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CRAMER-RAO LOWER BOUND FOR THE CLOCK OFFSET OF SILENT NODES SYNCHRONIZING THROUGH A GENERAL SENDER-RECEIVER PROTOCOL IN WIRELESS SENSORNETS

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ABSTRACT

Agreement on a common time reference among the nodes is one of the most critical issues in the successful operation of the Wireless Sensor Networks (WSNs). In a previous paper [1], the Maximum Likelihood Estimator (MLE) for the clock offset of the silent nodes overhearing a two-way timing message exchange between two nodes, assuming an exponential link delay model, was obtained. This paper targets the derivation of Cramer-Rao Lower Bound (CRLB) for any unbiased clock offset estimator of the silent nodes as a performance threshold. The CRLB is shown to be inversely proportional to the square of the number of observations, and hence the variance of the MLE decreases very rapidly with the increase in data points. However, it is proved that the MLE does not achieve the CRLB and hence it is not efficient for small number of observations (although MLE has optimal properties for a large number of observations, i.e., it tends to become unbiased and attains CRLB). In addition, the CRLB for the clock offset estimator of the active node, as estimated by the silent node, is also derived and shown to be slightly better, but on a similar scale as the CRLB for the clock offset of the inactive node.

1. INTRODUCTION

The advances in fabrication technology have enabled the development of tiny low power devices capable of performing sensing, computing and communication assignments through the equipped sensors, microprocessors and radio, respectively. These tiny devices can be deployed at any place forming a network of their own for observing and reporting a desired phenomenon to the information sink. Wireless Sensor Networks (WSNs) are a special kind of ad hoc networks, composed of such devices, functioning collectively as a network without any infrastructure. Sensor networks have their own characteristics, such as very limited energy sources, high density of node deployment and cheap and unreliable sensor nodes. With these extra limiting factors for their operation, sensor networks are designed to perform complex tasks such as flood detection, monitoring of forest fire, ecological and biological habitats, monitoring equipment and ammunition, deep sea exploration, etc.

As in all distributed systems, time synchronization is an important component of a sensor network. Time synchronization in a computer network aims to provide a common timescale for local clocks of nodes in the network. Since all hardware clocks are imperfect, local clocks of nodes may drift away from each other in time, so observed time or durations of time intervals may differ for each node in the network. However, for many application or networking protocols, it is required that a common view of time exists and be available to all or some of the nodes in the network at any particular instant.

Different protocols have been proposed for achieving unified notion of time across the WSN, most of which are based on packet synchronization techniques. Such protocols can be divided into two major approaches: sender-receiver synchronization (such as [2, 3]) and receiver-receiver synchronization (such as [4, 5]). Although the receiver-receiver protocols exploit the idea of multiple nodes receiving a timing beacon from the reference node to save a lot of redundant information flow, the same concept has not been utilized in sender-receiver protocols communicating through a wireless medium until recently [6]. When an active nodes synchronizes with a reference node, the inactive nodes lying within their common broadcast region can listen to this timing message exchange and hence can synchronize their local clocks with the reference node, without transmitting any information by themselves. Since the major shortcoming of sender-receiver protocols has always been the high number of timing message exchanges due to point-to-point nature of communication, this scheme enables such protocols to compete with the receiver-receiver protocols without being disadvantageous with respect to the high communication cost.

The Maximum Likelihood Estimator (MLE) for the clock offsets of these inactive or silent nodes under an exponential link delay model has been obtained in [1]. In this paper, the Cramer-Rao Lower Bound (CRLB) for any unbiased clock offset estimator of the inactive nodes has been derived. It has been shown that the CRLB is inversely proportional to the square of the number of observations and hence the variance of the MLE falls very rapidly with the number of message exchanges. When compared with the CRLB for the clock offset of the active node, it is evident that though slightly worse, both of them perform on a similar scale. In addition, it has been proved that the MLE does not achieve CRLB for a small number of observations and hence no efficient estimator (which attains CRLB) exists in this scenario.

2. MODELING ASSUMPTIONS

Fig. 1 illustrates a WSN with node r as the reference node. When the network synchronization is started (e.g., after a specific time at the reference node, or after sensing some event), node r undergoes a standard two-way timing message exchange procedure with an arbitrary node, say t. It

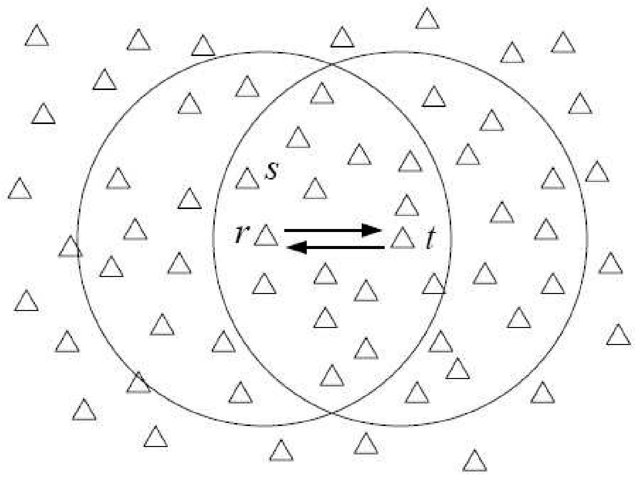

Figure 1: Active node t exchanging timing messages with reference node r with silent nodes like s, in their common broadcast region, listening to the conversation.

sends a timing message with its current timestamp m_1^r to the node t which records its arrival according to its own time as m_2^{rt}. Similarly, the timestamps m_1^t and m_2^{tr} are recorded by the nodes t and r, respectively. This process is illustated in detail in Fig. 2 and repeated N times to improve the quality of the the clock offset estimates, where the N is a function of the target synchronization accuracy and the maximum cost to be paid in the form of battery powers in the nodes.

Now notice in Fig. 1 that if the broadcast region of a node is modeled as a hexagon (or a circle), then a silent node, say s, lying in the common broadcast region of both nodes r and t, can overhear the complete timing message exchange between those two nodes. Without having to transmit any timing packets itself, it can synchronize its clock offset with the reference node through the procedure explained in [1]. As illustrated in Fig. 2, let m_2^{rs} and m_2^{ts} be the timestamps recorded at node s when it receives the timing messages m_1^r and m_1^t, respectively. Also, the silent nodes like s have the information m_2^{rt} since node t sends it back to node r with m_1^t, as required by the sender-receiver protocol.

There are two different kinds of message delays incurred from the transmitter to the receiver analyzed in detail by many researchers, who have divided the link delay uncertainties in deterministic and nondeterministic components (e.g., [7]). In this paper, it is assumed that the deterministic part of link delays is unknown but same for all the nodes receiving the messages from nodes r and t, because the nodes in a WSN share the same hardware characteristics and hence undergo similar transmission, reception, encoding, decoding and byte alignment times, which are all deterministic. In addition, the propagation time of RF waveforms is less than 1 μs for ranges under 300 meters which implies that for nodes present at short distances from each other, the difference in the propagation time of the same message will be even less than a few nano seconds. Finally, the nondeterministic delays have been modeled as originating from an exponential distribution with similar means. Some Probability Density Function (PDF) models, which have been proposed for random network delays, are Gamma, exponential and Log-Normal, but the exponential distribution has usually been the distribution of choice for random delays. A detailed discussion on the justifications behind this can be found in [8, 9]. Based on the above scenario, the related mathematical model can be expressed as follows.

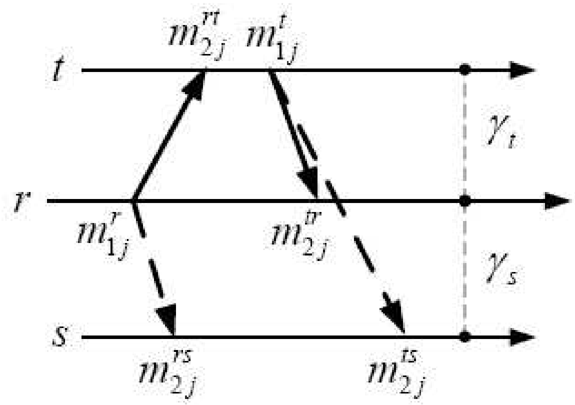

Figure 2: A sender-receiver message exchange with silent nodes also receiving the timing packets

$$\begin{aligned} m_{2j}^{rt} &= m_{1j}^{r} + \gamma_t + \eta + \varepsilon_j^{rt}, \\ m_{2j}^{rs} &= m_{1j}^{r} + \gamma_s + \eta + \varepsilon_j^{rs}, \\ m_{2j}^{ts} &= m_{1j}^{t} - \gamma_t + \gamma_s + \eta + \varepsilon_j^{ts}, \end{aligned}$$

where γ_t and γ_s are the clock offsets of the nodes t and s, respectively, η is the deterministic portion of link delays, and ε_j^{rt}, ε_j^{rs} and ε_j^{ts} are independent and identically distributed exponential random variables with similar means β. Let $A_j = m_{2j}^{rt} - m_{1j}^{r}$, $B_j = m_{2j}^{rs} - m_{1j}^{r}$ and $C_j = m_{2j}^{ts} - m_{1j}^{t}$, then rearranging the above equations implies

$$\begin{aligned} A_j &= \gamma_t + \eta + \varepsilon_j^{rt}, \\ B_j &= \gamma_s + \eta + \varepsilon_j^{rs}, \\ C_j &= \gamma_s - \gamma_t + \eta + \varepsilon_j^{ts}. \end{aligned}$$

Based on the above model, the MLE was derived in [1] as

$$\begin{aligned} \hat{\eta} &= A_{(1)} + C_{(1)} - B_{(1)} \\ \hat{\gamma}_s &= 2B_{(1)} - A_{(1)} - C_{(1)} \qquad (1) \\ \hat{\gamma}_t &= B_{(1)} - C_{(1)}. \qquad (2) \end{aligned}$$

Next, we proceed to deriving the CRLB for the clock offset estimators.

3. CRAMER-RAO LOWER BOUND

Provided that some regularity conditions are satisfied, the CRLB sets a lower bound on the variance of any unbiased estimator. It is important in estimation theory because it can provide a performance threshold against which the performance of any unbiased estimator can be compared. The CRLB theory also informs about the possibility of the existence of any unbiased estimator attaining that bound. In addition, if an estimator attains the CRLB, it is the minimum variance unbiased estimator. For the scenario studied here, it is desirable to set the benchmark through the CRLB for any unbiased estimator of clock offset $\hat{\gamma}_s$, when a node like s is silently listening to the timing cell exchange between a reference and an active node around. On the other hand, if the CRLB for $\hat{\gamma}_t$ can be obtained, then it can be compared with the CRLB for the clock offset in the general sender-receiver protocol derived in [8] and consequently identify the one with better achievable performance.

According to the CRLB theorem, if the regularity conditions are satisfied, i.e., $E[\partial \ln L(\theta)/\partial\theta] = 0$ for all θ, the variance of any unbiased estimator $\hat{\theta}$ must satisfy the relationship

$$\text{var}(\hat{\theta}) \geq I^{-1}(\theta),$$

where $I(\theta)$ is the quantity known as Fisher Information defined as

$$I(\theta) = -E\left[\frac{\partial^2 \ln L(\theta)}{\partial\theta^2}\right] = E\left[\left(\frac{\partial \ln L(\theta)}{\partial\theta}\right)^2\right].$$

It is evident that the domain of the likelihood function depends on both unknown parameters γ_t and γ_s due to which the order of differentiation and integration in the expression for regularity condition can not be interchanged. Therefore, CRLB can not be found by employing the likelihood function. However, utilizing the fact that the MLE (or in fact any estimator) is a random variable by itself, its Probability Density Function (PDF) can be derived from the form of its expression, where it is given as a function of data. Moreover, obtaining this random variable from the likelihood function is an equivalent problem to obtaining this random variable from its own PDF. Similarly, deriving its CRLB from the likelihood function is an equivalent problem to deriving its CRLB from its own PDF. In light of this, the CRLB for the clock offsets can be derived as follows.

Starting with $\hat{\gamma}_s$, note that from (1),

$$\begin{aligned} \hat{\gamma}_s &= 2B_{(1)} - A_{(1)} - C_{(1)} \\ &= 2\left(\eta + \gamma_s + \varepsilon^{rs}_{(1)}\right) - \left(\eta + \gamma_t + \varepsilon^{rt}_{(1)}\right) \\ &\quad - \left(\eta + \gamma_s - \gamma_t + \varepsilon^{ts}_{(1)}\right) \\ &= \gamma_s + 2\varepsilon^{rs}_{(1)} - \varepsilon^{rt}_{(1)} - \varepsilon^{ts}_{(1)}. \end{aligned} \tag{3}$$

It is simple to show that the PDF of the first order statistics $\varepsilon^{rt}_{(1)}$, $\varepsilon^{rs}_{(1)}$ and $\varepsilon^{ts}_{(1)}$ is given as

$$\begin{aligned} p_{\varepsilon_{(1)}}\left(\varepsilon_{(1)}\right) &= N\left[1 - F_{\varepsilon_j}\left(\varepsilon_{(1)}\right)\right]^{N-1} \cdot f_{\varepsilon_j}\left(\varepsilon_{(1)}\right) \\ &= \frac{N}{\beta} e^{-\frac{N}{\beta}\varepsilon_{(1)}} u\left[\varepsilon_{(1)}\right], \end{aligned} \tag{4}$$

where f_{ε_j} and F_{ε_j} are the PDF and CDF of the exponential random variables ε_j, respectively. Hence, the PDFs of the first order statistics $\varepsilon^{rt}_{(1)}$, $\varepsilon^{rs}_{(1)}$ and $\varepsilon^{ts}_{(1)}$ are also exponential with mean β/N.

Next, the mean and the variance, respectively, of $\hat{\gamma}_s$ can be written as

$$\begin{aligned} E[\hat{\gamma}_s] &= E\left[2B_{(1)} - A_{(1)} - C_{(1)}\right] \\ &= \gamma_s + \frac{2\beta}{N} - \frac{\beta}{N} - \frac{\beta}{N} = \gamma_s, \end{aligned}$$

and

$$\begin{aligned} \text{var}(\hat{\gamma}_s) &= E\left[(\hat{\gamma}_s - \gamma_s)^2\right] \\ &= E\left[\left(2\varepsilon^{rs}_{(1)} - \varepsilon^{rt}_{(1)} - \varepsilon^{ts}_{(1)}\right)^2\right] = \frac{6\beta^2}{N^2}, \end{aligned}$$

where the relation (4) above has been employed. Now the PDF of $\hat{\gamma}_s$ can be derived as follows. Note that (3) can be expressed as

$$\hat{\gamma}_s - \gamma_s = 2\varepsilon^{rs}_{(1)} - \left(\varepsilon^{rt}_{(1)} + \varepsilon^{ts}_{(1)}\right) = x - y,$$

where $x = 2\varepsilon^{rs}_{(1)}$ and $y = \varepsilon^{rt}_{(1)} + \varepsilon^{ts}_{(1)}$ for simplicity. From (4), it is evident that

$$p_X(x) = \frac{N}{2\beta} e^{-\frac{N}{2\beta}x} u[x]. \tag{5}$$

For finding the PDF of y, note that $\varepsilon^{rt}_{(1)}$ and $\varepsilon^{ts}_{(1)}$ are the first order statistics of independent data sets and hence these are also independent with the same distribution as (4), so

$$\begin{aligned} p_Y(y) &= \int_{-\infty}^{\infty} p_{\varepsilon^{rt}_{(1)}}\left(y - \varepsilon^{ts}_{(1)}\right) \cdot p_{\varepsilon^{ts}_{(1)}}\left(\varepsilon^{ts}_{(1)}\right) \cdot \\ &\qquad u\left[y - \varepsilon^{ts}_{(1)}\right] \cdot u\left[\varepsilon^{ts}_{(1)}\right] \, d\varepsilon^{ts}_{(1)} \\ &= \frac{N^2}{\beta^2} \int_0^y e^{-\frac{N}{\beta}\left(y - \varepsilon^{ts}_{(1)}\right)} \cdot e^{-\frac{N}{\beta}\varepsilon^{ts}_{(1)}} \, d\varepsilon^{ts}_{(1)} \\ &= \frac{N^2}{\beta^2}\, y\, e^{-\frac{N}{\beta}y} u[y], \end{aligned} \tag{6}$$

which is a Gamma distribution with scale parameter β/N and shape parameter 2. Clearly, $\hat{\gamma}_s - \gamma_s$ is actually the difference between an exponential random variable and a Gamma random variable, both of which are independent and positive valued. Therefore, the domain of $x - y$ is $(-\infty, \infty)$ and the PDF of $\hat{\gamma}_s$ can be derived using (5) and (6) as follows.

For $\hat{\gamma}_s \leq \gamma_s$,

$$\begin{aligned} p_{\hat{\gamma}_s}(\hat{\gamma}_s) &= \frac{N^3}{2\beta^3} \int_{-\infty}^{\infty} e^{-\frac{N}{2\beta}(\hat{\gamma}_s - \gamma_s + y)} y e^{-\frac{N}{\beta}y} \cdot \\ &\qquad u[\hat{\gamma}_s - \gamma_s + y]\, u[y]\, dh \\ &= \frac{N^3}{2\beta^3} e^{-\frac{N}{2\beta}(\hat{\gamma}_s - \gamma_s)} \int_{-(\hat{\gamma}_s - \gamma_s)}^{\infty} y e^{-\frac{3N}{2\beta}y} dh \\ &= \frac{N^3}{2\beta^3} e^{-\frac{N}{2\beta}(\hat{\gamma}_s - \gamma_s)} \left[e^{\frac{3N}{2\beta}(\hat{\gamma}_s - \gamma_s)} \left(-\frac{2\beta(\hat{\gamma}_s - \gamma_s)}{3N} \right.\right. \\ &\qquad \left.\left. + \frac{4\beta^2}{9N^2} \right) \right] \\ &= \frac{2N}{9\beta} e^{\frac{N}{\beta}(\hat{\gamma}_s - \gamma_s)} - \frac{N^2}{3\beta^2}(\hat{\gamma}_s - \gamma_s) e^{\frac{N}{\beta}(\hat{\gamma}_s - \gamma_s)} \\ &= \frac{N}{3\beta}\left[\frac{2}{3} - \frac{N}{\beta}(\hat{\gamma}_s - \gamma_s)\right] e^{\frac{N}{\beta}(\hat{\gamma}_s - \gamma_s)}. \end{aligned}$$

And for $\hat{\gamma}_s \geq \gamma_s$,

$$\begin{aligned} p_{\hat{\gamma}_s}(\hat{\gamma}_s) &= \frac{N^3}{2\beta^3}\int_{-\infty}^{\infty} e^{-\frac{N}{2\beta}(\hat{\gamma}_s-\gamma_s+y)} y e^{-\frac{N}{\beta}y}. \\ &\qquad u[y]\,u[\hat{\gamma}_s-\gamma_s+y]\,dh \\ &= \frac{N^3}{2\beta^3}e^{-\frac{N}{2\beta}(\hat{\gamma}_s-\gamma_s)}\int_{0}^{\infty} y e^{-\frac{3N}{2\beta}y} dh \\ &= \frac{N^3}{2\beta^3}e^{-\frac{N}{2\beta}(\hat{\gamma}_s-\gamma_s)}\left[\frac{4\beta^2}{9N^2}\right] \\ &= \frac{2N}{9\beta}e^{-\frac{N}{2\beta}(\hat{\gamma}_s-\gamma_s)}. \end{aligned}$$

Consequently, the PDF $p_{\hat{\gamma}_s}(\hat{\gamma}_s)$ is given by

$$p_{\hat{\gamma}_s}(\hat{\gamma}_s) = \begin{cases} \frac{N}{3\beta}\left[\frac{2}{3}-\frac{N}{\beta}(\hat{\gamma}_s-\gamma_s)\right]e^{\frac{N}{\beta}(\hat{\gamma}_s-\gamma_s)} & \hat{\gamma}_s \leq \gamma_s \\ \frac{2N}{9\beta}e^{-\frac{N}{2\beta}(\hat{\gamma}_s-\gamma_s)} & \hat{\gamma}_s \geq \gamma_s \end{cases}.$$

To confirm if it is indeed a valid PDF, note that

$$\begin{aligned} \frac{2N}{9\beta}\int_{-\infty}^{0} e^{\frac{N}{\beta}(\hat{\gamma}_s-\gamma_s)} d(\hat{\gamma}_s-\gamma_s) &= \frac{2}{9}, \\ -\frac{N^2}{3\beta^2}\int_{-\infty}^{0}(\hat{\gamma}_s-\gamma_s)e^{\frac{N}{\beta}(\hat{\gamma}_s-\gamma_s)} d(\hat{\gamma}_s-\gamma_s) &= \frac{1}{3}, \\ \frac{2N}{9\beta}\int_{0}^{\infty} e^{-\frac{N}{2\beta}(\hat{\gamma}_s-\gamma_s)} d(\hat{\gamma}_s-\gamma_s) &= \frac{4}{9}, \end{aligned}$$

which sum up to 1. Finally, to verify its unbiasedness, note that

$$\begin{aligned} E[\hat{\gamma}_s] &= \frac{N}{3\beta}\int_{-\infty}^{\gamma_s}\hat{\gamma}_s\left[\frac{2}{3}-\frac{N}{\beta}(\hat{\gamma}_s-\gamma_s)\right]e^{\frac{N}{\beta}(\hat{\gamma}_s-\gamma_s)}d\hat{\gamma}_s \\ &\quad + \frac{2N}{9\beta}\int_{\gamma_s}^{\infty}\hat{\gamma}_s e^{-\frac{N}{2\beta}(\hat{\gamma}_s-\gamma_s)}d\hat{\gamma}_s \\ &= \left(-\frac{N}{3\beta}\gamma_s{}^2+\frac{2}{3}\gamma_s-\frac{2\beta}{3N}+\frac{2}{9}\gamma_s+\frac{N}{3\beta}\gamma_s{}^2-\frac{1}{3}\gamma_s \right. \\ &\quad \left. -\frac{2\beta}{9N}\right)+\left(\frac{4}{9}\gamma_s+\frac{8\beta}{9N}\right) = \gamma_s. \end{aligned}$$

Now note that the PDF $p_{\hat{\gamma}_s}(\hat{\gamma}_s)$ is not differentiable at the point $\hat{\gamma}_s = \gamma_s$, but it is continuous at this point, which implies that its domain is independent of γ_s. Differentiating $\ln p_{\hat{\gamma}_s}(\hat{\gamma}_s)$ with respect to γ_s yields

$$\frac{\partial \ln p_{\hat{\gamma}_s}(\hat{\gamma}_s)}{\partial \gamma_s} = \begin{cases} \frac{N}{\beta\left[\frac{2}{3}-\frac{N}{\beta}(\hat{\gamma}_s-\gamma_s)\right]}-\frac{N}{\beta} & \hat{\gamma}_s \leq \gamma_s \\ \frac{N}{2\beta} & \hat{\gamma}_s \geq \gamma_s \end{cases}. \quad (7)$$

To check if the regularity conditions are satisfied, the expected value of $\partial \ln p_{\hat{\gamma}_s}(\hat{\gamma}_s)/\partial\gamma_s$ is

$$\begin{aligned} E\left[\frac{\partial \ln p_{\hat{\gamma}_s}(\hat{\gamma}_s)}{\partial \gamma_s}\right] &= \int_{-\infty}^{\gamma_s}\frac{N}{\beta}\frac{N}{3\beta}e^{\frac{N}{\beta}(\hat{\gamma}_s-\gamma_s)}d\hat{\gamma}_s \\ &- \int_{-\infty}^{\gamma_s}\frac{N}{\beta}\frac{N}{3\beta}\left[\frac{2}{3}-\frac{N}{\beta}(\hat{\gamma}_s-\gamma_s)\right]e^{\frac{N}{\beta}(\hat{\gamma}_s-\gamma_s)}d\hat{\gamma}_s \\ &+ \int_{\gamma_s}^{\infty}\frac{N}{2\beta}\frac{2N}{9\beta}e^{-\frac{N}{2\beta}(\hat{\gamma}_s-\gamma_s)}d\hat{\gamma}_s \\ &= \frac{N}{3\beta}\int_{-\infty}^{\gamma_s}\left[\frac{N}{3\beta}+\frac{N^2(\hat{\gamma}_s-\gamma_s)}{\beta^2}\right]e^{\frac{N}{\beta}(\hat{\gamma}_s-\gamma_s)}d\hat{\gamma}_s \\ &+ \frac{N^2}{9\beta^2}\int_{\gamma_s}^{\infty}e^{-\frac{N}{2\beta}(\hat{\gamma}_s-\gamma_s)}d\hat{\gamma}_s = -\frac{2N}{9\beta}+\frac{2N}{\beta} \\ &= 0. \end{aligned}$$

Since the clock offset estimator is unbiased and the regularity conditions are satisfied, the CRLB exists in this case. Differentiating (7) again with respect to γ_s,

$$\frac{\partial^2 \ln p_{\hat{\gamma}_s}(\hat{\gamma}_s)}{\partial \gamma^{s2}} = \begin{cases} -\frac{N^2}{\beta^2}\left[\frac{2}{3}-\frac{N}{\beta}(\hat{\gamma}_s-\gamma_s)\right]^{-2} & \hat{\gamma}_s \leq \gamma_s \\ 0 & \hat{\gamma}_s \geq \gamma_s \end{cases}.$$

Taking the expectation on both sides,

$$E\left[\frac{\partial^2 \ln p_{\hat{\gamma}_s}(\hat{\gamma}_s)}{\partial \gamma^{s2}}\right] = -\frac{N^3}{3\beta^3}\int_{-\infty}^{\gamma_s}\left[\frac{2}{3}-\frac{N}{\beta}(\hat{\gamma}_s-\gamma_s)\right]^{-1} e^{\frac{N}{\beta}(\hat{\gamma}_s-\gamma_s)}d\hat{\gamma}_s\,.$$

Let $z = N/\beta\,(\hat{\gamma}_s-\gamma_s)-2/3$, which implies

$$\begin{aligned} E\left[\frac{\partial^2 \ln p_{\hat{\gamma}_s}(\hat{\gamma}_s)}{\partial \gamma^{s2}}\right] &= \frac{N^2}{3\beta^2}\,e^{2/3}\int_{-\infty}^{-2/3} z^{-1}e^{z}dz \\ &= \frac{N^2}{3\beta^2}\,e^{2/3}\,\mathrm{Ei}(-2/3) \quad (8) \\ &= -0.258\,\frac{N^2}{\beta^2}, \end{aligned}$$

where $\mathrm{Ei}(a)$ is the well known *Exponential Integral Function* defined as

$$\mathrm{Ei}(a) = \begin{cases} -\int_{-a}^{\infty} z^{-1}e^{-z}\,dz = \int_{-\infty}^{a} z^{-1}e^{z}\,dz, & a<0 \\ -\lim\limits_{\delta+0}\left[\int_{-a}^{-\delta}+\int_{\delta}^{\infty}\right]z^{-1}e^{-z}\,dz, & a>0 \end{cases}.$$

In the relation (8) above, the value of $\mathrm{Ei}(-2/3)$ has been computed as -0.398 and $e^{2/3} = 1.948$. Therefore, CRLB for γ_s is given by the expression

$$\mathrm{CRLB}(\hat{\gamma}_s) = \frac{3.869\beta^2}{N^2}.$$

Notice that the variance of $\hat{\gamma}_s$ is inversely proportional to the square of the number of data points N^2 and hence decreases very rapidly as the nodes exchange more and more messages. Continuing with the CRLB theory, observe from (7) that

$$\begin{aligned} \frac{\partial \ln p_{\hat{\gamma}_s}(\hat{\gamma}_s)}{\partial \gamma_s} &\neq I(\gamma_s)(\hat{\gamma}_s-\gamma_s), \\ &= \frac{N^2}{3.866\beta^2}\left(2B_{(1)}-A_{(1)}-C_{(1)}-\gamma_s\right). \end{aligned}$$

Consequently, since the MLE does not satisfy the above relation, it is not efficient (i.e., it does not attain the CRLB) for small N, although it attains this bound for large N due to its optimal properties for large number of observations. Moreover, an efficient estimator for the scenario targeted here does not exist due to the rule: if an efficient estimator exists, the maximum likelihood procedure will produce it.

Turning out attention towards finding the CRLB of $\hat{\gamma}_t$, again the likelihood function can not be used in it, in view of the fact that the domain of the likelihood function depends on γ_t. Instead, employing the PDF of $\hat{\gamma}_t$ using (2),

$$\begin{aligned} \hat{\gamma}_t &= B_{(1)} - C_{(1)} = \eta + \gamma_s + \varepsilon^{rs}_{(1)} - \left(\eta + \gamma_s - \gamma_t + \varepsilon^{ts}_{(1)}\right), \\ &= \gamma_t + \varepsilon^{rs}_{(1)} - \varepsilon^{ts}_{(1)}. \end{aligned}$$

The mean and variance of $\hat{\gamma}_t$, are given respectively by

$$E\left[\hat{\gamma}_t\right] = E\left[\gamma_t + \varepsilon^{rs}_{(1)} - \varepsilon^{ts}_{(1)}\right] = \gamma_t + \frac{\beta}{N} - \frac{\beta}{N} = \gamma_t,$$

$$E\left[\left(\hat{\gamma}_t - \gamma_t\right)^2\right] = E\left[\left(\varepsilon^{rs}_{(1)} - \varepsilon^{ts}_{(1)}\right)^2\right] = \frac{2\beta^2}{N^2},$$

which confirms the unbiasedness of $\hat{\gamma}_t$. Utilizing the fact that the difference between two exponential random variables with mean β/N is a Laplacian random variable with mean 0, $p_{\hat{\gamma}_t}(\gamma_t)$ can be expressed as

$$p_{\hat{\gamma}_t}(\gamma_t) = \begin{cases} \frac{N}{2\beta} e^{\frac{N}{\beta}(\hat{\gamma}_t - \gamma_t)} & \hat{\gamma}_t \leq \gamma_t \\ \frac{N}{2\beta} e^{-\frac{N}{\beta}(\hat{\gamma}_t - \gamma_t)} & \hat{\gamma}_t \geq \gamma_t \end{cases}.$$

Notice that the PDF is symmetric around γ_t and hence $E[(\partial \ln p_{\hat{\gamma}_t}(\gamma_t)/\partial \gamma_t)] = 0$. Differentiating both sides with respect to γ_t and taking the expectation of its square,

$$E\left[\left(\frac{\partial \ln p_{\hat{\gamma}_t}(\gamma_t)}{\partial \gamma_t}\right)^2\right] = \frac{N^2}{\beta^2},$$

and hence the CRLB for $\hat{\gamma}_t$ can be expressed as

$$\mathrm{CRLB}\left(\hat{\gamma}_t\right) = \frac{\beta^2}{N^2},$$

where again the variance is inversely proportional to N^2. This is the case due to the positive or one-sided nature of the link delays, assumed from an exponential distribution here.

Observe that due to the dependance of the CRLB on N^2 instead of N, the variance of the MLE of both γ_s and γ_t rapidly decreases with the number of observations. This is due to the optimal properties of the MLE for a large number of observations, i.e., when N increases, the MLE tends to become unbiased and its variance reaches the CRLB.

4. CONCLUSIONS

For a silent node overhearing a general sender-receiver two-way timing message exchange between two nodes, the CRLB has been derived for the unbiased estimator of its clock offset. The CRLB is shown to be inversely proportional to the square of the number of observations and hence the variance of the MLE quickly falls with the increase in data points, which in addition to its zero cost, makes it very attractive for synchronizing with the reference node. As a future work, using a higher order model, involving the clock skew, for the relationship between the clocks of inactive nodes and the reference node will increase the estimation accuracy and keep the network synchronized for a longer period of time.

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