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MODELING ERROR SENSITIVITY OF THE MUSIC ALGORITHM CONDITIONED ON RESOLVED SOURCES

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ABSTRACT

When the correlation matrix is known, the resolution power of subspace algorithms is infinite. In the presence of modelling errors, even if the correlation matrix is known, sources can no longer be resolved with certainty. Focusing on the MUSIC algorithm [1], the purpose of this work is to provide closed form expression of bias and variance versus the model mismatch (these errors can be different for each source). Unlike previous work, these performance measures are derived conditioned on the success of a certain source resolution test. Among the resolution definitions proposed in [2], we investigate which one is more suitable for our purposes. Numerical results support the theoretical investigations. Our findings are of a great interest for the determination of the necessary antenna calibration accuracy to achieve specifications on the estimator performance.

1. INTRODUCTION

Estimating the direction of arrival (DOA) of a signal incident on a planar array of sensors has applications in many fields including radar, sonar, telecommunications. A lot of works have already been done to characterize the performance in the asymptotic region (in SNR and in the number of snapshots) ([3][4][5][6]). A major reason for the interest in so called high resolution methods (e.g. [1]) is on their asymptotic infinite resolution power if the assumed model of reception is correct. Unfortunately, in any operational context the assumed reception model is different from the true one, even after a calibration procedure. Some efforts have recently been done in this sense to characterize the algorithms in presence of model mismatch.

In the presence of modeling errors, some authors have studied the performance degradation, in terms of variance (e.g. [7, 8, 9, 10],...). For small model mismatch, these results are correct in the sense that the resolution probability is close to one. This is no more the case when the mismatch increases, because even if the correlation matrix is known, the resolution power is no more infinite [2].

Some other authors have investigated the performance degradation versus the model mismatch, under the point of view of Resolution threshold for equipowered sources. A heuristic definition of the resolution threshold ([11][12][13][14],...) has been used. The performances in the region threshold have also been analysed [15].To the best of our knowledge, there is no general result regarding performances (bias, variance and probability of resolution) under a given resolution condition. In the particular case of resolution probability study, only Zhang in [16] and Richmond [17] have analyzed this performance criterion but without deriving closed form expressions for bias and variance under a resolution condition.

In this paper we propose to fill these gaps with a more detailed investigation of the phenomenon. For this we apply two out of the three resolution definitions proposed in [2], and derive the associated closed form expressions of bias and variance for the MUSIC algorithm conditioned on resolution success.

The organization of the paper is as follows. In Section 2 we formulate the problem an introduce the notation. In Section 3, we provide some background for the mathematical derivation. In Section 4, the bias and variance conditioned on the sources resolution are established for two of the definitions in [2]. In Section 5, a numerical simulation is used to illustrate the theoretical results and to investigate the usefulness of the different resolution definitions. A conclusion is given in Section 6.

2. SIGNAL MODELLING AND PROBLEM FORMULATION

Assume a noisy mixture of M narrow-band sources of DOAs θ_m, $1 \leq m \leq M$, is received by an array of N sensors. The associated observation vector, $\mathbf{x}(t)$, whose components $x_n(t)$ ($1 \leq n \leq N$) are the complex envelopes of the signals at the output of the sensors, is then given by

$$\mathbf{x}(t) = \sum_{m=1}^{M} \tilde{\mathbf{a}}(\theta_m)\ s_m(t) + \mathbf{n}(t) = \tilde{\mathbf{A}}\ \mathbf{s}(t) + \mathbf{n}(t) \qquad (1)$$

Here, $\tilde{\mathbf{a}}(\theta)$ is the true steering vector of a signal source in the direction θ, and $\tilde{\mathbf{A}}$=[$\tilde{\mathbf{a}}_1 \cdots \tilde{\mathbf{a}}_M$] is the steering matrix, with $\tilde{\mathbf{a}}_m = \tilde{\mathbf{a}}(\theta_m)$. Further, $\mathbf{n}(t)$ is an additive noise vector, which is supposed to be spatially white, and $s_m(t)$ is the complex envelope of the mth source. Denoting $\mathbf{a}(\theta)$ the assumed (incorrect) steering vector, the MUSIC [1] DOA estimates are given by the M smallest local minima of the criterion function

$$c(\theta) = \mathbf{a}(\theta)^H\, \hat{\mathbf{\Pi}}(T)\ \mathbf{a}(\theta) \qquad (2)$$

where $\hat{\mathbf{\Pi}}(T)$ is the orthogonal projection matrix onto the noise subspace, extracted from the $N-M$ minor eigenvectors of the estimated correlation matrix $\hat{\mathbf{R}}_x(T)$. The true correlation matrix is denoted $\mathbf{R}_x = \mathbb{E}\left[\mathbf{x}(t)\mathbf{x}(t)^H\right]$, and

$$\hat{\mathbf{R}}_x(T) = \frac{1}{T}\sum_{t=0}^{T} \mathbf{x}(t)\mathbf{x}(t)^H,$$

where $(\cdot)^H$ denotes complex conjugate transpose. In this paper, the MUSIC algorithm is analyzed in the presence of steering vector modelling errors only. Thus, the asymptotic case ($T \to \infty$) is considered. Assuming non-coherent signals, the noise projector is given by

$$\lim_{T\to\infty} \hat{\mathbf{\Pi}}(T) = \mathbf{\Pi} = \mathbf{I}_N - \tilde{\mathbf{A}}\tilde{\mathbf{A}}^{\#}$$

where $\tilde{\mathbf{A}}=[\tilde{\mathbf{a}}_1 \cdots \tilde{\mathbf{a}}_M]$ and $\tilde{\mathbf{A}}^{\#}=(\tilde{\mathbf{A}}^{\mathbf{H}}\tilde{\mathbf{A}})^{-1}\tilde{\mathbf{A}}^H$ is the Moore-Penrose pseudo-inverse. We assume additive modelling errors $\boldsymbol{\epsilon}_m$ to the steering vectors, defined by

$$\boldsymbol{\epsilon}_m = \tilde{\mathbf{a}}(\theta_m) - \mathbf{a}(\theta_m) = \tilde{\mathbf{a}}_m - \mathbf{a}_m \ . \tag{3}$$

The asymptotic projector $\boldsymbol{\Pi}$ depends on the modelling error vector, which we define as the $(2NM+1)\times 1$ vector

$$\boldsymbol{\epsilon} = \left[\begin{array}{ccc} 1 & vec(\boldsymbol{\epsilon}_1 \cdots \boldsymbol{\epsilon}_M)^T & vec(\boldsymbol{\epsilon}_1 \cdots \boldsymbol{\epsilon}_M)^H \end{array}\right]^T, \tag{4}$$

We write $\boldsymbol{\Pi}(\boldsymbol{\epsilon})$ to stress this dependence. Consequently, the associated MUSIC criterion depends on θ and $\boldsymbol{\epsilon}$ according to $c(\theta,\boldsymbol{\epsilon}) = \mathbf{a}(\theta)^H\,\boldsymbol{\Pi}(\boldsymbol{\epsilon})\,\mathbf{a}(\theta)$. When $\tilde{\mathbf{a}}_m \neq \mathbf{a}_m$, the $m-th$ local minimum of $c(\theta,\boldsymbol{\epsilon})$ occurs at $\hat{\theta}_m$, which is different from the true θ_m. The DOA estimation error $\Delta\theta_m=\hat{\theta}_m-\theta_m$ then depends only on the modelling error $\boldsymbol{\epsilon}$. In this paper, the random vectors $\boldsymbol{\epsilon}_m$ are assumed to be circular and Gaussian distributed.

Two sources are said "resolved" when the MUSIC criterion exhibits two minima at $\hat{\theta}_1$ and $\hat{\theta}_2$, "close to" the true θ_1 and θ_2 respectively. Rigorously speaking, a sound definition of a resolution condition should be built from the statistical properties of the shape of $c(\theta,\boldsymbol{\epsilon})$. This realistic definition appears impractical for a performance analysis.

In order to obtain tractable derivations of conditional resolution performance, we focus on a small number of MUSIC criterion features η_i $(1 \le i \le I)$ for a discrete set of directions of arrival to obtain a wide class of resolution definitions. Thus we propose a general framework for conditional resolution performance derivation. This approach has been adopted in [2] for the derivation of the resolution probability. More precisely, the resolution conditions rely on the comparison between features η_i, taken at the i^{th} DOA θ_i $(1 \le i \le I)$ of the $c(\theta,\epsilon)$ shape, and some threshold values, α_i, leading to different resolution definitions. In this paper we focus on two resolution conditions from [2] and conclude that the practical validity of each definition is different and scenario-dependent.

For a given resolution definition, the sources are said to be resolved when the I conditions on η_i verify $\eta_i < \alpha_i$. Consequently, the associated mathematical bias and RMS (Root Mean Square) error needs the evaluation of $\mathbb{E}[\Delta\theta_m \mid \eta_i < \alpha_i, 1 \le i \le I]$ and $RMS_m = \sqrt{\mathbb{E}[\Delta\theta_m^2 \mid \eta_i < \alpha_i, 1 \le i \le I]}$, where $\mathbb{E}[.\mid .]$ is the conditional expectation. The probability of resolution $Pr = \Pr(\eta_i < \alpha_i \ \ , 1 \le i \le I)$ is derived in [2]. The main purpose of this work is to provide conditional performance results for two resolution definitions.

3. PRELIMINARY MATHEMATICAL RESULTS

In this section we introduce recent approximations ([9]) of the DOA error $\Delta\theta_m$ and the MUSIC criterion $c(\theta,\epsilon)$, which remain valid in presence of closely spaced sources. We also provide some definitions and results for incomplete statistical moments that are necessary for computing the desired conditional performance measures.

3.1 Approximation of DOA error and MUSIC criterion

According to [9], the MUSIC criterion and its derivatives can be approximated to the second order by

$$c(\theta,\boldsymbol{\epsilon}) \approx \boldsymbol{\epsilon}^H \mathbf{Q}(\mathbf{a}(\theta),\mathbf{a}(\theta))\,\boldsymbol{\epsilon} \tag{5}$$
$$\dot{c}(\theta,\boldsymbol{\epsilon}) \approx \boldsymbol{\epsilon}^H\ \dot{\mathbf{Q}}(\boldsymbol{\theta})\boldsymbol{\epsilon}$$
$$\ddot{c}(\theta,\boldsymbol{\epsilon}) \approx \boldsymbol{\epsilon}^H\ \ddot{\mathbf{Q}}(\boldsymbol{\theta})\boldsymbol{\epsilon}$$

where

$$\dot{\mathbf{Q}}(\theta) \overset{def}{=} \mathbf{Q}(\mathbf{a}(\theta),\dot{\mathbf{a}}(\theta)) + \mathbf{Q}(\dot{\mathbf{a}}(\theta),\mathbf{a}(\theta)),$$
$$\ddot{\mathbf{Q}}(\theta) \overset{def}{=} \mathbf{Q}(\mathbf{a}(\theta),\ddot{\mathbf{a}}(\theta)) + \mathbf{Q}(\ddot{\mathbf{a}}(\theta),\mathbf{a}(\theta)) + 2\ \mathbf{Q}(\dot{\mathbf{a}}(\theta),\dot{\mathbf{a}}(\theta))$$

Here, $\dot{\mathbf{a}}(\theta)$ and $\ddot{\mathbf{a}}(\theta)$ are, respectively, the first and second derivatives of $\mathbf{a}(\theta)$ with respect to θ, and the matrix $\mathbf{Q}(\mathbf{u},\mathbf{v})$, as a function of two vectors $\mathbf{u}$ and $\mathbf{v}$, is defined by

$$\mathbf{Q}(\mathbf{u},\mathbf{v}) = \begin{bmatrix} q & -\mathbf{q}_{12}^H & \mathbf{0}^T \\ -\mathbf{q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ \mathbf{0} & \mathbf{Q}_{32} & \mathbf{Q}_{33} \end{bmatrix}, \tag{6}$$

In the above expression, $\boldsymbol{\Pi}_0=\boldsymbol{\Pi}(\boldsymbol{\epsilon}=\mathbf{0})$, $q=\mathbf{v}^H\ \boldsymbol{\Pi}_0\ \mathbf{u}$, $\mathbf{q}_{12}=\Phi(\mathbf{u},\mathbf{v})$ and $\mathbf{q}_{21}=\Phi(\mathbf{v},\mathbf{u})$ with $\Phi(\mathbf{u},\mathbf{v})=\left(\mathbf{A}^{\#}\mathbf{u}\right)^* \otimes (\boldsymbol{\Pi}_0\mathbf{v})$. The remaining blocks are given by

$$\begin{aligned} \mathbf{Q}_{22} &= \Psi\left(\mathbf{A}^{\#},\mathbf{A}^{\#},\boldsymbol{\Pi}_0\right), \\ \mathbf{Q}_{23} &= \Psi\left(\mathbf{A}^{\#},\boldsymbol{\Pi}_0,\left(\mathbf{A}^{\#}\right)^H\right)\mathbf{P}, \\ \mathbf{Q}_{32} &= \mathbf{P}^H\ \Psi\left(\boldsymbol{\Pi}_0,\mathbf{A}^{\#},\mathbf{A}^{\#}\right), \\ \mathbf{Q}_{33} &= -\mathbf{P}^H\ \Psi\left(\boldsymbol{\Pi}_0,\boldsymbol{\Pi}_0,\mathbf{A}^{\#}\left(\mathbf{A}^{\#}\right)^H\right)\mathbf{P}, \end{aligned}$$

where

$$\Psi(\mathbf{X},\mathbf{Y},\mathbf{Z}) = \left((\mathbf{X}\mathbf{v})^*\,(\mathbf{Y}\mathbf{u})^T\right)\otimes\mathbf{Z}$$

Here, $\otimes$ is the Kronecker product and $\mathbf{P}$ the permutation matrix defined by $vec\left(\mathbf{E}^T\right)=\mathbf{P}vec(\mathbf{E})$ where $\mathbf{E}=[\begin{array}{ccc}\boldsymbol{\epsilon}_1 & \cdots & \boldsymbol{\epsilon}_M\end{array}]$. According to [7, 9] a first-order approximation of the DOA estimation error $\Delta\theta_m=\hat{\theta}_m-\theta_m$ is

$$\Delta\theta_m = -\frac{\dot{c}(\theta_m,\boldsymbol{\epsilon})}{\ddot{c}(\theta_m,\boldsymbol{\epsilon})} \tag{7}$$

where $\dot{c}(\theta,\boldsymbol{\epsilon})$ and $\ddot{c}(\theta,\boldsymbol{\epsilon})$, respectively, indicate the first and second derivatives of the criterion $c(\theta,\boldsymbol{\epsilon})$ at θ as given in (5).

3.2 Incomplete Moments

In this section, x, y and z are assumed to be scalar random variables. The correlation and the covariance are defined by

$$R_{xy} = \mathbb{E}[xy]\ , \quad C_{xy} = \mathbb{E}[xy] - \mathbb{E}[x]\,\mathbb{E}[y] \tag{8}$$

The nth Cumulative Moments Function (CMF) is

$$\gamma(x^n, p(x), \alpha) = \int_{-\infty}^{\alpha} x^n p(x)dx \tag{9}$$

where $p(x)$ is the probability density function of x. In the particular case of a Gaussian variable, let us denote the function $\gamma(x^n, p(x), \alpha)$ as $\gamma_G(x^n, \left[\mu\ \sigma^2\right], \alpha)$, where μ and σ are, respectively, the mean and the standard deviation. The CMF function is written as $\gamma_N(u^n,\alpha) = \gamma_G(u^n, [0\ 1], \alpha)$ in the standardized Gaussian case. According to [2], the incomplete moment $\mathbb{E}[x^n \mid x < \alpha]$, for any n, is given by

$$\mathbb{E}[x^n \mid x < \alpha] = \frac{\gamma(x^n, p(x), \alpha)}{\gamma(x^0, p(x), \alpha)} \tag{10}$$

When x is a Gaussian random variable of standard deviation σ and mean μ, we have

$$\mathbb{E}[x^n \mid x < \alpha] = \frac{\gamma_G(x^n, \left[\mu\ \sigma^2\right], \alpha)}{\gamma_G(x^0, \left[\mu\ \sigma^2\right], \alpha)} \tag{11}$$

The closed form expressions of the CMF function in the general and standardized Gaussian cases can be expressed as

$$\gamma_G(x^n, \left[\mu\ \sigma^2\right], \alpha) = \sum_{k=0}^{n} C_n^{n-k} \sigma^{n-k} \mu^k \gamma_N(u^{n-k}, \alpha_N) \tag{12}$$

$$\begin{aligned}\gamma_N(u^n, \alpha) = & -\tfrac{(\alpha_N)^{n-1}}{\sqrt{2\pi}} \exp\left(-\tfrac{(\alpha_N)^2}{2}\right) \\ & +(n-1)\,\gamma_N(u^{n-2}, \alpha_N)\end{aligned} \tag{13}$$

where $x = \mu + \sigma u$, $\alpha = \mu + \sigma\alpha_N$, $\gamma_N(u, \alpha_N) = -\left(1/\sqrt{2\pi}\right)\exp\left(-(\alpha_N)^2/2\right)$ and $C_n^{n-k} = n!/\left((n-k)!\,k!\right)$. These last expressions are proved in [2]. In particular, the first and second incomplete moments of the Gaussian variable x are

$$\mathbb{E}\left[x \mid x<\alpha\right] = -\sqrt{C_{xx}}\Psi_x(\alpha) + \mathbb{E}[x] \tag{14}$$

$$\begin{aligned}\mathbb{E}\left[x^2 \mid x<\alpha\right] - \mathbb{E}\left[x^2\right] &= -\sqrt{C_{xx}}\left(\alpha + \mathbb{E}[x]\right)\Psi_x(\alpha) \\ &= \left(\alpha + \mathbb{E}[x]\right)\left(\mathbb{E}\left[x \mid x<\alpha\right] - \mathbb{E}[x]\right)\end{aligned} \tag{15}$$

where $\mathbb{E}\left[(x-\mathbb{E}[x])^2\right] = C_{xx}$ and

$$\Psi_x(\alpha) = \Psi\left(\frac{\alpha - \mathbb{E}[x]}{\sqrt{C_{xx}}}\right) \tag{16}$$

$$\Psi(y) = \frac{\left(1/\sqrt{2\pi}\right)\exp\left(-y^2/2\right)}{\gamma_N(u^0, y)}$$

Thus, according to (14)

$$\mathbb{E}\left[x^2 \mid x<\alpha\right] - \mathbb{E}\left[x \mid x<\alpha\right]^2 = -\sqrt{C_{xx}}\Psi_{xx}(\alpha) + C_{xx} \tag{17}$$

$$\Psi_{xx}(\alpha) = \Psi_x(\alpha)\left(\alpha + \sqrt{C_{xx}}\Psi_x(\alpha) - \mathbb{E}[x]\right) \tag{18}$$

3.3 Incomplete Conditional Moments

Introduce the notation $\bar{x} = x - \mathbb{E}[x]$ and $\bar{\alpha} = \alpha - \mathbb{E}[x]$. According to [2] and (14) the first order conditional moment is in the Gaussian case given by

$$\begin{aligned}\mathbb{E}\left[y \mid x<\alpha\right] &= \frac{C_{xy}}{C_{xx}}\mathbb{E}\left[\bar{x} \mid \bar{x}<\bar{\alpha}\right] + \mathbb{E}[y] \\ &= -\frac{C_{xy}}{\sqrt{C_{xx}}}\Psi_x(\alpha) + \mathbb{E}[y]\end{aligned} \tag{19}$$

As proved in [18], the expression for the second order conditional moment is

$$\begin{aligned}C_{yz}(x<\alpha) &= \mathbb{E}\left[yz \mid x<\alpha\right] - \mathbb{E}\left[y \mid x<\alpha\right]\mathbb{E}\left[z \mid x<\alpha\right] \\ &= \frac{C_{xy}C_{xz}}{(C_{xx})^2}C_{xx}(x<\alpha) + \frac{C_{yz}C_{xx} - C_{xz}C_{xy}}{C_{xx}} \\ &= -\frac{C_{xy}C_{xz}}{(C_{xx})^{3/2}}\Psi_{xx}(\alpha) + C_{yz}\end{aligned} \tag{20}$$

where $C_{xx}(x<\alpha) = \mathbb{E}\left[x^2 \mid x<\alpha\right] - \mathbb{E}\left[x \mid x<\alpha\right]^2$ is given by (17).

4. CONDITIONAL PERFORMANCES

This paper requires a resolution definition to evaluate the performance given that the sources are resolved. We use definitions based on the MUSIC pseudo-spectrum $c(\theta, \boldsymbol{\epsilon})$, taken at the true position θ_1 and θ_2 and at the mean value $\bar{\theta} = (\theta_1 + \theta_2)/2$ proposed in [2]. From these definitions this paper establishes the corresponding expressions for the bias and RMS error conditioned on resolution.

The derivations of this paper are based on the fact that the real random variables $\boldsymbol{\epsilon}^H\mathbf{Q}(\boldsymbol{\theta})\boldsymbol{\epsilon}$, $\boldsymbol{\epsilon}^H\dot{\mathbf{Q}}(\theta)\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}^H\ddot{\mathbf{Q}}(\boldsymbol{\theta})\boldsymbol{\epsilon}$ are asymptotically Gaussian distributed when the modelling error vector $\boldsymbol{\epsilon}$ is Gaussian and circular. This assumption has been justified in [2] by the Lyapunov Central Limit Theorem. To summarize, we can expect a good Gaussian approximation of $\boldsymbol{\epsilon}^H\mathbf{Q}\ \boldsymbol{\epsilon}$ even for $M = 2$, provided N is not "too small". In this paper the DOA estimation error $\Delta\theta_m = -\dot{c}(\theta_m, \boldsymbol{\epsilon})/\ddot{c}(\theta_m, \boldsymbol{\epsilon})$ is approximated by the following first order Taylor expansion with respect to (u_i, v_i)

$$\Delta\theta_m = -\frac{\dot{c}(\theta_m, \boldsymbol{\epsilon})}{\mathbb{E}\left[\ddot{c}(\theta_m, \boldsymbol{\epsilon})\right]} + o\left(\|u_i\|, \|v_i\|\right) \tag{21}$$

where $u_i = -\dot{c}(\theta_m, \boldsymbol{\epsilon})/\mathbb{E}\left[\ddot{c}(\theta_m, \boldsymbol{\epsilon})\right]$, $v_i = \ddot{c}(\theta_m, \boldsymbol{\epsilon})/\mathbb{E}\left[\ddot{c}(\theta_m, \boldsymbol{\epsilon})\right] - 1$. According to (5), $\dot{c}(\theta_m, \boldsymbol{\epsilon})$ and $\ddot{c}(\theta_m, \boldsymbol{\epsilon})$ can be approximated by $\dot{c}(\theta_m, \boldsymbol{\epsilon}) \approx \boldsymbol{\epsilon}^H\dot{\mathbf{Q}}_m\boldsymbol{\epsilon}$ and $\ddot{c}(\theta_m, \boldsymbol{\epsilon}) \approx \boldsymbol{\epsilon}^H\ddot{\mathbf{Q}}_m\boldsymbol{\epsilon}$ respectively. Noting that $\mathbb{E}\left[\left(\boldsymbol{\epsilon}^H\mathbf{Q}\boldsymbol{\epsilon}\right)\right] = trace\left(\mathbf{Q}\mathbf{R}_{\boldsymbol{\epsilon}}\right)$, the DOA estimation error $\Delta\theta_m$ becomes

$$\Delta\theta_m \approx -\frac{\boldsymbol{\epsilon}^H\dot{\mathbf{Q}}_m\boldsymbol{\epsilon}}{trace\left(\ddot{\mathbf{Q}}_m\mathbf{R}_{\boldsymbol{\epsilon}}\right)} \tag{22}$$

where $\dot{\mathbf{Q}}_m = \dot{\mathbf{Q}}(\theta_m)$ and $\ddot{\mathbf{Q}}_m = \ddot{\mathbf{Q}}(\theta_m)$ and $\mathbf{R}_{\boldsymbol{\epsilon}} = \mathbb{E}[\boldsymbol{\epsilon}\boldsymbol{\epsilon}^H]$. As the hermitian form $\boldsymbol{\epsilon}^H\dot{\mathbf{Q}}_m\boldsymbol{\epsilon}$ is Gaussian due to the Lyapunov Central Limit Theorem according to [2], the approximation (22) of $\Delta\theta_m$ is also Gaussian.

4.1 First resolution definition

In this approach firstly introduced in [19], the resolution of two sources of DOAs θ_1 and θ_2 is defined by

$$\ddot{c}\left(\bar{\theta}, \boldsymbol{\epsilon}\right) < 0. \tag{23}$$

This condition means that the MUSIC null spectrum presents a negative concavity between θ_1 and θ_2, ensuring the presence of two minima. Using (5), the approximate expression for $\ddot{c}\left(\bar{\theta}, \boldsymbol{\epsilon}\right)$ is

$$\ddot{c}\left(\bar{\theta}, \boldsymbol{\epsilon}\right) = \boldsymbol{\epsilon}^H\ddot{\bar{\mathbf{Q}}}\boldsymbol{\epsilon}$$

$$\ddot{\bar{\mathbf{Q}}} = \mathbf{Q}\left(\mathbf{a}\left(\bar{\theta}\right), \ddot{\mathbf{a}}\left(\bar{\theta}\right)\right) + \mathbf{Q}\left(\ddot{\mathbf{a}}\left(\bar{\theta}\right), \mathbf{a}\left(\bar{\theta}\right)\right) + 2\,\mathbf{Q}\left(\dot{\mathbf{a}}\left(\bar{\theta}\right), \dot{\mathbf{a}}\left(\bar{\theta}\right)\right)$$

where $\dot{\mathbf{a}}\left(\bar{\theta}\right)$, $\ddot{\mathbf{a}}\left(\bar{\theta}\right)$ are, respectively, the first and second derivatives of $\mathbf{a}(\theta)$ evaluated at $\bar{\theta}$. The resolution condition given by (23) is therefore expressed as

$$\eta_1 < 0 \text{ with } \eta_1 = \boldsymbol{\epsilon}^H\ddot{\bar{\mathbf{Q}}}\boldsymbol{\epsilon}$$

where η_1 is a real random variable. The resolution probability $Pr = \Pr(\eta_1 < 0)$ has been established in [2] as

$$\begin{aligned}Pr &= \Pr(\eta_1 < 0) = \gamma_N\left(u^0, -\frac{\mathbb{E}[\eta_1]}{\sqrt{C_{\eta_1\eta_1}}}\right) \\ &= \int_{-\infty}^{-\mathbb{E}[\eta_1]/\sqrt{C_{\eta_1\eta_1}}} \frac{\exp\left(-u^2/2\right)}{\sqrt{2\pi}}du\end{aligned}$$

The conditional bias $\mathbb{E}[\Delta\theta_m \mid \eta_1 < 0]$ is

$$\mathbb{E}[\Delta\theta_m \mid \eta_1 < 0] = -\frac{C_{\eta_1\Delta\theta_m}}{\sqrt{C_{\eta_1\eta_1}}}\Psi_{\eta_1}(0) + \mathbb{E}[\Delta\theta_m] \tag{24}$$

using (16) and(19), the associated RMS error $\mathbb{E}[\Delta\theta_m^2 \mid \eta_1 < 0]$ is given by

$$\begin{aligned} \mathbb{E}\left[(\Delta\theta_m)^2 \mid \eta_1 < 0\right] = & -\frac{\left(C_{\eta_1\Delta\theta_m}\right)^2}{\left(C_{\eta_1\eta_1}\right)^{3/2}} \Psi_{\eta_1\eta_1}(0) + C_{\Delta\theta_m\Delta\theta_m} \\ & + \mathbb{E}[\Delta\theta_m \mid \eta_1 < 0]^2 \end{aligned} \tag{25}$$

according to (18) and (20) where

$$\begin{aligned} \Psi_{\eta_1\eta_1}(0) &= \Psi_{\eta_1}(0)\left(\sqrt{C_{\eta_1\eta_1}}\Psi_{\eta_1}(0) - \mathbb{E}[\eta_1]\right) \\ \Psi_{\eta_1}(0) &= \frac{(1/\sqrt{2\pi})\exp\left(-\frac{\mathbb{E}[\eta_1]^2}{C_{\eta_1\eta_1}}\right)}{\Pr(\eta_1 < 0)} \end{aligned}$$

The above incomplete moments require the following results

$$\begin{aligned} C_{\Delta\theta_m\Delta\theta_m} &= \frac{F_2(\dot{\mathbf{Q}}_m, \dot{\mathbf{Q}}_m) - \left(trace(\dot{\mathbf{Q}}_m\mathbf{R}_\epsilon)\right)^2}{trace\left(\ddot{\mathbf{Q}}_m\mathbf{R}_\epsilon\right)^2} \qquad (26) \\ C_{\eta_1\Delta\theta_m} &= \frac{trace\left(\dot{\mathbf{Q}}_m\mathbf{R}_\epsilon\right) trace\left(\ddot{\bar{\mathbf{Q}}}\mathbf{R}_\epsilon\right) - F_2\left(\dot{\mathbf{Q}}_m, \ddot{\bar{\mathbf{Q}}}\right)}{trace\left(\ddot{\mathbf{Q}}_m\mathbf{R}_\epsilon\right)} \\ C_{\eta_1\eta_1} &= F_2\left(\ddot{\bar{\mathbf{Q}}}, \ddot{\bar{\mathbf{Q}}}\right) - trace\left(\ddot{\bar{\mathbf{Q}}}\mathbf{R}_\epsilon\right)^2 \\ \mathbb{E}[\Delta\theta_m] &= -\frac{trace\left(\dot{\mathbf{Q}}_m\mathbf{R}_\epsilon\right)}{trace\left(\ddot{\mathbf{Q}}_m\mathbf{R}_\epsilon\right)} \\ \mathbb{E}[\eta_1] &= trace\left(\ddot{\bar{\mathbf{Q}}}\mathbf{R}_\epsilon\right) \end{aligned}$$

where $\Delta\theta_m$ is approximated by $-\boldsymbol{\epsilon}^H\dot{\mathbf{Q}}_m\boldsymbol{\epsilon}/trace\left(\ddot{\mathbf{Q}}_m\mathbf{R}_\epsilon\right)$ according to (22) and the expression of $F_2(\mathbf{A}, \mathbf{B}) = \mathbb{E}\left[\left(\boldsymbol{\varepsilon}^H\mathbf{A}\boldsymbol{\varepsilon}\right)\left(\boldsymbol{\varepsilon}^H\mathbf{B}\boldsymbol{\varepsilon}\right)\right]$ is

$$\begin{aligned} F_2(\mathbf{A}, \mathbf{B}) = & \; trace(\mathbf{A}\ \mathbf{R}_\epsilon)\, trace(\mathbf{B}\ \mathbf{R}_\epsilon) \\ & + trace(\mathbf{A}\ \mathbf{R}_\epsilon\mathbf{B}\ \mathbf{R}_\epsilon) - 2\ [\mathbf{A}]_{11}[\mathbf{B}]_{11} \\ & + trace(\mathbf{A}^T\mathbf{C}_\epsilon^*\mathbf{B}\ \mathbf{C}_\epsilon) \end{aligned} \tag{27}$$

Above, $[\mathbf{A}]_{11}$ is the upper left element of the matrix $\mathbf{A}$, $\mathbf{R}_\epsilon = \mathbb{E}[\boldsymbol{\epsilon}\boldsymbol{\epsilon}^H]$ and $\mathbf{C}_\epsilon = \mathbb{E}[\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T]$. The proof is given in [2].

4.2 Second resolution definition

In this second approach, similar to Zhang [16] in the case of finite number of snapshots without modelling error, two sources of DOAs θ_1 and θ_2 are considered resolved if

$$\frac{c(\theta_1, \boldsymbol{\epsilon}) + c(\theta_2, \boldsymbol{\epsilon})}{2} < c\left(\bar{\theta}, \boldsymbol{\epsilon}\right). \tag{28}$$

Using (5), $c(\theta, \boldsymbol{\epsilon})$ is approximated by $c(\theta, \boldsymbol{\epsilon}) \approx \boldsymbol{\epsilon}^H \mathbf{Q}(\mathbf{a}(\theta), \mathbf{a}(\theta))\boldsymbol{\epsilon}$. Denoting $\mathbf{Q}_m \stackrel{def}{=} \mathbf{Q}(\mathbf{a}(\theta_m), \mathbf{a}(\theta_m))$, $\bar{\mathbf{Q}} \stackrel{def}{=} \mathbf{Q}\left(\mathbf{a}(\bar{\theta}), \mathbf{a}(\bar{\theta})\right)$, the resolution condition given by (28) can be rewritten as

$$\begin{aligned} \eta_1 < 0 \text{ with } \eta_1 = \boldsymbol{\epsilon}^H\tilde{\mathbf{Q}}\boldsymbol{\epsilon} \\ \tilde{\mathbf{Q}} = \frac{\mathbf{Q}_1 + \mathbf{Q}_2}{2} - \bar{\mathbf{Q}}. \end{aligned}$$

Again, assuming η_1 is Gaussian distributed, the conditional bias and RMS error according to the second resolution definition is obtained directly from (24)-(26) where the matrix $\ddot{\bar{\mathbf{Q}}}$ must be replaced by $\tilde{\mathbf{Q}}$.

5. SIMULATIONS

In this simulation we use a N=5 sensors uniform circular array of radius R=λ (wavelength), in which the 3 dB beamwidth is $\Delta\theta = \pm 14°$. Two sources, separated less than one beamwidth are impinging on the antenna; the first from the direction $\theta_1 = 0°$ and the second from $\theta_2 = -5.84°$. All simulations are conducted with 4000 independent trials. The random modeling errors of these simulations are circularly symmetric Gaussian distributed with $\mathbb{E}[\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_j^H]$=$\delta_{i-j}$ $\sigma^2\mathbf{I}_N$ and $\mathbb{E}[\boldsymbol{\epsilon}_i]$=0. In the simulations, two sources of direction θ_1 and θ_2 are said to be resolved when the criterion function has two minima $\hat{\theta}_1$ and $\hat{\theta}_2$, verifying $c(\hat{\theta}_m, \epsilon) < \eta N$ (Assuming that $\|\mathbf{a}(\theta)\|$ is normalized to N). The coefficient η is fixed to 0.1, following practical considerations concerning the ambiguity property of the uniform circular array of radius R=λ with N=5 sensors used for simulation. This process avoids angular estimation associated to array outliers (due to quasi-ambiguities).

In Figure 1, the theoretical probabilities of resolution with the definitions (23) and (28) are compared to the empirical blind one. The empirical is closer to the 2^{nd} definition and is between both theoretical predictions.

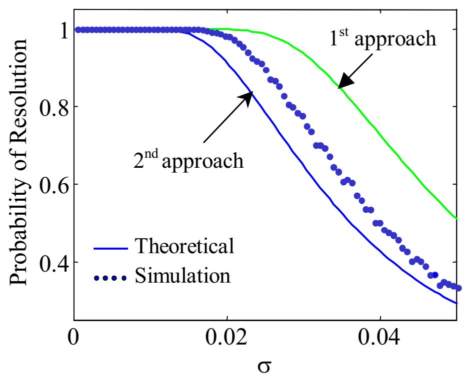

Figure 1: Probability of resolution for the first source versus the level σ of Gaussian modelling errors. $\theta_2 - \theta_1 = -5.84°$.

In Figures 2-3, the bias and RMS error conditioned on empirical resolution are compared to the corresponding theoretical result using the two resolution definitions, and to the second order unconditional performance results ("*classical*") recently introduced in [9]. The first order unconditional performance, traditionally used, ([7, 10]) gives RMS error linear with respect to σ and null bias. In the particular scenario of these simulations, the first and second order unconditional performances gives the same RMS error but two different solution for bias. If $\sigma < 0.02$, where the probability of resolution is higher than 0.9, the theoretical performances are very close to the empirical one. When $0.02 < \sigma < 0.04$, the RMS error of the 1^{st} definition is the closest and for the bias it is the 2^{nd} . As the classical performance, which are unconditional, take into account the unresolved case, they give an incorrect result in terms of bias. In contrast, the proposed conditional performance results give a good prediction of bias and RMS error.

6. CONCLUSION

This paper applies two resolution definitions from [2], where the associated probability of resolution is derived. For each definition, we derive the bias and variance of the MUSIC DOA estimates given that the sources are resolved. To the best of our knowledge, a theoretical expression of the condi-

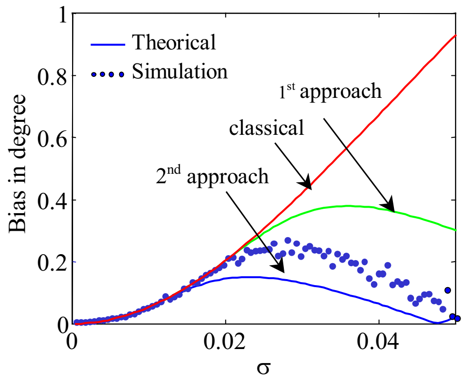

Figure 2: Empirical conditional **Bias** for the first source versus the level σ of Gaussian modelling errors. $\theta_2 - \theta_1 = -5.84°$.

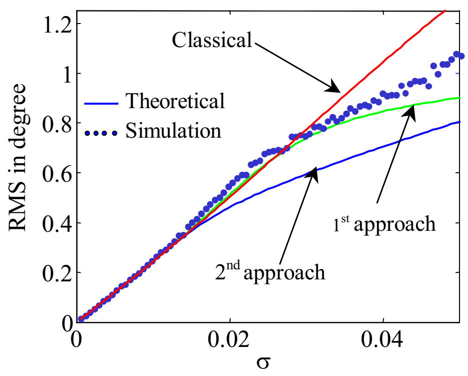

Figure 3: Empirical conditional **RMS** error for the first source versus the level σ of Gaussian modelling errors. $\theta_2 - \theta_1 = -5.84°$.

tional (resolution) bias and variance is available for the first time. Simulations show that the resolution condition that best matches empirical results is somewhat between the definitions $(c(\theta_1, \boldsymbol{\epsilon}) + c(\theta_2, \boldsymbol{\epsilon})) < 2\ c\left(\bar{\theta}, \boldsymbol{\epsilon}\right)$ and $\ddot{c}\left(\bar{\theta}, \boldsymbol{\epsilon}\right) < 0$. Using these criteria and our proposed theoretical resolution results leads to a practical tool for setting the calibration requirements for an experimental bearing estimation system.

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