structures. An efficient analysis of such signal should highlight the time-frequency energy of signal structures despite of artefacts that inherently appear when using time-frequency representations (TFR). Hence, in the case of linear TFRs, the well-known trade-off between time and frequency resolutions has to be considered. This topic has been subject to a large number of works. An alternative to linear TFR is the concept of bilinear TFRs [2]. One of the major research directions concerns the interference (generated by the bi-linearity) control in order to focus on time-frequency components of the signals. There are two types of interferences: inner interferences, generated in the case of non-linear time-frequency components and cross-terms, generated by the multi-component structures. The first interferences type is usually addressed by non-linear TFR designed with help of time or frequency warping concept [1], [3]. Recently, complex time distribution concept has been introduced in [4] as an efficient way to produce almost completely concentrated representations along the polynomial instantaneous frequency laws (IFL) of order 4 or less. In [5] we propose the generalization of the complex time distribution producing, in the mono component case, highly concentrated distributions around arbitrary polynomial IFLs.

In this paper we will show that the complex time distribution concept is able to deal with multi-component signals. This property is achieved by an appropriate choice of sets of complex lags. Combining the complex-lag moments computed for these sets produces a significant attenuation of cross terms with respect of auto-terms.

The paper is organized as follows. In Section 2 a brief presentation of the complex time distribution concept is done. The modified version of this concept having multi-component capabilities is presented in Section 3. The theoretical benefits of the new concept are analyzed via numerical examples in Section 4. We conclude in Section 5.

2. TIME-FREQUENCY DISTRIBUTION BASED ON COMPLEX LAGS ARGUMENTS

The concept of complex lag distributions has been introduced in [4] as a way for inner interferences reduction with respect of Wigner distribution. Recently, this concept has been generalized in order to focus on arbitrary instantaneous phase derivate of a signal [5]. Let consider the signal defined as $s(t) = Ae^{j\phi(t)}$. The case of A depending of t can also be addressed since the effect of slowly varying amplitude is "visible" on the instantaneous phase.

By using the Cauchy's integral formula [6] it is possible to compute the K^{th} order derivative of the instantaneous phase as

$$\phi^{(K)}(t) = \frac{K!}{2\pi j} \oint_{\gamma} \frac{\phi(z)}{(z-t)^{K+1}} dz \qquad (1)$$

This relation shows the interest of the complex time concept: the K^{th} order derivate of function ϕ at instant t can be computed as the complex integral over the integration path γ defined around this point. Applying the theory of Cauchy's integral theorem [6] and after few computations the expression (1) becomes [5]:

$$\phi^{(K)}(t) = \frac{K!}{2\pi\tau^K} \int_0^{2\pi} \phi\left(t+\tau e^{j\theta}\right) e^{-jK\theta} d\theta \qquad (2)$$

The discrete version of (2) is defined for $\theta = 2\pi p/N;\ p = 0,..,N-1$ (see the figure 1) where N is the number of discrete values of the angle θ.

$$\phi^{(K)}(t) = \frac{K!}{N\tau^K} \sum_{p=0}^{N-1} \phi\left(t+\tau e^{j\frac{2\pi p}{N}}\right) e^{-j\frac{2\pi pK}{N}} + \varepsilon \qquad (3)$$

where ε is the discretization error term.

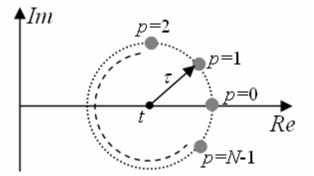

Figure 1 - Definition of complex lag coordinates

Using the properties of the roots of unity, $\omega_{N,p} = e^{j2\pi p/N}$ and the variable change $\tau \to \sqrt[K]{\tau \frac{K!}{N}}$ the expression (3) becomes:

$$\sum_{p=0}^{N-1} \phi\left(t + \omega_{N,p} \sqrt[K]{\tau \frac{K!}{N}}\right) \omega_{N,p}^{N-K} = \phi^{(K)}(t)\tau + Q(t,\tau) \qquad (4)$$

where Q is the spread function defined as [6] :

$$Q(t,\tau) = N \sum_{p=1}^{+\infty} \phi^{(Np+K)}(t) \frac{\tau^{\frac{Np}{K}+1}}{(Np+K)!} \left(\frac{K!}{N}\right)^{\frac{Np}{K}+1} \qquad (5)$$

As indicated by (4) and (5), the sum of the phase samples defined in the complex coordinates (left side of 4) is **linear** depending on τ if the ϕ's derivates of orders greater than $N+K$ are 0. In order to exploit this property we define the generalized complex-lag moment (GCM) of s as the operation leading to (4):

$$GCM_N^K[s](t,\tau) = \prod_{p=0}^{N-1} s^{\omega_{N,p}^{N-K}} \left(t + \omega_{N,p} \sqrt[K]{\frac{K!}{N}\tau}\right) \overset{with\,(4)}{=} e^{j\phi^{(K)}(t)\tau + jQ(t,\tau)} \qquad (6)$$

The computation of GCMs implies the evaluation of signal samples at complex coordinates. This abstract notion is defined with help of analytical continuation of a signal defined as [6]:

$$s(t+jm) = \int_{-\infty}^{\infty} S(f) e^{-2\pi mf} e^{j2\pi ft} df \qquad (7)$$

where $S(f)$ is the Fourier transform of signal s.

Taking the Fourier transform of GCM with respect of τ we define the generalized complex-lag distribution (GCD):

$$\begin{aligned} GCD_N^K[s](t,\omega) &= \Im_\tau\left[GCM_N^K[s](t,\tau)\right] = \\ &= \delta\left(\omega - \phi^{(K)}(t)\right) \underset{\omega}{*} \Im_\tau\left[Ae^{jQ(t,\tau)}\right] \end{aligned} \qquad (8)$$

As stated by this definition, the K^{th} order distribution of the signal, obtained for N complex-lags, highly concentrates the energy around the K^{th}-order derivate of the phase law. This concentration is optimal if the ϕ's derivates of orders greater than $N+K$ are 0, exactly like in the case of chirps represented by Wigner distribution.

The general definition (8) leads to a large number of TFRs, part of them well known in literature. For example, for K=1; N=2 the WVD is obtained whereas the case K=1; N=4 corresponds to the complex-time distribution (CTD) [4]. In [5] we have shown that increasing the number of complex lags leads to an attenuation of inner interference due to the time-frequency non-linearity. This is illustrated by the next example for a test signal

$$s_1(t) = \exp\left(j2\pi\left(0.1t + 4\cos(0.025t)\right)\right); t = 0,...,511.$$

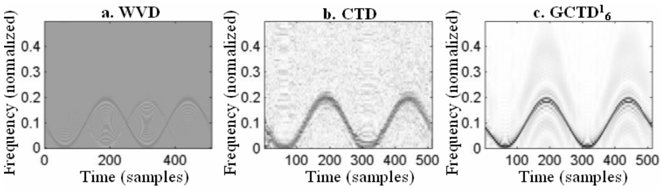

Figure 2 - Inner interference for WVD, CTD and GCD_6^1

We remark the better concentration of time-frequency energy in the case of GCD^1_6 than in the case of the other TFRs. This is analytically proved by the spread function expression (5) and illustrated by this example.

The next example points out on the derivability property of GCD. We consider a 4^{th} order polynomial phase signal defined by

$$s_2(t) = \exp\left(j2\pi\left(0.32t + 10^{-5}t^2 - 4.8\cdot 10^{-5}t^3 + 8.6\cdot 10^{-10}t^4\right)\right); t = 0,...,511.$$

The figure 3 plots in the top the analytic derivatives of first, second and third orders of the IPL of this signal. We plot, in the bottom of the figure 3, the GCDs of the same orders.

We remark that the theoretical derivatives are correctly represented by the GCDs of corresponding order, justifying the derivation property of the complex-lag distribution. By derivation property it is possible to reduce a high order polynomial phase signal to a simpler one. That is, in the case of signal s_2, the estimation of 4^{th} order polynomial coefficient could be done by computing the Fourier transform of GCM of order 4.

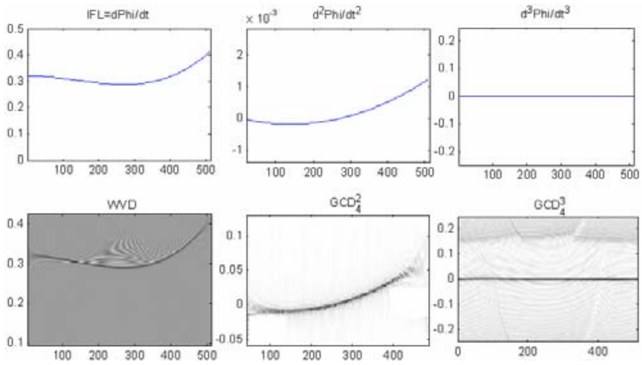

Figure 3 - Derivation of instantaneous phase by GCDs

This property, detailed in [5], has been studied so far only in the case of a single component. The next section contains an extension of GCD to multi-component signals.

3. COMPLEX LAGS MOMENT FOR MULTI-COMPONENTS SIGNALS

As proved in [5] and illustrated in the previous section, the GCD is an interesting tool for analysis of arbitrary derivates of instantaneous phase of a signal. So far, only mono-component signals have been considered. In the case of multi-component signals cross-terms will appear due to the non-linear form of the GCMs (6). That is, considering a signal composed by two components, $s(t)=s_1(t)+s_2(t)=e^{j\phi_1(t)}+e^{j\phi_2(t)}$, by analogy with the CTD case [4], the GCM can be expressed as :

$$GCM_N^K[s_1+s_2](t,\tau)=$$
$$=\prod_{p=0}^{N-1}\left[s_1\left(t+\omega_{N,p}\sqrt[K]{\frac{K!}{N}}\tau\right)+s_2\left(t+\omega_{N,p}\sqrt[K]{\frac{K!}{N}}\tau\right)\right]^{\omega_{N,p}^{N-K}}$$
$$\overset{with\,(4)}{=} e^{j\phi_1^{(K)}(t)\tau+jQ_1(t,\tau)}+e^{j\phi_2^{(K)}(t)\tau+jQ_2(t,\tau)}+CT_{\phi_1,\phi_{12}}(N,K) \quad (9)$$

where CT is the terms indicating the cross terms. Because of the complicate GCM's definition, an analytical calculation of cross terms is a very difficult task. From (9) we remark that the auto-terms are sinusoids in τ with frequencies given by the K^{th} order derivative of instantaneous phase laws ϕ_1 et ϕ_2. The spread functions, Q_1 and Q_2, can be neglected if ϕ's derivates of orders greater than $N+K$ are 0. Obviously, the cross terms generated by combination of signal's components according to the GCM definition affect the visibility of auto-terms. This is illustrated, via the next example, in the case of a two-component signal defined as:

$$s_3(t)=\exp\left(j2\pi\left(0.32t+10^{-5}t^2-4.8\cdot10^{-5}t^3+8.6\cdot10^{-10}t^4\right)\right)+\exp\left(j2\pi\left(0.12t+3\cdot10^{-3}t^2\right)\right).$$

We remark that the GCD_4^2 of the two-component signal contains the structures corresponding to the second order derivative and having a correct shape with respect of theoretical derivation (top side of figure 4).

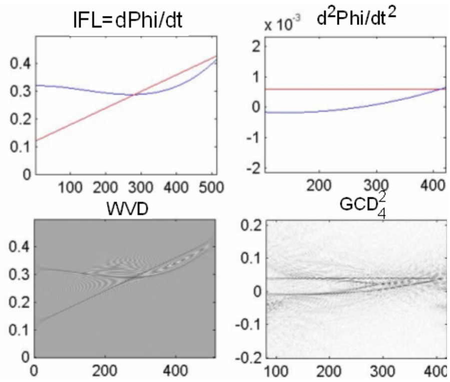

Figure 4 - Derivation IPLs for a two-component signal

Nevertheless, the cross terms have the same energetic level and they have a more complicate shape than in the case of WVD. In order to reduce the level of cross terms the starting point is the remark that the auto-terms of (9), excepting Q function and lag set $\tau\rightarrow\sqrt[K]{\tau K!/N}$ (however, this dependence can be avoided by a simple warping procedure as we will see further), **don't** depend of number of lags and of their structure. It means that, computing GCMs for several lag sets, the auto-terms will have the same structure whereas the cross-terms will be differently located because of their dependence of N. Furthermore, the summation of these several GCMs will highlight the auto-terms decreasing in the same time the level of cross terms. Considering the sets of GCMs computed for several lags sets,

$$\left\{GCM_{N_i}^K[s]\left(t,\tau^{(i)}\right)\right\}_{i=1,..,P};\tau^{(i)}=\sqrt[K]{K!/N_i} \quad (10)$$

the *multi-lags sets* GCM (mlsGCM) is defined as

$$mlsGCM_{\{N_i\}}^K[s](t,\tau)=\sum_{i=1}^{P}GCM_{N_i}^K[s](t,\tau) \quad (11)$$

Practically, the first step is the computation of GCMs (10). The second step is the warping of lag sets of each GCM according to $\tau\rightarrow\left\{\tau^{(i)}\right\}^K\dfrac{N_i}{K!}$. This step is very important in order to ensure that all GCMs (10) are expressed in the same lag axis. Finally, the summation (11) is computed. For the two-component signal used in (9) the mls-GCM shows analytically how the auto-terms are amplified:

$$mlsGCM^K[s](t,\tau)=P\sum_{i=1}^{2}e^{j\phi_i^{(K)}(t)\tau+jQ_i(t,\tau)}+\sum_{i=1}^{P}CT_{\phi_1,\phi_{12}}(N_i,K) \quad (12)$$

The Fourier transform of *mlsGCM*, called *mlsGCD*,

$$mlsGCD_{\{N_i\}}^K[s](t,\omega)=\Im_\tau\left[mlsGCM_{\{N_i\}}^K[s](t,\tau)\right] \quad (13)$$

produces a cross-terms reduced distribution. As an illustrative example, for the signal used in figure 4, the results obtained by *mlsGCD* are illustrated in figure 5. As this figure shows, using many lag sets leads to a cross-terms level reduction. The attenuation of the cross-terms is more significant as the number of lag sets increases. For this example, eight set of lags is appropriate for an accurate representation of auto-terms.

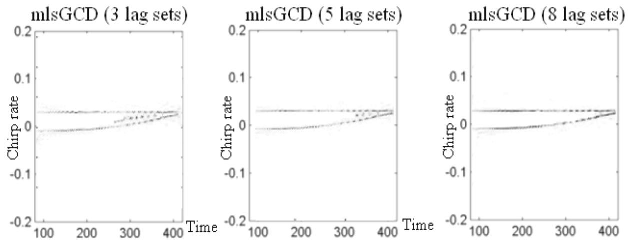

Figure 5 - Cross-terms reduction using *mlsGCM*

4. RESULTS

The capability of mlsGCD, defined in the previous section, to perform in a multi-component case can be successfully exploited in a large number of applications dealing with complex time-frequency modulations. The first example is the estimation of a harmonically frequency modulation mixed with coherent signals. This could be the case of audio signals or micro-Doppler radar signals (corresponding to radar signal returned by helicopter blades) corrupted by coherent attacks as tones or chirps. To illustrate such application let consider a sinusoidal frequency modulation corrupted by a chirp and sinusoid (theoretical IFLs plotted in the figure 6.a).

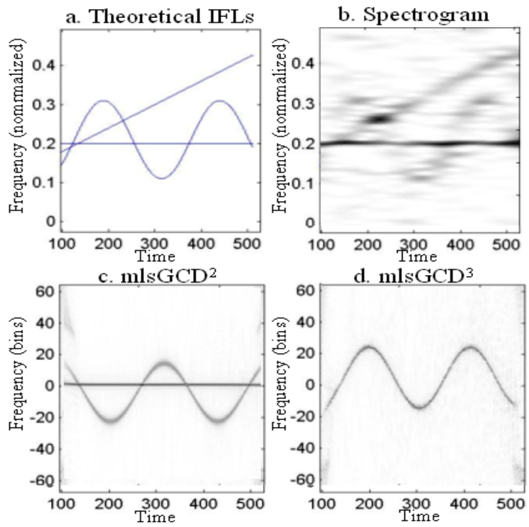

Figure 6 - Separation of a sinusoidal frequency modulation (SFM)

Clearly, in the spectrogram domain (figure 6.b), the SFM is masked by the chirp and the tone as well as the noise (SNR=10 dB). The mlsGCD^2 (computed for 9 lag sets, figure 6.c) removes the effect of tone and stationaries the chirp effect. Finally, only the SFM is "visible" (figure 6.d) if the third order derivate is done via mlsGCD^3. In this domain, the filtering or parameter estimation of SFM can be easily done.

Another application of the concept of mlsGCD is the extraction of transient impulses corrupted by some coherent signals. Hence, using the mlsGCD concept, we take advantage on the infinite derivability of the transient signals.

Transient signals can be globally defined as impulsive or very short duration signals often with oscillations. These signals exist in many different applications and systems from underwater acoustic to audio signals with sound attacks or electrical systems with partial discharges and commutation switches, for example. The main processing done on this type of signal aims to detect them, characterize them and localize their source. Several signal processing tools such as higher-order statistics and wavelets coefficients have been already used and upgraded to perform the goals of detection and localization but they do not lead to real characterization of the transient signals. The concept of GCD and mlsGCD shows its convenience to perform on transient signals in so far as, for an impulse, the successive order derivates of the phase law remain impulsive in a certain way and with no end. Therefore, the distinction of the transient is marked in the instantaneous law whatever the derivative order is.

The capability of GCD and mlsGCD to perform with transient signals can be exploited in several applications. Let us consider a multi-component signal composed by a chirp and a BPSK (Binary Phase-Shift Keying) modulation.

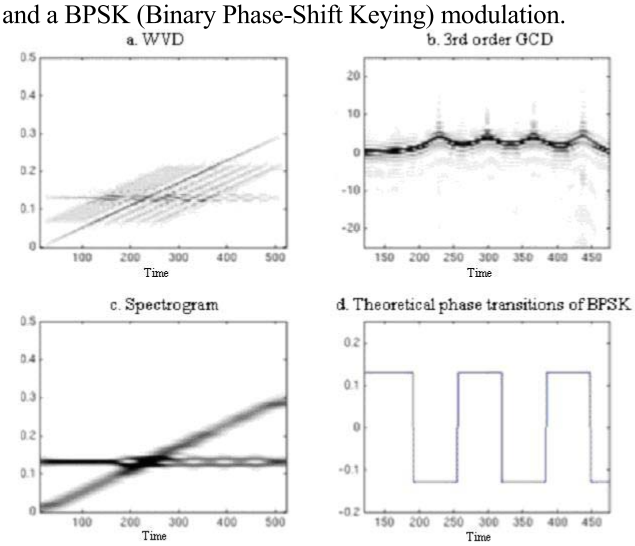

Figure 7 – GCD Performance to detect transient information

As shown in the figure 7, the chirp crossing the BPSK hides, in the case of spectrogram and the WVD, the phase transitions corresponding to the BPSK. Using the IPL's derivation provided by GCD (figure 7.b) we can remove the chirp effect highlighting also the transitions. We can remark that the transitions detected in GCD domain are coherent with the theoretical ones illustrated in figure 7.d.

Another application is illustrated in figure 9 where a simulated train of impulses (plotted in figure 8) is corrupted by a chirp (spectrogram plotted in figure 9.a). The GCD^3 theoretically removes the chirp effect and keeps the mark of the impulses in so far as the successive phase derivatives of a transient remain impulsive. The result of the GCD^3 plotted in figure 9.b consequently shows the successful separation of the four transients present in the signal.

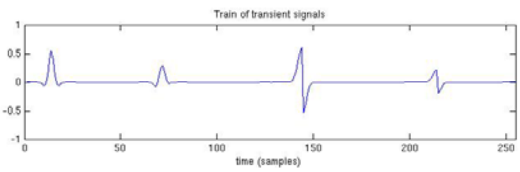

Figure 8 - Train of impulses

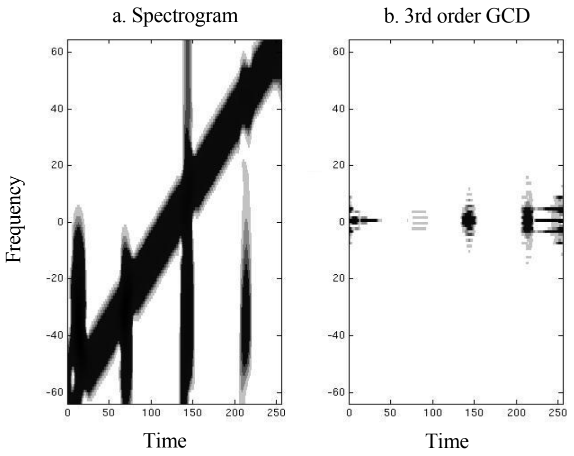

Figure 9 - Separation of the transient signals in a chirp

The previous example illustrates also the mlsGCD performing in the case of variable amplitudes signals. Even if the pulses have different amplitude their representation in the GCD^3 domain is correctly done.

As well as GCD, the mlsGCD concept shows its utility for transient signals analysis, especially in the case of a signal corrupted by noise. As explained in section 3, the mlsGCD for multi-components signals leads to amplify the auto-terms and reduce the level of cross terms. In the case of transients embedded in noise, the idea is the same. A high order GCD of such a signal gives a high value signature of each transient but the phase derivative of the noise strongly corrupts the distribution. By using mlsGCD, the high value signature of the transients is amplified whereas the noise effect is reduced. Figure 10 illustrates this property of mlsGCD; the same train of impulses (defined in the figure 8) is used and complex white gaussian noise is added (SNR=12 dB). On the third order GCD, the transient signatures are clearly masked by the noise phase derivative, even after thresholding to keep the highest amplitude values. The distribution resulted from the third order mlsGCD (computed for 8 lag sets) makes the transient signatures more visible.

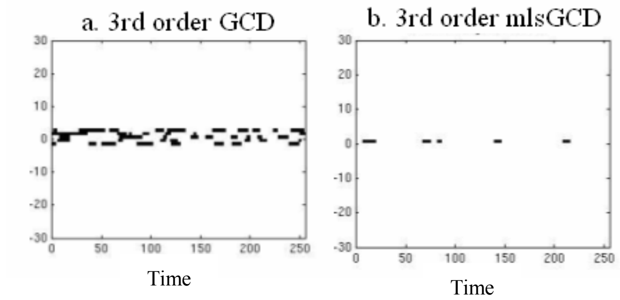

Figure 10 - Noise effect reduction using mlsGCD

This example shows the capability of the mlsGCD to extract, via derivability property, the transient signals corrupted by both noise and coherent perturbations. It performs also in the case of pulses having different amplitudes.

As shown in figure 10.b, the detection of transient signal can be accurately done in the mlsGCD domain.

5. CONCLUSION

In this paper we address the problem of complex-lag representation of multi-component signals. The derivation property of the phase of a signal has been extended to multi-component signals. This extension is based on the property of the auto-terms of the GCM which don't depend to the complex lag sets. Thus, the summation of GCMs evaluated for several lag sets, can produce the amplification of auto-terms. The new defined distribution allows focusing on arbitrary derivate of phase in a multi-component context. This property can be of great helpful to separate signals with infinitely derivate phase laws from embedded observation.

Otherwise, mlsGCD can be seen as a new methodology for polynomial phase modelling. These applications as well as theoretical subjects, related to signal-dependent choice of lag sets and 2D extension of complex lag concept, will be address in our future works.

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