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ENHANCING ERROR LOCALIZATION OF DFT CODES BY WEIGHTED ℓ_1-NORM MINIMIZATION

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ABSTRACT

We consider the problem of decoding of real BCH discrete Fourier transform codes (RDFT) which are considered for joint source channel codes to provide robustness against errors in communication channels. In this paper, we propose to combine the subspace based algorithm like MUSIC algorithm with ℓ_1-norm minimization algorithm, which is promoted as a sparsity solution functional, to enhance the error localization of RDFT codes. Simulation results show that the combined algorithm performs better over the performances of these individual algorithms.

1. INTRODUCTION

Error correction coding over real and complex fields, as opposed to finite-fields was introduced by Marshall [1]. Within the class of codes defined for error correction over real fields, the discrete Fourier transform (DFT) codes forms an important class. DFT codes are being considered for joint source channel coding for providing robustness to errors over communication channel [2, 3] and in particular in hiperlan2 [4].

An (N, K) DFT code is a linear block code whose generator matrix consists of any K columns of an IDFT matrix of order N [1]. The parity check matrix consists of the remaining $(N-K)$ columns. Within the class of DFT codes, real number Bose-Chaudhuri-Hocquengem (BCH) DFT codes (RDFT) are possible if the spacing of parity frequencies are relatively prime to N and complex conjugate columns are selected for the generator matrix. The generator matrix of such a code can be defined as $G = \sqrt{(N/K)} W_N^h \Sigma W_K$, and the parity check matrix as $H = W_{N\times d}^h$, where W_N^h denotes inverse DFT matrix of order N, Σ denotes an $N \times K$ binary matrix whose nonzeros elements are $\Sigma_{00} = 1, \Sigma_{ii} = \Sigma_{N-i,k-i} = 1$ for $i = 1, ..., (K-1)/2$, and $d = (N-K)$ which denotes the indexes $(K+1)/2, ..., N-(K+1)/2$ of an IDFT matrix of order N [5]. A RDFT code is a maximum distance separable code. The minimum distance of this code is $d+1$ and hence it can correct up to $\lfloor d/2 \rfloor$ errors.

Several algorithms have been proposed for decoding of DFT codes [3] - [8]. Rioul [6] modified the Peterson - Gorenstein - Zeiler (PGZ) algorithm which is used for decoding RS codes over finite fields, for decoding real BCH codes in the presence of background noise. The authors in [3], [7] showed that the problem of error correction of RDFT codes is analogous to the problem of complex sinusoidal estimation. Subspace based approaches like MUSIC, ESPRIT were applied in [5] for estimating these complex frequencies. All these proposed algorithms decode perfectly under no background noise. But in the presence of background noise like the quantization noise which affects all locations, the decoding is not exact and depends on the algorithm. In general, subspace based algorithms performs better compared to other algorithms in the presence of noise.

The problem of sampling and recovering sparse signals [9, 10] is analogous to the problem of decoding RDFT codes. The techniques that have been proposed by using the annhilating filter in [10] and by ℓ_1-norm minimization in [11] can also be used for decoding RDFT codes. However in the presence of the noise these techniques perform worse than the subspace based techniques.

Recently it is shown in [12] that using weighted ℓ_1-norm minimization, which is achieved by the application of an appropriate weighting matrix, the performance of ℓ_1-norm minimization can be enhanced, and the enhancement is significant. The weighting should reflect the solution, for which an *a priori* information is required.

In this paper we propose to use a two step algorithm. In the first step a subspace based algorithm like MUSIC will be applied. An application of *a posteriori* test followed by MUSIC algorithm will confirm whether the decoding is proper. If the *a posteriori* test fails then in the second step a sub matrix of the parity check matrix will be formed, which includes those columns which may correspond to the potential error locations. This sub matrix will be formed with the help of the *a priori* output obtained by the MUSIC algorithm. A weighted ℓ_1-norm minimization algorithm will be applied upon this sub matrix which will be of a smaller dimension than the original parity check matrix. Simulation results in Section 4 shows that the proposed two step algorithm performs much better than the performance obtained by the application of each of these individual algorithms.

The organization of the paper is as follows: In Section 2 we give a brief overview of the subspace algorithm for error localization, Section 3 describes the proposed two step algorithm, simulation results are provided in Section 4 and finally, Section 5 concludes the paper.

2. MUSIC LIKE SUBSPACE BASED ALGORITHM: A REVIEW

In this section we briefly review the MUSIC like subspace based algorithm for decoding RDFT codes [5]. Classically, the decoder works in three steps. i) Detect the numbers of errors, ii) Localize the error locations, iii) Apply corrections to these locations. Let $\mathbf{r} = \mathbf{c} + \mathbf{e}$ denote the received codeword, where $\mathbf{c}$ denotes the codeword generated with a message vector $\mathbf{m}$ as $\mathbf{c} = G\mathbf{m}$, and $\mathbf{e}$ denotes the combined impulse channel errors which affects few locations and quantization error which affects all locations. The syndrome vector $\mathbf{s}$ is computed as

$$\mathbf{s} = H^h\mathbf{r} = H^h(\mathbf{c}+\mathbf{e}) = H^h\mathbf{e} \tag{1}$$

The syndrome depends only on the error.

The syndrome covariance matrix is defined as $R_m = \frac{1}{(d-m+1)} S_m S_m^h$, where S_m is a matrix of syndromes [5, pg. 2117]. It can be shown that the rank of S_m is equal to ν, the number of impulse channel errors under no quantization error [5]. R_m is a hermitian matrix of size $m \times m$,$\nu + 1 \leq m \leq d - \nu + 1$. R_m can be decomposed using eigenvalue decomposition as

$$R_m = U_\nu \Lambda_\nu U_\nu^h + U_e \Lambda_e U_e^h \tag{2}$$

where Λ_ν is a $\nu \times \nu$ diagonal matrix having ν largest eigenvalues and U_ν is a $m \times \nu$ eigenvector matrix corresponding to the ν eigenvalues. The columns of U_ν can be shown to span the channel error subspace [5] in the case when there is no quantization noise.
Let $V^{m \times \nu}$ denote a error locator matrix whose k^{th} column is given by $[1, \omega_k, \omega_k^2, ..., \omega_k^{m-1}]$, $\omega_k = \exp(-2\pi jk/N)$ and $k = 1, 2, ..., \nu$. The columns of V_x defines the ν dimensional channel error subspace on a space of dimension m. Let $\phi(x)$ denote the following function

$$\phi(x) = \mathbf{v}_x^h U_\nu U_\nu^h \mathbf{v}_x \tag{3}$$

where $\mathbf{v}_x$ denotes the x^{th} column of V. When there is no quantization noise, this function goes to zero at error locations. In the presence of quantization noise, the error locations can be found out by minimizing the above equation over the set of N^{th} roots of unity. Knowing the error locations, the error values say $\hat{\mathbf{e}}_{mu}$, can be found out by solving eq.(1) in the least square sense by retaining only those columns of H^h, which corresponds to error locations.

MUSIC algorithm performs better for fewer number of errors and for more errors the performance is better only at higher channel to quantization noise power (CNR). The effect on error localization performance at moderate CNR can be observed in fig.(1), where we plot $|\phi(x)|$ versus the position x of a $(18, 9)$ RDFT code. The error locations at 2, 4 and 10, which are marked by small circles correspond to the actual impulse channel error locations. Because of the quantization noise, $|\phi(x)|$ do not go to zeros at these channel locations. Indeed at positions $x \in \{2, 3, 4\}$ we observe that $|\phi(x)|$ are almost equal and close to zero, which results in an ambiguity for the decoding algorithm to select the proper error locations. Blindly selecting three locations will end up in the selection of locations 2, 3 and 10. With these wrong error localizations, the distortion in the decoder will increase due to the additional errors being introduced.

To avoid this kind of ambiguity, we propose to use a two step algorithm. In the first step MUSIC like algorithm will be used and in the next step a weighted ℓ_1-norm minimization algorithm will be used. In the following section we elobarate the proposed approach.

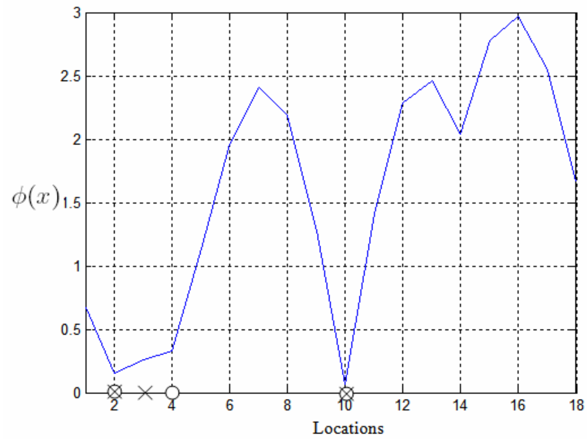

Figure 1: $\phi(x)$ plotted Vs locations for a (18,9) RDFT code. The positions denoted by small circle are the actual error locations, while the positions decoded by the MUSIC algorithm are denoted by X

3. PROPOSED METHOD

The system of linear equations in eq.(1), forms a under determined set i.e., the matrix H^h, which is a $d \times N$ has fewer rows than columns ($d < N$). Such kind of problems has infinitely many solutions and thus, impossible to identify which of the them is indeed the correct solution.

The impulse channel errors affect only few locations and hence most of the values in $\mathbf{e}$ will be zero (or close to zero, because of the quantization noise, which affects all locations). Hence we can assume the sparseness (or approximately sparse) structure of the error vector $\mathbf{e}$. With the assumption of sparseness on $\mathbf{e}$, the solution for eq.(1) can also be obtained by solving the convex optimization problem [9]

$$(P_1) \quad \min_{\hat{\mathbf{e}} \in \mathbb{R}^N} \|\hat{\mathbf{e}}\|_{\ell_1} \quad \text{subject to} \quad \|H^{\mathrm{h}}\hat{\mathbf{e}} - \mathbf{s}\|_{\ell_2} \leq \epsilon \tag{4}$$

where the factor ϵ bounds the amount of quantization noise in $\mathbf{e}$.

Intuitively, performance of (P_1) can be enhanced, by incorporating any *a priori* knowledge about the impulse channel error locations. In this paper we derive the *a priori* information from the MUSIC algorithm. The proposed weighted ℓ_1-norm minimization method can be outlined as follows:
Step 1:Apply a threshold 'δ' on $|\phi(x)|$ for all $x \in \{1, 2, ..., N\}$ and form a subset $x_s = \{x : |\phi(x)| < \delta\}$.Let

the $|x_s|$ be β.
Step 2: A sub matrix H_s^h is created by deleting those columns of H^h that do not belong to the set x_s.
Step 3: Let W denote a diagonal matrix of size $\beta \times \beta$, whose diagonal elements are $|\phi(x_s)|$. W is known as weighting matrix.
Step 4: Solve the following convex optimization problem:

$$(WP_1) \quad \min_{\mathbf{e}_s \in \mathbb{R}^\beta} \|W\mathbf{e}_s\|_{\ell_1} \quad \text{subject to} \quad \|H_{\mathrm{s}}^{\mathrm{h}}\mathbf{e}_{\mathrm{s}} - \mathbf{s}\|_{\ell_2} \leq \epsilon \tag{5}$$

Step 5: The error locations $\mathbf{e}_{loc} = \{i : |e_{s_i}| > \sigma\}$, σ is a quantity which is proportional to the quantization noise. The decoded error vector $\hat{\mathbf{e}}$ can be obtained by inserting the values of $\mathbf{e}_s$ at the locations from the set x_s onto a zero vector of length of $N \times 1$.

The simulation results in Section 4 shows that (WP_1) performs better than (P_1). The justification for the improved performance can be explained with the help of re-weighted ℓ_1-norm minimization proposed recently [12]. It is shown in [12], the performance of (P_1) can be improved significantly by appropriately weighting the minimizing variable. The weights can be chosen with some *a priori* information, which reflects the solution. In our formulation these *a priori* information are obtained from the MUSIC algorithm. The elimination of columns in our algorithm is equivalent to weighting the eliminated positions by ∞, since we know *a priori*, these positions cannot be error positions. The other positions are weighted by $|\phi(x)|$, which are approximately proportional to the magnitude of the error values.

The performance enhancement of the proposed algorithm comes at the cost of increase in the decoder complexity. Since the proposed method makes use of a second step, which involves solving an optimization program after the MUSIC algorithm, the method is a bit costlier compared to MUSIC. However, the proposed optimization problem (WP_1), operates on a smaller dimension compared to (P_1) and hence is less complex compared to solving (P_1).

Solving (WP_1) is costlier than MUSIC algorithm and in practice, most of the times, the MUSIC algorithm alone would be sufficient to decode, which will eliminate the need for (WP_1) in most cases. Thus the RDFT decoder involving (WP_1) would work in the following two steps: In the first step, a MUSIC like algorithm will be applied to find $\hat{\mathbf{e}}_{mu}$. A *a posteriori* check will be performed by testing the following inequality $\|H^h(\mathbf{r} - \hat{\mathbf{e}}_{mu}) - \mathbf{s}\|_{\ell_2} \leq \epsilon$. If the inequality is satisfied then the decoder declares a success. If the inequality fails then we move to the second step, where we solve the optimization function (WP_1) as outlined above.

4. SIMULATION RESULTS

In order to test the performance of the proposed algorithm, we performed simulations on a $(18, 9)$ RDFT code. This code is capable of detecting and correcting up to four channel errors. In our simulations, we assume the number of channel errors and concentrate only on the error localization performance of these channel error locations in presence of the quantization noise. The impulse channel errors were added manually at random positions. These error magnitudes are generated from a normal distribution $\mathcal{N}(\mu_c, \sigma_c^2)$ with mean μ_c and variance σ_c^2. The standard deviation σ_c is chosen to be $0.25\mu_c$, which ensures that the magnitude of these channel errors generated are within $\pm 0.75\mu_c$ around μ_c with a confidence interval of 99%.

The codewords are uniformly quantized, with a step size of $\sqrt{12}$, which ensures the quantization noise variance (σ_q^2) of unity and zero mean. Statistically, due to the quantization noise, each of the elements in the syndrome vector $\mathbf{s}$ can be assumed to be Gaussian (by central limit theorem, since it is linear sum of few uniformly distributed random variables) with zero mean. The variance of each of the elements of $\mathbf{s}$ is equal to the sum of the squares of the elements of corresponding row of the matrix H^h. The factor ϵ is set such that $\|H^h\mathbf{q}\|_{\ell_2} \leq \epsilon$, with 95% probability, where $\mathbf{q}$ is the quantization noise vector. $\|H^h\mathbf{q}\|_{\ell_2}^2$ follows a chi-square distribution with d degrees of freedom χ_d^2, hence $\epsilon^2 = \chi_d^2(0.95)$. Numerically, this factor was computed by Monte Carlo Simulation.

With these settings in the encoder and decoder, the error localization performances of the following cases are compared.
i) The localization performance when only MUSIC algorithm is used.
ii) The localization performance when only ℓ_1-norm minimization is employed i.e., solving the optimization problem (P_1).
iii) The localization performance of the proposed two step algorithm, i.e., solving MUSIC algorithm in the first step and then solving the optimization problem (WP_1).
The factor δ required for the case iii) was chosen such that, β, the size of subset $\mathbf{x}_s$ was eight in all our experiments.

The optimization programs (WP_1) and (P_1) is a second order cone program and can be solved very efficiently using standard algorithms [13]. In our work, to solve (WP_1) and (P_1), we used `CVX`, a package for specifying and solving convex programs [14].

Fig.(2)-(5) provides the simulation results of one, two , three and four channel localization results. In addition to the three cases mentioned above, an additional result of solving (WP_1) with identity weighting matrix W is also provided. The simulation plots show the relative frequency of correct localization of all the errors versus channel error power to quantization noise power, i.e., the ratio of $(\mu_c^2 + \sigma_c^2)/\sigma_q^2$. The relative frequency of correct localization denotes the number of correct localizations of all error locations to the total number of codewords generated which is 1000, i.e., we generate 1000 codewords and add the channel errors manually at random locations for each of these 1000 codewords, and then find how many of them in that set of 1000 erroneous codewords are correctly localized.

First we can compare the localizing performance between MUSIC algorithm and using (P_1) i.e., between case i) and ii). From all the plots we can clearly see that MUSIC algorithm exceeds the performance of (P_1). The MUSIC algorithm performs well for fewer errors (fig.(2) and (3)), but for more errors the performance decreases.

However, as the impulse noise power increases the performance increases, unlike in the case of (P_1) which saturates. This behaviour is visible in the fig.(4) and (5) for three and four channel error cases, where one can clearly see that as the impulse noise power increases the performance of MUSIC increases, while the performance of (P_1) saturates. This also suggests that the performance of (P_1) is almost independent on the channel noise power, and is a function of the number of the errors.

The performance plot corresponding to solving (WP_1) with identity weighing matrix is in a sense equivalent to solving the problem (P_1) on a smaller dimension. The dimension of H^h is reduced by keeping only those columns which are potential error location candidates, which is achieved with the help of MUSIC algorithm. The optimization algorithm now works on this smaller dimension with the length of the syndrome vector $\mathbf{s}$ being unchanged which results in enhanced performance compared to the solving (P_1). The best performance is achieved by solving (WP_1) with appropriate weighting matrix chosen as outlined in the Section (3). From all the plots we can observe that (WP_1) clearly performs better than all other methods, thus showing the power of weighting. The gap between the performances of the proposed method and other methods is significant with more number of errors.

Thus we can conclude that by combining the MUSIC algorithm with weighted ℓ_1-norm minimization algorithm, a performance better than the performances of the individual algorithms is achieved, which is evident from the simulation results.

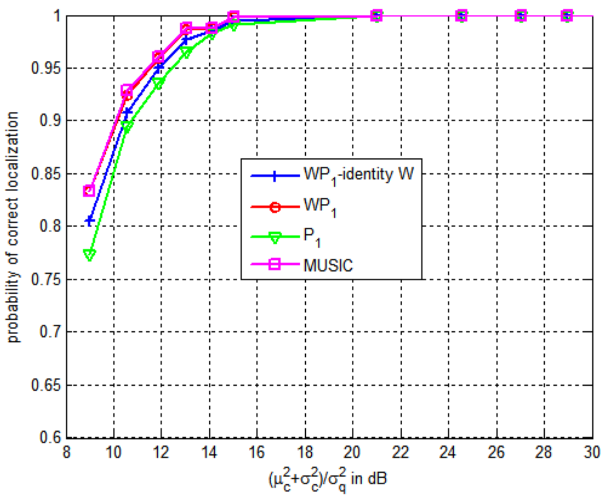

Figure 2: Relative frequency of correct localization of one channel error

5. CONCLUSION

This paper has considered the error localization of real BCH DFT codes, which are used for joint - source channel coding. In this context, we have proposed a two step algorithm by combining the subspace based algorithm like MUSIC, with ℓ_1-norm minimization algo-

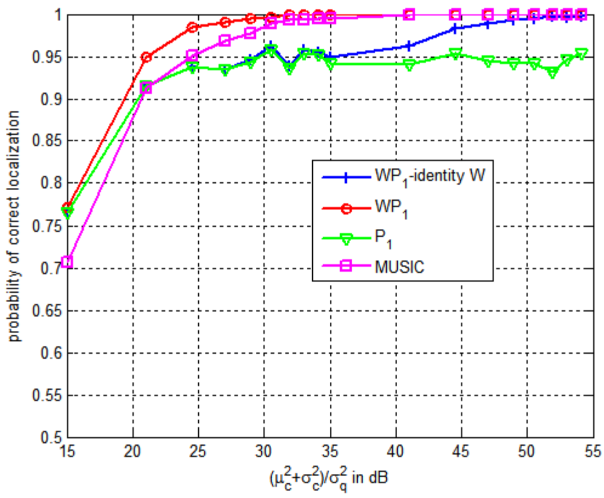

Figure 3: Relative frequency of correct localization of two channel errors

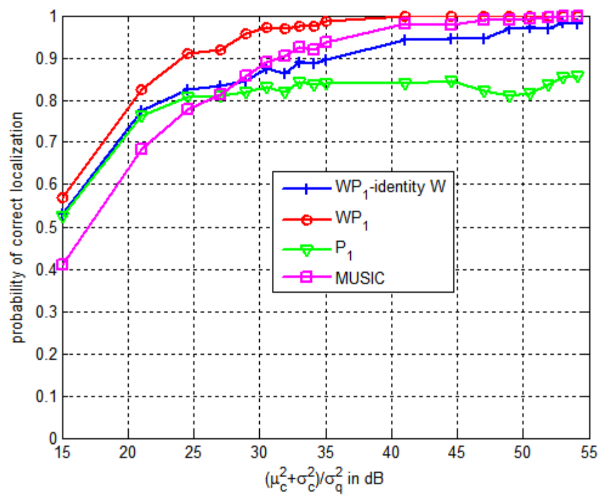

Figure 4: Relative frequency of correct localization of three channel errors

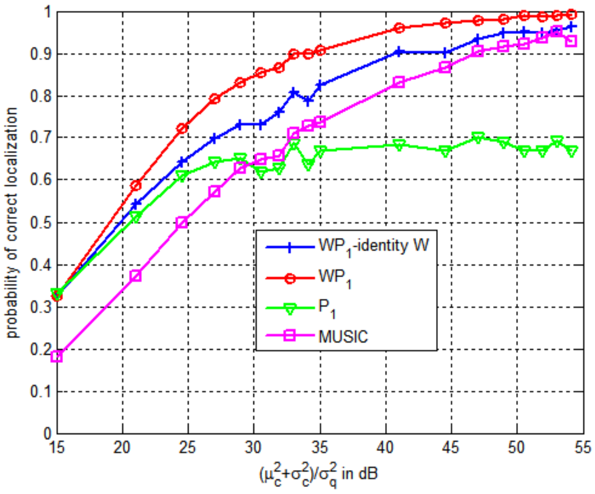

Figure 5: Relative frequency of correct localization of four channel errors

rithm, which is promoted as a sparsity solution functional. The simulation results are provided which shows the proposed two step algorithm performs clearly better than the individual performances of both these algorithms, with a decoding complexity a bit more than solving MUSIC alone but clearly much less than solving (P_1).

The sampling of sparse diracs and error localization problems are analogous. There are results which shows that with a set of cardinality 2ν, it is able to reconstruct any ν-sparse signal [10], but in practice in the presence of quantization noise much more samples are required. Though not considered in this paper, the proposed method can be easily extended for such sampling applications which improves the reconstruction performance in the presence of noise with lesser number of samples.

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