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THE $\mathbb{H}$-ANALYTIC SIGNAL

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ABSTRACT

We consider the extension of the analytic signal concept known for real valued signals to the case of complex signals. This extension is based on the Quaternion Fourier Transform (QFT) and leads to the so-called $\mathbb{H}$-analytic signal. After defining the $\mathbb{H}$-analytic signal and giving some of its properties, we present a new notation for quaternions, named the polar Cayley-Dickson form, which allows the extension of instantaneous phase and amplitude for the $\mathbb{H}$-analytic signal. Identification of the components of a complex signal are then performed through the analysis of its $\mathbb{H}$-analytic signal. We illustrate these new ideas on simulations.

1. INTRODUCTION

The definition of an analytic signal for general complex signals is still an open question. When considering complex signals, the class of *proper* (or analytic, in the sense originally stated by [1]) signals contain the signals with real and imaginary parts having the same amplitude and being decorrelated, while the *improper* class contains the remaining complex signals. While the *proper* signal can be identified as the analytic signal (in the sense defined by Ville [1]) of a real signal (in fact, its real part), the *improper* signal has no such link with real signals. However, *improper* signals arise in different areas in signal processing such as communications, for example [2, 3, 4]. The aim of this paper is to propose an extension of the analytic signal concept for *improper* complex signals, and this requires the use of a Quaternion Fourier Transform. It must be noticed that previous extensions of the analytic signal concept already exist [5, 6], some based on Quaternion Fourier Transforms as well, but they all considered multidimensional real signals, while our approach here is about complex signals.

In previous work [7], Sangwine and Le Bihan proposed the use of the biquaternion Fourier Transform [8] to define a hyperanalytic signal. This previous approach was motivated by the definition of the complex envelope which had most of the "classical" properties and thus was an obvious candidate for the extension of the analytic signal to complex signals. In this paper, we demonstrate that the Quaternion Fourier Transform, as defined in [9], is "sufficient" to construct the so-called $\mathbb{H}$-analytic signal. Provided that the axis of the Quaternion Fourier Transform is correctly chosen, it is possible to construct the $\mathbb{H}$-analytic signal which exhibits the same properties as the "classical" analytic signal. In order to extend the concept of instantaneous phase and amplitude to *improper* complex signals, we also introduce a new quaternion representation, named the *polar Cayley-Dickson* form, which is helpful to interpret the $\mathbb{H}$-analytic signal. Simulations illustrate the concept introduced in this paper.

2. PRELIMINARY CONCEPTS

We present here some useful concepts used in the definition of the $\mathbb{H}$-analytic signal.

2.1 Quaternions

We review shortly some facts about quaternions. Details can be found for example in [10]. A quaternion q is a 4D hypercomplex number classicaly written in its Cartesian form as: $q = a + b\mathtt{i} + \mathtt{j}c + \mathtt{k}d$, where $a, b, c, d \in \mathbb{R}$ are its components and where $\mathtt{i}, \mathtt{j}$ and $\mathtt{k}$ are roots of -1 and multiply together like: $\mathtt{ij} = -\mathtt{ji}$ and $\mathtt{ijk} = -1$. The norm of q is $|q| = (a^2 + b^2 + c^2 + d^2)^{\frac{1}{2}}$, its conjugate is $\overline{q} = a - b\mathtt{i} - \mathtt{j}c - \mathtt{k}d$ and its inverse is $q^{-1} = \overline{q}/|q|^2$. Any quaternion q can be expressed in the polar form: $q = |q|(\sin(\theta) + \mu\cos(\theta)) = |q|\exp(\mu\theta)$. Another notation, called Cayley-Dickson notation, represents a quaternion as a complex number with complexified components (with a different imaginary unit), the following way: $q = s + r\mathtt{j}$ where $s = a + \mathtt{i}b$ and $r = c + \mathtt{i}d$. A quaternion is called unitary if $|q| = 1$ and any unitary quaternion can be written as: $\exp(\mu\theta)$. A pure quaternion q is such that $a = 0$. A pure unit quaternion is a square root of -1. A quaternion basis is a 4D basis such as $\{1, \mu, \xi, \mu\xi\}$ where μ, ξ are two orthogonal pure unit quaternions[1]. Over the set of quaternions $\mathbb{H}$, it is possible to define some involutions and we present here one of use in the sequel. Given a quaternion $q \in \mathbb{H}$ and a pure unit quaternion $p \in \mathbb{H}$, then $q_p = -pqp$ is an involution[2]. Such involutions are useful in quaternion components identification, see for example [5]. More on quaternion involutions can be found in [11].

2.2 Generalized Cayley-Dickson form

Consider a quaternion valued signal $q(t)$ that can be expressed in a (generalized) Cayley-Dickson form:

$$q(t) = z_1(t) + z_2(t)\mu \tag{1}$$

where $z_1(t) = \Re(z_1(t)) + \xi\Im(z_1(t))$, $z_2(t) = \Re(z_2(t)) + \xi\Im(z_2(t))$ are complex signals and $\{1, \mu, \xi, \mu\xi\}$ is a quaternion basis. Such signals are representative of polarized signals for example [12, 13]. The two components of the Cayley-Dickson decomposition can be expressed as follows:

$$\begin{cases} z_1(t) = \dfrac{1}{2}\left(q_\xi(t) + q(t)\right) \\ \\ z_2(t) = \dfrac{1}{2}\left(q_\xi(t) - q(t)\right)\mu \end{cases} \tag{2}$$

[1] Among all the possible quaternion basis, the most widely used is $\{1, \mathtt{i}, \mathtt{j}, \mathtt{k}\}$

[2] Involution means here that $(q_p)_p = q$ and $(qm)_p = q_p m_p$ where $q, m \in \mathbb{H}$

where $q_\xi(t) = -\xi q(t)\xi$. In this notation, $z_1(t)$ is called the simplex part of $q(t)$ while $z_2(t)$ is called the perplex part of $q(t)$ (see [9] for details). Thus, any quaternion signal can be seen as a pair of complex signals in any quaternion basis.

2.3 Quaternion Fourier transform

Here, we present an important property of the Quaternion Fourier Transform (QFT). We make use of the right QFT defintion given given by Sangwine and Ell in [9].

Consider the complex signal $z(t) = \Re(z(t)) + \xi\Im(z(t))$, *i.e.* $z(t)$ takes values in $\mathbb{C}^\xi$. Before trying to build its $\mathbb{H}$-analytic signal, we examine how it is transformed using a QFT of axis μ (noted as QFT_μ in the sequel), when $(1,\xi,\mu,\xi\mu)$ is a quaternion basis. The QFT_μ of $z(t)$ is thus:

$$\begin{aligned} Z(\nu) &= QFT_\mu[z(t)] = \int_{-\infty}^{+\infty} z(t)e^{-\mu 2\pi\nu t}dt \\ &= \int_{-\infty}^{+\infty} \Re(z(t))[\cos(2\pi\nu t) - \mu\sin(2\pi\nu t)]dt \\ &\quad +\xi\int_{-\infty}^{+\infty} \Im(z(t))[\cos(2\pi\nu t) - \mu\sin(2\pi\nu t)]dt \\ &= \int_{-\infty}^{+\infty} \Re(z(t))\cos(2\pi\nu t)dt \\ &\quad -\mu\int_{-\infty}^{+\infty} \Re(z(t))\sin(2\pi\nu t)dt \\ &\quad +\xi\int_{-\infty}^{+\infty} \Im(z(t))\cos(2\pi\nu t)dt \\ &\quad -\xi\mu\int_{-\infty}^{+\infty} \Im(z(t))\sin(2\pi\nu t)dt \end{aligned} \tag{3}$$

This last equality shows that the QFT_μ of $z(t)$ naturally makes the following decomposition/association:

- Even part of $\Re(z(t)) \longrightarrow \Re(Z(\nu))$.
- Odd part of $\Re(z(t)) \longrightarrow \Im_\mu(Z(\nu))$
- Even part of $\Im(z(t)) \longrightarrow \Im_\xi(Z(\nu))$.
- Odd part of $\Im(z(t)) \longrightarrow \Im_{\xi\mu}(Z(\nu))$

where $\Im_\eta$ (when η is a pure unit quaternion) stands for the η imaginary component of the quaternion. So, the QFT_μ of a complex signal $z(t) = \Re(z(t)) + \xi\Im(z(t))$ allows us to isolate the odd and even parts of its real and imaginary parts in the four different components of its $Z(\nu)$. This point guarantees that the symmetries of the real and imaginary parts are not mixed.

Now, consider two functions g and f such that: $g : \mathbb{R} \to \mathbb{C}$ and $f : \mathbb{R} \to \mathbb{R}$. Then, consider the QFT of their convolution:

$$\begin{aligned} QFT_\mu[g * f(t)] &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(\tau)f(t-\tau)d\tau e^{-2\mu\pi\nu t}dt \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(\tau)e^{-\mu 2\pi\nu(t'+\tau)}f(t')d\tau dt' \\ &= \int_{-\infty}^{+\infty} g(\tau)e^{-2\mu\pi\nu\tau}d\tau \int_{-\infty}^{+\infty} f(t')e^{-2\mu\pi\nu t'}dt' \end{aligned} \tag{4}$$

and so:

$$\begin{aligned} QFT_\mu[g * f(t)] &= QFT_\mu[g(t)]QFT_\mu[f(t)] \\ &= QFT_\mu[f(t)]QFT_\mu[g(t)] \end{aligned} \tag{5}$$

Thus, the definition of the QFT we use here has the property of "verifying" the convolution theorem in the considered case of functions g and f. This will be of use for the extension of the analytic signal (definition of the Hilbert transform). Futhermore, the QFT of $f(t) = \frac{1}{\pi t}$ is given by:

$$F(\nu) = -\mu\,\mathrm{sign}(\nu) \tag{6}$$

This is obvious from the possiblity of calculating the QFT with axis μ from two complex Fourier transforms in an appropriate basis [9]. Here, as f is real valued, the change of basis has no effect.

3. THE $\mathbb{H}$-ANALYTIC SIGNAL

We now give the definition and properties of the $\mathbb{H}$-analytic signal based on the QFT.

3.1 Definition and properties

The $\mathbb{H}$-analytic signal of $z(t)$ presented here has been worked out with an approach similar to the one originally developed by Ville [1]. The following definitions give the details of the construction of this signal. Note that the signal $z(t)$ is considered to be an *improper* complex signal, *i.e.* for example $\Re(z(t))$ and $\Im(z(t))$ are not orthogonal.

Definition 1. *Consider a complex signal $z(t) = \Re(z(t)) + \xi\Im(z(t))$ and its Quaternion Fourier Transform $Z(\nu)$ given by:*

$$Z(\nu) = QFT_\mu[z(t)] = \int_{-\infty}^{+\infty} z(t)e^{-\mu 2\pi\nu t}dt \tag{7}$$

where μ, the axis of the transform, is taken such that $(1,\xi,\mu,\xi\mu)$ is a quaternion basis. Then, the "Hilbert transform" of $z(t)$, noted $z_\mathrm{h}(t)$, has the following QFT_μ:

$$Z_\mathrm{h}(\nu) = -\mu\ \mathrm{sign}(\nu)Z(\nu) \tag{8}$$

where the Hilbert transform is defined as: $HT[z(t)] = p.v.\left(z * \dfrac{1}{\pi t}\right)$

The principal value (p.v.) is understood in its classical sense here (see [1] for example).

This definition of the Hilbert transform based on QFT_μ is derived thanks to the convolution property given in Section 2.3.

Definition 2. *Given a complex valued signal $z(t)$ that can be expressed as $z(t) = \Re(z(t)) + \xi\Im(z(t))$, and given a pure unit quaternion μ such that $(1,\xi,\mu,\xi\mu)$ is a quaternion basis, then the $\mathbb{H}$-analytic signal of $z(t)$, noted $z_\mathrm{a}(t)$ is given by:*

$$z_\mathrm{a}(t) = z(t) + z_\mathrm{h}(t)\mu \tag{9}$$

where $z_\mathrm{h}(t)$ is the "Hilbert transform" of $z(t)$ given in Definition 1. The QFT of the $\mathbb{H}$-analytic signal is thus:

$$Z_\mathrm{a}(\nu) = Z(\nu) - \mu\ \mathrm{sign}(\nu)Z(\nu)\mu \tag{10}$$

which is a direct extension of the "classical" analytic signal.

With this definition of the $\mathbb{H}$-analytic signal given in Definition 2, we now investigate some of its properties.

Property 1. *The spectrum of the* $\mathbb{H}$*-analytic signal is right-sided,* i.e. $Z_{\mathrm{a}}(\nu) = 0,\ \forall\ \nu < 0$.

Proof. The QFT of $z_{\mathrm{a}}(t)$ is given by:

$$\begin{aligned}
Z_{\mathrm{a}}(\nu) &= Z(\nu) - \mu\ \mathrm{sign}(\nu) Z(\nu)\mu \\
&= \int_{-\infty}^{+\infty} z(t) e^{-\mu 2\pi\nu t} dt \\
&\quad -\mu\ \mathrm{sign}(\nu) \left(\int_{-\infty}^{+\infty} z(t) e^{-\mu 2\pi\nu t} dt \right) \mu \\
&= \int_{-\infty}^{+\infty} \Re(z(t)) \cos(2\pi\nu t) dt - \\
&\quad \mu \int_{-\infty}^{+\infty} \Re(z(t)) \sin(2\pi\nu t) dt + \\
&\quad \xi \int_{-\infty}^{+\infty} \Im(z(t)) \cos(2\pi\nu t) dt \\
&\quad -\xi\mu \int_{-\infty}^{+\infty} \Im(z(t)) \sin(2\pi\nu t) dt \\
&\quad -\mu\ \mathrm{sign}(\nu) \left(\int_{-\infty}^{+\infty} \Re(z(t)) \cos(2\pi\nu t) dt \right) \mu \\
&\quad +\mu\ \mathrm{sign}(\nu) \left(\mu \int_{-\infty}^{+\infty} \Re(z(t)) \sin(2\pi\nu t) dt \right) \mu \\
&\quad -\mu\ \mathrm{sign}(\nu) \left(\xi \int_{-\infty}^{+\infty} \Im(z(t)) \cos(2\pi\nu t) dt \right) \mu \\
&\quad +\mu\ \mathrm{sign}(\nu) \left(\xi\mu \int_{-\infty}^{+\infty} \Im(z(t)) \sin(2\pi\nu t) dt \right) \mu
\end{aligned}$$

Noting that μ and ξ commute with all the other terms (sin and cos, sign, $\Re$ and $\Im$) and remembering that $\xi\mu = -\mu\xi$, then the QFT of $z_{\mathrm{a}}(t)$ takes the following simple expression:

$$Z_{\mathrm{a}}(\nu) = (1 + \mathrm{sign}(\nu)) Z(\nu)$$

which completes the proof. □

Property 1 together with the property of the QFT_μ given in section 2.3 show that the $\mathbb{H}$-analytic signal is right-sided and at the same time, keeps the different part of the original signal in different imaginary components of the transform.

Property 2. *The original signal* $z(t)$ *is the* simplex *part of its corresponding* $\mathbb{H}$*-analytic signal* $z_{\mathrm{a}}(t)$*. It is obtained by:*

$$z(t) = \frac{1}{2} \left(z_{\mathrm{a}}(t) - \xi z_{\mathrm{a}}(t) \xi \right)$$

Note that if the original signal $z(t)$ is expressed in the classical complex basis $\{1, \mathrm{i}\}$ and if the axis of the QFT is taken as j, then the $\Im_{\mathrm{j}}$ and $\Im_{\mathrm{k}}$ parts of $z_{\mathrm{a}}(t)$ contain the Hilbert transform of $z(t)$. This property is a direct consequence of the way we have defined the $\mathbb{H}$-analytic signal given in eq. (9), and this allows us to recover the original complex signal from its quaternion valued $\mathbb{H}$-analytic signal. Note that this is the counterpart of the fact that, in the “classical” case, the original real signal is the real part of the analytic signal [1].

Also, note that our definition of the $\mathbb{H}$-analytic signal includes the classical definition of Ville [1] as a special case. If the signal $z(t)$ is real, then the $\mathbb{H}$-analytic signal is simply complex with the imaginary axis being the one chosen for the QFT.

4. THE POLAR CAYLEY-DICKSON FORM

We now look at the definition of the amplitude and phase concepts for the $\mathbb{H}$-analytic signal. In order to so, we introduce a new notation for quaternions. It is different from the classical polar form and the polar form introduced in [5]. It is based on the Cayley-Dickson notation. Details about this new quaternion representation can be found in [14].

Definition 3. *Any quaternion* $q \in \mathbb{H}$ *with Cartesian form as:* $q = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ *can be expressed in a* polar Cayley-Dickson *form:*

$$q = A e^{B\mathrm{j}} \tag{11}$$

where $A = \Re(A) + \mathrm{i}\Im(A) \in \mathbb{C}$ *and* $B = \Re(B) + \mathrm{i}\Im(B) \in \mathbb{C}$.

This form of a quaternion q is the counterpart of the polar form of complex numbers. Here, the modulus and phase are complex valued. A method for finding A and B is detailed in [14].

Now, in the case of the $\mathbb{H}$-analytic signal $z(t)$, its polar Cayley-Dickson form is given as:

$$z_{\mathrm{a}}(t) = A_{\mathrm{a}}(t) e^{B_{\mathrm{a}}(t)\mathrm{j}} \tag{12}$$

The values of the components (as well as the information they provide on the original signal) of this polar Cayley-Dickson form of the $\mathbb{H}$-analytic signal are illustrated in the following section.

5. SIMULATIONS

We illustrate here the $\mathbb{H}$-signal concept on a simple simulation example. Consider a complex signal $z(t)$ made up in the following way:

$$z(t) = f(t).(s_1(t) + \mathrm{i} s_2(t)) \tag{13}$$

where $s_1(t) = \sin(2\pi\nu_1 t)$; and $s_2(t) = \sin(2\pi\nu_2 t + \xi)$ and $f(t) = \sin(2\pi\nu_f t)$ and with $\nu_f > \nu_1 > \nu_2$. The $\mathbb{H}$-analytic signal of $z(t)$, *i.e.* $z_{\mathrm{a}}(t)$, is computed, using j as the axis of the QFT, and expressed in its polar Cayley-Dickson form as in (12).

Then, from the polar Cayley-Dickson form of the $\mathbb{H}$-analytic signal, and remembering that $A_{\mathrm{a}}(t)$ and $B_{\mathrm{a}}(t)$ are complex valued and can be expressed as $A_{\mathrm{a}}(t) = |A_{\mathrm{a}}(t)| \exp\left(\Psi_{A_{\mathrm{a}}(t)}\right)$ and $A_{\mathrm{a}}(t) = |B_{\mathrm{a}}(t)| \exp\left(\Psi_{B_{\mathrm{a}}(t)}\right)$, the following information is available:

$$\left\{ \begin{array}{rcl}
s_1(t) &=& \Re(A_{\mathrm{a}}(t)) \\
s_2(t) &=& \Im(A_{\mathrm{a}}(t)) \\
\Phi_2(t) - \Phi_1(t) &=& \tan\left(\Psi_{A_{\mathrm{a}}(t)}\right) \\
\dfrac{|z(t)|}{|f(t)|} &=& |A_{\mathrm{a}}(t)| \\
f(t) &=& \mp\cos(|B_{\mathrm{a}}(t)|)
\end{array} \right. \tag{14}$$

where $\Phi_2(t)$ and $\Phi_1(t)$ are the instantaneous amplitudes of $s_2(t)$ and $s_1(t)$ respectively. In figure (1) we present, as a function of time, the complex amplitude $A_{z_{\mathrm{a}}}(t)$ as well as the original signal $z(t)$. It shows that the complex envelope of the $\mathbb{H}$-analytic signal, namely $A_{z_{\mathrm{a}}}(t)$ is covering the original signal.

As presented above, $A_{z_{\mathrm{a}}}(t)$ allows to recover parts of the original signal: $\Re\left(A_{z_{\mathrm{a}}}(t)\right) = s_1(t)$ and $\Im\left(A_{z_{\mathrm{a}}}(t)\right) = s_2(t)$. This interesting property could be of interest for example in finding the modulation frequency of an improper complex signal (or a common component shared by the real and imaginary parts of an improper signal), as it allows a simple way of identifying the real and imaginary base band signals (here $s_1(t)$ and $s_2(t)$). This is a consequence of theorem 1 in [14].

In figure (2) the modulus of the original is compared to $|A_{z_a}(t)|$. It can be seen that the modulus of $A_{z_a}(t)$ is the envelope of the modulus of the original signal $z(t)$, which illustrates the concept of instantaneous amplitude to the case of improper complex signals. In Figure (3), the signal $f(t)$ is compared with the cosine of the modulus of the instantaneous complex phase $B_{z_a}(t)$. It can be seen that there is an ambiguity sign on some cycles, however, from an estimation point of view it can be seen that estimation of the frequency of $f(t)$ directly from $\cos(|B_{z_a}(t)|)$ is an easy task. Note that this could be used as a easy estimator of the correlation between real and imaginary components of an improper complex signal $z(t)$. Future work could investigate a comparison with the work proposed in [3]. Finally, in figure (4), the difference between the instantaneous frequencies of $s_1(t)$ and $s_2(t)$ (computed using the classical analytic signal) is compared to the tangent of the phase of the complex envelope, *i.e.* $\tan(\Psi_{A_a}(t))$. The perfect match between the two curves also suggest that it is possible to estimate the *relative instantaneous phase* between the real and imaginary components of an improper complex signal by inspection of the phase of the modulus of its $\mathbb{H}$-analytic signal.

6. DISCUSSION AND CONCLUSIONS

We have introduced a new extension of the concept of analytic signal to the case of improper complex signals. The $\mathbb{H}$-analytic signal is based on the use of the Quaternion Fourier transform. Some of its properties have been presented, that generalize in a straightforwad manner the known results in the "classical" case. In order to access the information provided by the $\mathbb{H}$-analytic signal, we have introduced a new representation for quaternions and linked the components of this representation to useful information on the improper original signal. In particular, the $\mathbb{H}$-analytic signal allows direct access to common parts, relative instantaneous frequencies and uncorrelated components of the complex original signal. Applications of the $\mathbb{H}$-analytic signal may be expected in the numerous applications dealing with improper complex signals. In particular, some estimators of the mentioned caracteristics of the improper signal could be based on the $\mathbb{H}$-analytic signal and allow fast identification of, for example, the parameters of an unknown improper complex signal. Such estimators should be compared to existing work. Also, the possible definition of a $\mathbb{H}$-analytic signal for improper complex signals suggests the possiblity of defining some time-frequency representations for such signals, based on the quaternion Fourier transform.

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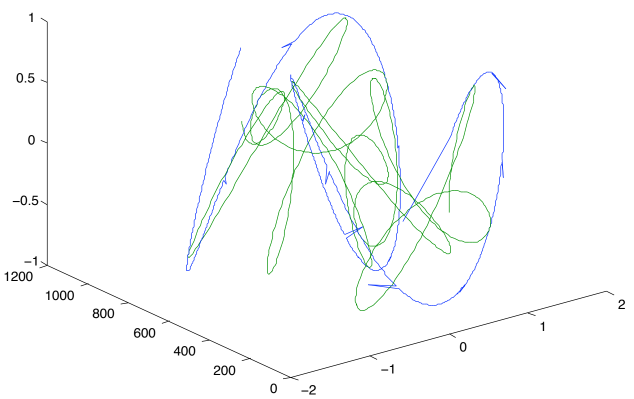

Figure 1: Original complex signal $z(t) = f(t).(s_1(t) + \mathrm{i}s_2(t))$ (green) and complex envelope $A_{z_a}(t)$ (blue).

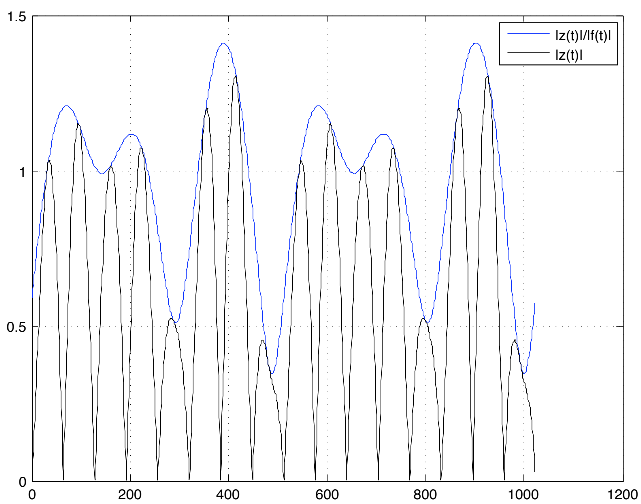

Figure 2: Modulus of the original signal $z(t)$ (black) and modulus of the complex envelope $A_{z_a}(t)$ (blue).

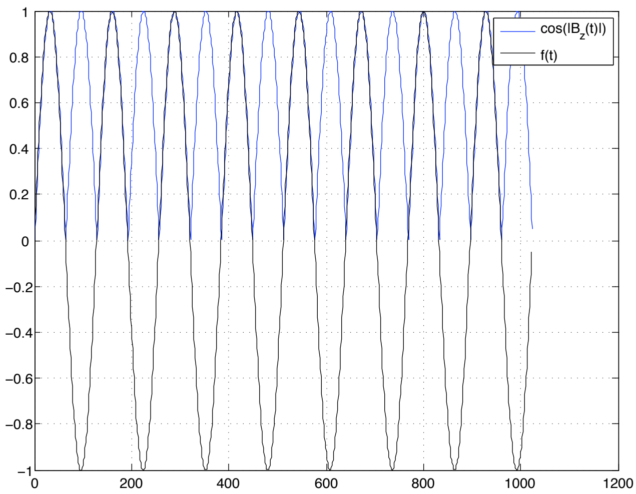

Figure 3: Cosine of the modulus of the complex phase of the $\mathbb{H}$-analytic signal $|B_{z_a}(t)|$ (blue) and the $f(t)$ signal (black).

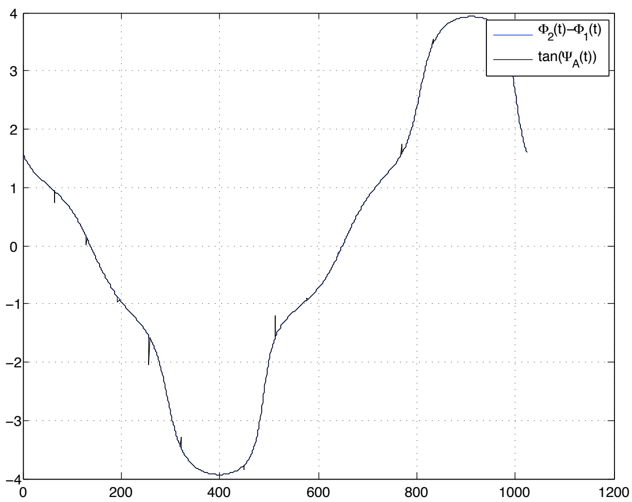

Figure 4: Tangent of the phase of the complex modulus of the $\mathbb{H}$-analytic signal $A_{z_a}(t)$ (blue) and the difference of instantaneous phases between original signals $s_1(t)$ and $s_2(t)$, namely $f\Phi_2(t) - \Phi_1(t)$ (black).

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