A FAMILY OF REAL SINGLE-TONE FREQUENCY ESTIMATORS USING HIGHER-ORDER SAMPLE COVARIANCE LAGS

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ABSTRACT

In this paper, the problem of frequency estimation of a single sinusoid immersed in white noise is addressed. Inspired by the Pisarenko Harmonic Decomposer (PHD) estimator, which makes use of the sample covariance of the observations with lags 1 and 2, we propose a family of estimators using higher lags p and $2p$ of the sample covariance. The proposed estimator outperforms the PHD, except for frequencies close to the edges $k\pi/p$. As a means to sidestep this problem, two different estimators with different lags can be appropriately combined. A statistical analysis of the proposed method in terms of the Mean Square Error of the frequency estimate is presented. Computer simulations are included to validate the theoretical analysis of the novel estimator and to compare its performance to that of PHD and to the Cramer-Rao Lower Bound.

1. INTRODUCTION

Detection of sinusoidal components and estimation of their frequencies in the presence of broadband noise are common problems in signal processing with a broad range of areas of application [1], and numerous techniques have been developed for their treatment [1, 2].

Among these methods, it is well known that the Maximum Likelihood (ML) method is statistically efficient in the sense that the estimator variance achieves Cramer-Rao Lower Bound (CRLB) asymptotically, but its computational requirements is extremely demanding. In fact, this estimator requires the maximization of a highly nonlinear and multimodal cost function [3, 4]. On the other hand, simple estimators can be obtained using the LP of sinusoidal. Among these, the Pisarenko Harmonic Decomposer (PHD) [5] is of historical interest since it was the first to exploit the eigenstructure of the covariance matrix. Although the PHD method constitutes a simple approach to frequency estimation, a number of statistical analysis have shown its inefficiency [6, 7].

A variety of alternative schemes have been developed to improve the performance of PHD estimator. In [8], a Reformed PHD estimator is derived, using the linear prediction property of sinusoidal signals. Also, in [9], alternative sample covariance expressions for PHD estimator, which are inspired by the modified covariance method, are derived. Although these two frequency estimation methods are computationally simple, they are outperformed by the original PHD method in the medium Signal to Noise Ratio (SNR) range. On the other hand, a constrained weighted least squares frequency estimator is presented in [10] which improves the performance of the PHD estimator, but at the cost of extensive computations. This will be prohibitive in applications were rapid frequency estimation is required.

In this paper we show that the performance of the PHD estimator can be improved by using lags p and $2p$ in the sample covariance of the input signal. This approach results in a family of unbiased schemes, which we refer to as p-estimators. A statistical analysis shows that the p-estimator outperforms the original PHD method (corresponding to $p = 1$) except in the case where the unknown frequency is close to the 'edges' $k\pi/p$ for $k = 0, 1, \ldots, p$. In order to avoid this edge problem, we propose using a weighted sum of two p-estimators with different lags. The expression of the Mean Square Error (MSE) of the resulting estimate is developed.

The paper is organized as follows. In Section 2, the formal statement of the single-tone frequency estimation problem is given and the PHD estimator is reviewed. In Section 3, the family of p-estimators is introduced and its performance is analyzed in terms of MSE, showing the corresponding edge problem. A possible fix to this problem is presented in Section 4. Numerical examples are presented in Section 5 to validate the theoretical results, comparing the performance of the proposed estimator to that of the original PHD scheme and the CRLB. Finally, conclusions are drawn in Section 6.

2. PROBLEM FORMULATION AND PHD ESTIMATOR

Consider the problem of estimating the unknown frequency w of a real-valued sine wave $s(n)$ immersed in white noise $u(n)$. The observed signal, $y(n)$, is given by:

$$\begin{aligned} y(n) &= s(n) + u(n) \\ &= \alpha \sin(wn + \varphi) + u(n), \qquad 1 \le n \le N, \end{aligned} \tag{1}$$

where α is the sinusoid amplitude, φ is a random phase uniformly distributed in the interval $[-\pi, \pi[$, $u(n)$ is a zero mean white noise with variance σ_u^2 and N is the number of observations. We assume that $u(n)$ is independent of $s(n)$. The SNR is defined as SNR $\doteq \alpha^2/(2\sigma_u^2)$.

The PHD estimate [5, 6, 7] is obtained in terms of the unit-norm eigenvector $[v_0\ v_1\ v_2]^T$ corresponding to the smallest eigenvalue of the 3×3 sample covariance matrix:

$$\begin{pmatrix} \hat{r}(0) & \hat{r}(1) & \hat{r}(2) \\ \hat{r}(1) & \hat{r}(0) & \hat{r}(1) \\ \hat{r}(2) & \hat{r}(1) & \hat{r}(0) \end{pmatrix},$$

where $\hat{r}(\tau)$ denotes the lag-τ sample covariance coefficient of the signal $y(n)$, given by:

$$\hat{r}(\tau) = \frac{1}{N-\tau} \sum_{i=\tau+1}^{N} y(i) y(i-\tau), \tag{2}$$

which, is an unbiased estimate of the true coefficient $r(\tau) = \mathrm{E}\{y(n)y(n-\tau)\}$. The corresponding eigenvector is symmetric ($v_2 = v_0$), and the frequency estimate is taken as the angular position $\hat{w}$ of the zero of the transfer function $v_0 + v_1 z^{-1} + v_0 z^{-2}$ [6]:

$$\hat{w} = \arccos\left(\frac{\hat{r}(2) + \sqrt{\hat{r}^2(2) + 8\hat{r}^2(1)}}{4\hat{r}(1)} \right). \tag{3}$$

Alternatively, this estimator can be obtained directly by matching the theoretical and sample covariance coefficients of lags 1 and 2

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[11]. The former are given by:

$$r(\tau) = \sigma_u^2 \delta_\tau + \frac{\alpha^2}{2}\cos(w\tau). \quad (4)$$

Hence, eliminating the unknown amplitude α from the expressions of $r(1)$, $r(2)$ the following equation is obtained:

$$2r(1)\cos^2 w - r(2)\cos w - r(1) = 0. \quad (5)$$

After replacing $r(1)$, $r(2)$ by their estimates $\hat{r}(1)$, $\hat{r}(2)$ in (5), we can solve for the frequency value, thus obtaining the estimate $\hat{w}$ in (3).

3. EXPLOITING HIGHER-ORDER SAMPLE COVARIANCE LAGS

3.1 *p*-estimator

In this section, we show how the performance of the PHD estimator can be improved by using lags p and $2p$ of the sample covariance. From (4), for $p \geq 1$ one has:

$$r(p) = \frac{\alpha^2}{2}\cos(pw), \qquad r(2p) = \frac{\alpha^2}{2}\cos(2pw). \quad (6)$$

Eliminating now the unknown amplitude α from these two equations and substituting $\cos(2pw) = 2\cos^2(pw) - 1$, the following relation is obtained:

$$2r(p)\cos^2(pw) - r(2p)\cos(pw) - r(p) = 0. \quad (7)$$

Similarly to the PHD estimator, we can replace $r(p)$, $r(2p)$ by their sample estimates $\hat{r}(p)$, $\hat{r}(2p)$ in (7), and then solve for the frequency estimate $\hat{w}$. In this way, we obtain a family of estimators as follows:

1. Compute the sample covariance coefficients $\hat{r}(p)$, $\hat{r}(2p)$.
2. Solve for $\hat{w}$ in $f_p(\hat{w}) = 0$, where:

$$f_p(\hat{w}) \doteq 2\hat{r}(p)\cos^2(p\hat{w}) - \hat{r}(2p)\cos(p\hat{w}) - \hat{r}(p). \quad (8)$$

When $p = 1$, we recover the original PHD estimator (3).

Observe that (8) has p solutions in $[0, \pi]$: In fact, solving $f_p(\hat{w}) = 0$ in terms of $\hat{a} \doteq \cos(p\hat{w})$, one obtains:

$$\hat{a} = \frac{\hat{r}(2p) + \sqrt{\hat{r}^2(2p) + 8\hat{r}^2(p)}}{4\hat{r}(p)}. \quad (9)$$

Hence there exist p possible solutions $\hat{w}_k$ for the frequency estimate, namely:

$$\hat{w}_k = \frac{1}{p}\arccos(\hat{a}) + \frac{2k\pi}{p} \quad \text{for} \quad k = 0, \ldots, p-1. \quad (10)$$

A possible means to resolve this ambiguity is to use the $p = 1$ (i.e. PHD) scheme as a first step, in order to obtain a coarse estimate $\hat{w}_{\text{coarse}}$. Then the solution $\hat{w}_k$ in (10) that is closer to $\hat{w}_{\text{coarse}}$ is selected, thus obtaining a refined estimate.

In the following, we refer to the estimator using lags p and $2p$ of the sample covariance of $y(n)$ as *p-estimator,* and denote the corresponding estimate as $\hat{w}^{(p)}$.

3.2 Performance analysis and edge problem

In order to study the statistical properties of the frequency estimate $\hat{w}^{(p)}$, related to the estimate $\hat{a}^{(p)}$ by $\hat{a}^{(p)} = \cos(p\hat{w}^{(p)})$, we derive first the expression of the MSE of $\hat{a}^{(p)}$.

The estimate $\hat{a}^{(p)}$ must satisfy:

$$F_N\left(\hat{a}^{(p)}\right) = 2\hat{r}(p)\left(\hat{a}^{(p)}\right)^2 - \hat{r}(2p)\hat{a}^{(p)} - \hat{r}(p) = 0. \quad (11)$$

For sufficiently large N, a small error approximation applies. Hence, a first-order Taylor expansion of $F_N\left(\hat{a}^{(p)}\right)$ around $a^{(p)} \doteq \cos(pw)$ yields

$$F_N\left(\hat{a}^{(p)}\right) \approx F_N\left(a^{(p)}\right) + \beta\left(\hat{a}^{(p)} - a^{(p)}\right), \quad (12)$$

where

$$\beta = \left.\frac{\partial\left[F_N(a)\right]}{\partial a}\right|_{a=a^{(p)}}.$$

The terms neglected in (12) go to zero faster than $\left|\hat{a}^{(p)} - a^{(p)}\right|$ when N tends to infinity.

It is important to note that the following analysis holds only if $N - 2p$ is sufficiently large. In this condition, using the weak law of large numbers, the evaluation of β shows that:

$$\beta = \frac{\alpha^2}{2}\left(2\left(a^{(p)}\right)^2 + 1\right). \quad (13)$$

Then, it follows from (11) and (12) that:

$$\text{MSE}\left(\hat{a}^{(p)}\right) \doteq \text{E}\left\{\left(\hat{a}^{(p)} - a^{(p)}\right)^2\right\} \approx \text{E}\left\{F_N^2\left(a^{(p)}\right)\right\}/\beta^2.$$

The evaluation of the numerator $\text{E}\left[F_N^2\left(a^{(p)}\right)\right]$ is done in Appendix A, where it is shown that, for $N - 2p$ large enough,

$$\text{E}\left\{F_N^2\left(a^{(p)}\right)\right\} \approx \frac{\sigma_u^4\left(4(a^{(p)})^4 - 3(a^{(p)})^2 + 1\right)}{N}. \quad (14)$$

Therefore, $\text{MSE}\left(\hat{a}^{(p)}\right)$ satisfies:

$$\text{MSE}\left(\hat{a}^{(p)}\right) \approx \frac{\left(4\left(a^{(p)}\right)^4 - 3\left(a^{(p)}\right)^2 + 1\right)}{N \cdot \text{SNR}^2\left(2\left(a^{(p)}\right)^2 + 1\right)^2}. \quad (15)$$

On the other hand, the relationship between the mean square errors in the estimation of $a^{(p)}$ and w satisfies:

$$\begin{aligned}
\text{MSE}\left(\hat{a}^{(p)}\right) &= \text{E}\left\{\left(\cos\left(p\hat{w}^{(p)}\right) - \cos(pw)\right)^2\right\} \\
&= 4\text{E}\left\{\sin^2\left(\frac{p\left(\hat{w}^{(p)} + w\right)}{2}\right)\sin^2\left(\frac{p\left(\hat{w}^{(p)} - w\right)}{2}\right)\right\} \\
&\approx p^2\sin^2(pw)\text{E}\left\{\left(\hat{w}^{(p)} - w\right)^2\right\}.
\end{aligned}$$

Therefore, we have:

$$\text{MSE}\left(\hat{w}^{(p)}\right) = \text{E}\left\{\left(\hat{w}^{(p)} - w\right)^2\right\} \approx \frac{\text{MSE}\left(\hat{a}^{(p)}\right)}{p^2\sin^2(pw)}. \quad (16)$$

From (15) and (16), the expression of $\text{MSE}\left(\hat{w}^{(p)}\right)$ is obtained:

$$\text{MSE}\left(\hat{w}^{(p)}\right) \approx \frac{4\cos^4(pw) - 3\cos^2(pw) + 1}{Np^2\text{SNR}^2\sin^2(pw)\left(2\cos^2(pw) + 1\right)^2}. \quad (17)$$

For $p = 1$, this expression coincides with a known approximation for the MSE of the PHD estimate [6]. The two important differences between the MSE of the p-estimator and that of the PHD are:

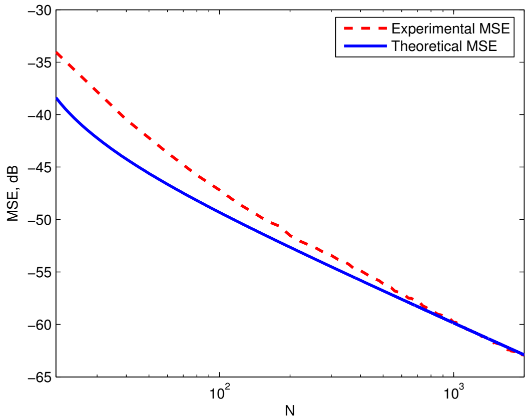

Figure 1: MSE $\left(\hat{w}^{(p)}\right)$ versus N, $w = 0.4\pi$, SNR=10 dB, $p = 2$.

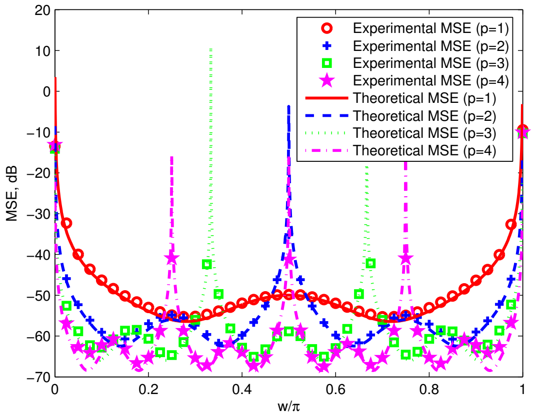

Figure 2: MSE $\left(\hat{w}^{(p)}\right)$ versus w, $N = 1000$, SNR=10 dB.

- The frequency variable w is replaced by pw. Due to the factor $\sin^2(pw)$ in the denominator of (17), performance can be expected to degrade at the 'edge frequencies' $w = k\pi/p$, $k = 0, 1, \ldots, p$. This is a known effect for the PHD estimate, whose performance is poor when $w = 0$ or $w = \pi$.
- A factor of p^2 appears in the denominator of (17). This suggests that, as long as the unknown frequency is not close to one of the edge frequencies, the p-estimator may outperform the PHD method. Of course, if p is increased so as to reduce the MSE, the number of edge frequencies will increase as well, and the minimum distance between the true frequency w and an edge frequency $k\pi/p$ will decrease.

Computer simulations have been carried out to evaluate the performance of the p-estimators for a single real sinusoid in white Gaussian noise. The sinusoid amplitude is taken as $\alpha = \sqrt{2}$ and a random phase is used, whereas different SNRs were obtained by properly scaling the noise variance σ_u^2. All simulation results were averaged over 1000 independent runs.

In order to validate the theoretical MSE (17), Fig. 1 shows the empirical MSE of the frequency estimate and the theoretical approximation (17) as a function of N, for $p = 2$, SNR=10 dB and $w = 0.4\pi$. A good agreement is observed for a sufficiently large value of N.

Fig. 2 shows the variation of MSE($\hat{w}^{(p)}$) with frequency for SNR=10 dB, $N = 1000$, and $1 \leq p \leq 4$. The theoretical and the empirical MSE agree reasonably well. As predicted by the analysis, a noticeable improvement can be achieved compared to the PHD estimator except for the frequencies close to the edges $k\pi/p$ for $k = 1, \ldots, p-1$. We also note that the higher the order p, the better the performance of the p-estimator for the frequencies far from the edges, whereas the edge problem becomes more severe.

4. ESTIMATOR USING INFORMATION FROM TWO *P*-ESTIMATORS

In order to alleviate the edge problem, we note that the use of two p-estimators with consecutive lags p and $p+1$ permits to detect accurately all frequencies by at least one of the two p-estimators (with exception of frequencies very close to $w = 0$ or $w = \pi$). This solution provides two possible frequency estimates, so the question that arises is how to use these two p-estimators in order to come up with a single estimator $\hat{w}$ with good behavior in terms of MSE. One possibility is to obtain $\hat{w}$ by means of a weighted sum of the individual estimates $\hat{w}^{(p)}$ and $\hat{w}^{(p+1)}$:

$$\hat{w} = \mu_0 \hat{w}^{(p)} + \mu_1 \hat{w}^{(p+1)} \quad \text{with} \quad \mu_0 + \mu_1 = 1. \tag{18}$$

The condition $(\mu_0 + \mu_1 = 1)$ ensures that the weighted sum estimate is unbiased.

We assume that the estimation errors $\hat{w}^{(p)} - w$ and $\hat{w}^{(p+1)} - w$ are uncorrelated. Under this assumption, the MSE of the frequency estimate $\hat{w}$ is evaluated as

$$\text{MSE}(\hat{w}) \approx \sum_{k=0}^{1} \mu_k^2 \text{MSE}\left(\hat{w}^{(p+k)}\right). \tag{19}$$

The optimal values of the weights μ_k^*, in the sense that they minimize (19), are given by:

$$\mu_k^* = \frac{\text{MSE}^{-1}\left(\hat{w}^{(p+k)}\right)}{\sum_{l=0}^{1} \text{MSE}^{-1}\left(\hat{w}^{(p+l)}\right)} \quad \text{for} \quad k = 0, 1. \tag{20}$$

Since w is unknown, the exact values of MSE $\left(\hat{w}^{(p+k)}\right)$, given by (17), are not available. We propose to use the estimates $\hat{w}^{(p+k)}$ instead of w in order to obtain an approximation to the optimal weights μ_k^*. We refer to the proposed weighted sum estimator as $\{p, p+1\}$-estimator, and the corresponding frequency estimate is denoted by:

$$\hat{w}^{(p,p+1)} = \sum_{k=0}^{1} \mu_k^* \hat{w}^{(p+k)}. \tag{21}$$

The mean-squared error associated to the $\{p, p+1\}$-estimator is given by:

$$\text{MSE}(\hat{w}^{p,p+1}) \approx \frac{1}{\sum_{k=0}^{1} \text{MSE}^{-1}(\hat{w}^{(p+k)})} \approx \frac{1}{N\text{SNR}^2 h(p,w)}, \tag{22}$$

where

$$h(p,w) \doteq \sum_{k=0}^{1} (p+k)^2 \frac{\sin^2((p+k)w)\left(2\cos^2((p+k)w)+1\right)^2}{4\cos^4((p+k)w) - 3\cos^2((p+k)w)+1}. \tag{23}$$

5. SIMULATION RESULTS

To study the performance of the $\{p, p+1\}$-estimator, we consider the same simulation conditions as in Section 3.2.

In Fig. 3, we compare the performance of the $\{p, p+1\}$-estimator to that of the individual p-estimators. We evaluate the

MSE of the frequency estimate versus w for $N = 1000$ and SNR=10 dB with $p = 2,3$. Besides the unavoidable degradation near the endpoints $w = 0$ and $w = \pi$, these constituent p-estimators $\hat{w}^{(2)}$, $\hat{w}^{(3)}$ present an edge problem at $w = \pi/2$ and $w \in \{\pi/3, 2\pi/3\}$ respectively. Using the $\{p, p+1\}$-estimator with $p = 2$, this problem is avoided: the novel proposed estimator consistently outperforms the PHD estimator over the whole frequency range. In Fig. 3, expression (22) is plotted as well in order to check the validity of the theoretical MSE of the $\{p, p+1\}$-estimator. It can be seen that the empirical MSE agrees well with the theoretical value from (22).

Fig. 4 shows the behavior of $\mathrm{MSE}(\hat{w}^{(p,p+1)})$ in terms of p for different sequence lengths N. The frequency w is fixed to 0.4π, whereas SNR=10 dB. As can be seen, when $N = 100$, a lag of $p = 20$ is required to attain the minimum of $\mathrm{MSE}(\hat{w}^{(p,p+1)})$, which implies that no noticeable improvement in performance is achieved if higher lags are employed. We also note that the higher N is, the higher is the lag p required to minimize $\mathrm{MSE}(\hat{w}^{(p,p+1)})$.

Fig. 5 displays $\mathrm{MSE}(\hat{w}^{(p,p+1)})$ versus w for $N = 100$ and SNR=10 dB, using a lag $p = 20$, together with the ML estimator and the CRLB for frequency estimation of a single sinusoid in additive Gaussian noise [12]:

$$\mathrm{MSE}_{\mathrm{cr}}(\hat{w}) \cong \frac{24\sigma_u^2}{N^3\alpha^2}. \tag{24}$$

In addition to outperforming the PHD scheme, the $\{p, p+1\}$-estimator performance approaches the ML and the CRLB for $w \in [0.12, 0.88]\pi$. In view of this, This result indicates the potential of the the $\{p, p+1\}$-estimator becomes an appealing choice with short data records in high SNR.

Fig. 6 shows the behavior of $\mathrm{MSE}(\hat{w}^{p,p+1})$ with p for different SNR levels, using $N = 100$ and for $w = 0.4\pi$. It is seen that the adequate choice of lag p depends not only on the sequence length N but also on the SNR level: the higher the SNR level, the higher the optimum lag p for a given N. In fact, using the PHD estimator to obtain $\hat{w}_{\mathrm{coarse}}$ provides an inaccurate first frequency estimate in low SNR environment, and thus it becomes more difficult to resolve the inherent p-fold ambiguity in (10) for large values of p. This shows that, in low SNR settings, the performance of the $\{p, p+1\}$-estimator is sensitive to the choice of the initial estimate $\hat{w}_{\mathrm{coarse}}$.

Fig. 7 plots $\mathrm{MSE}(\hat{w}^{(p,p+1)})$ versus SNR for different lags p, with $N = 100$ and for $w = 0.4\pi$. It is observed that with a low lag $p = 2$, the $\{p, p+1\}$-estimator outperforms the PHD estimator even in low SNR environments. In addition, the proposed estimator approached the ML and the CRLB for SNR$\in [7.5, 20]$ dB, when using a high lag $p = 20$. Hence some tradeoff is needed in the choice of p, depending on the expected operating SNR.

Fig. 8 displays $\mathrm{MSE}(\hat{w}^{(p,p+1)})$ versus w for $N = 100$, SNR=0 dB and $p = 2$. It is seen that the proposed estimator still yields better performance than PHD, even in this short record, low SNR level.

From Figs. $3-8$, we can conclude that the $\{p, p+1\}$-estimator is superior to the PHD estimator and can achieve very good estimation performance. It can approach the CRLB with short data lengths when the SNR is sufficiently high.

6. CONCLUSION

In this paper, we derive a family of estimators, referred to as to *p-estimators*, which uses the sample covariance with lags p and $2p$ of a noisy sinusoid to estimate its unknown frequency. Since the p-estimator presents an edge problem, a solution is presented taking information from two p-estimators with consecutive lags, an approach referred to as $\{p, p+1\}$-*estimator*. The performance of the $\{p, p+1\}$-estimator is theoretically analyzed and evaluated via computer simulations, which showed its superiority over Pisarenko's method, approaching in some cases the CRLB with high SNR.

These results motivate the extension of the proposed estimator to the case of colored noise, since using high lags in sample covariance permits to decorrelate the noise components.

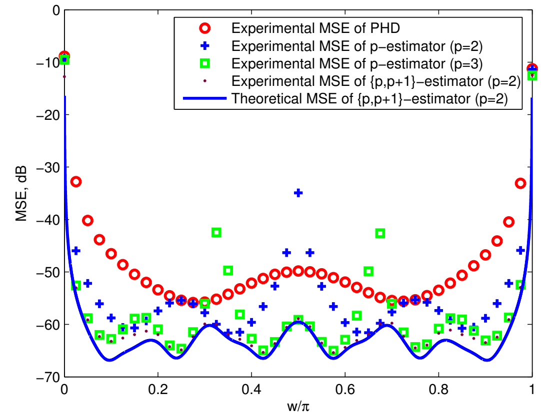

Figure 3: MSE versus w, $N = 1000$, SNR=10 dB.

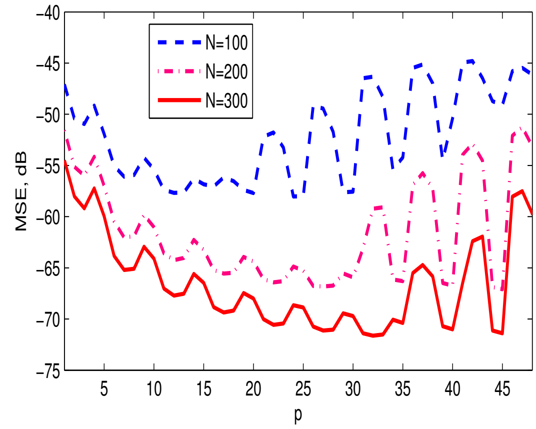

Figure 4: $\mathrm{MSE}(\hat{w}^{(p,p+1)})$ versus p, $w = 0.4\pi$, SNR=10 dB.

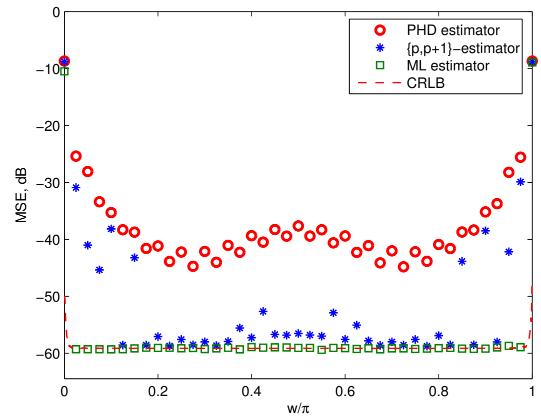

Figure 5: $\mathrm{MSE}(\hat{w}^{(p,p+1)})$ versus w, $p = 20$, $N = 100$, SNR=10 dB.

Appendix A

Evaluation of $\mathrm{E}\{F_N^2(a^{(p)})\}$

The derivation of (14) is as follows. From (11), it is easy to show

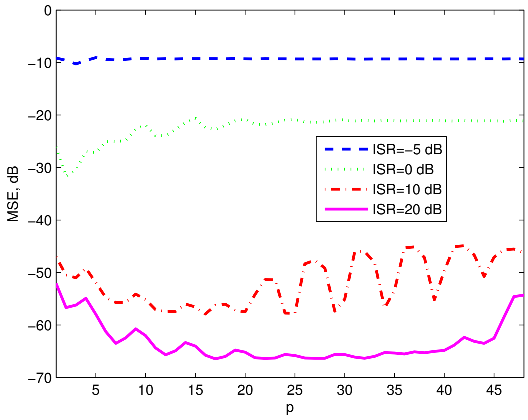

Figure 6: MSE($\hat{w}^{(p,p+1)}$) versus p, $w = 0.4\pi$, $N = 100$.

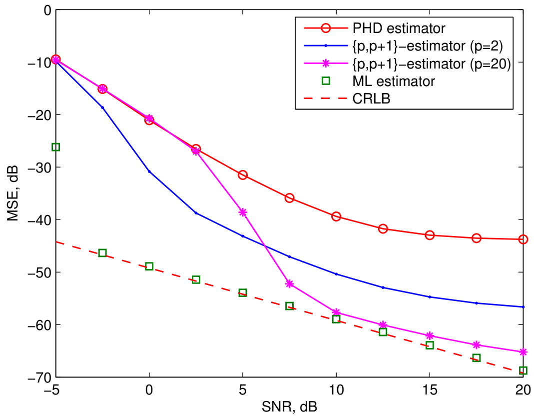

Figure 7: MSE($\hat{w}^{(p,p+1)}$) versus SNR, $w = 0.4\pi$, $N = 100$.

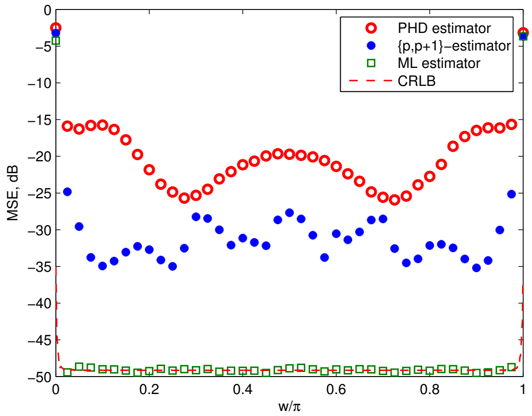

Figure 8: MSE($\hat{w}^{(p,p+1)}$) versus w, $p = 2$, $N = 100$, SNR=0 dB.

that:

$$\begin{aligned} \mathrm{E}\{F_N^2(a^{(p)})\} &= [1-2(a^{(p)})^2]^2\mathrm{E}\{\hat{r}^2(p)\}+(a^{(p)})^2\mathrm{E}\{\hat{r}^2(2p)\} \\ &\quad +2a^{(p)}(1-2(a^{(p)})^2)\mathrm{E}\{\hat{r}(p)\hat{r}(2p)\}. \end{aligned}$$

For sufficiently large $N-2p$, the terms $\mathrm{E}\{\hat{r}^2(p)\}$, $\mathrm{E}\{\hat{r}^2(2p)\}$ and $\mathrm{E}\{\hat{r}(p)\hat{r}(2p)\}$ are approximately given by:

$$\mathrm{E}\{\hat{r}^2(p)\} \approx \frac{\alpha^4(a^{(p)})^2}{4}+\frac{2\alpha^2\sigma_u^2(a^{(p)})^2}{N-p}+\frac{\sigma_u^4}{N-p}$$

$$\mathrm{E}\{\hat{r}^2(2p)\} \approx \frac{\alpha^4(2(a^{(p)})^2-1)^2}{4}+\frac{2\alpha^2\sigma_u^2(2(a^{(p)})^2-1)^2}{N-2p}+\frac{\sigma_u^4}{N-2p}$$

$$\mathrm{E}\{\hat{r}(p)\hat{r}(2p)\} \approx \frac{\alpha^4 a^{(p)}(2(a^{(p)})^2-1)}{4}+\frac{2\alpha^2\sigma_u^2 a^{(p)}(2(a^{(p)})^2-1)}{N-p}$$

Therefore, (14) is obtained asymptotically:

$$\begin{aligned} \mathrm{E}\left\{F_N(a^{(p)})^2\right\} &\approx \sigma_u^4\left(\frac{\left(2(a^{(p)})^2-1\right)^2}{N-p}+\frac{(a^{(p)})^2}{N-2p}\right)+ \\ &\quad \frac{2\alpha^2\sigma_u^2 p(a^{(p)})^2(2(a^{(p)})^2-1)^2}{(N-p)(N-2p)} \\ &\approx \frac{\sigma_u^4\left(4(a^{(p)})^4-3(a^{(p)})^2+1\right)}{N} \end{aligned}$$

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