A NEW DEFINITION OF DISCRETE QUADRATIC TIME–FREQUENCY DISTRIBUTIONS

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ABSTRACT

Discrete time–frequency distributions (DTFDs) of discrete signals differ from their continuous counterparts. The DTFD is usually aliased and lacks some desirable mathematical properties inherent to the continuous distribution. The DTFD should, ideally, be free from aliasing and conserve all desirable properties. Two existing DTFD definitions, namely the generalised DTFD (GDTFD) and the alias-free GDTFD (AF-GDTFD), approximate this ideal: the GDTFD does not satisfy all desirable properties but is alias free; the AF-GDTFD satisfies most properties but is not always alias free. We propose, in this article, a new DTFD definition which is alias free and satisfies all desirable properties.

1. INTRODUCTION

Time–frequency distributions (TFDs) are two dimensional representations of signal energy in the joint time–frequency domain. The most commonly used TFDs belong to the quadratic class [1]. The Wigner–Ville distribution (WVD) is an important member of this class—it satisfies more desirable mathematical properties compared with other distributions in the class. Its definition, for signal $z(t)$, is

$$W_z(t,f) = \int_{-\infty}^{\infty} z(t+\tfrac{\tau}{2})z^*(t-\tfrac{\tau}{2})\mathrm{e}^{-\mathrm{j}2\pi\tau f}\mathrm{d}\tau. \tag{1}$$

In addition, the WVD is a fundamental member of the quadratic class as all other quadratic TFDs are related to the WVD as follows:

$$\rho_z(t,f) = W_z(t,f) \underset{t}{*}\underset{f}{*} \gamma(t,f) \tag{2}$$

where $\gamma(t,f)$ is the time–frequency kernel and $*$ represents the convolution operation.

A digital device, such as a computer, requires a sampled TFD. This sampled TFD must be discrete in both time and frequency—that is, a discrete-time, discrete-frequency TFD, or discrete TFD (DTFD) for short. We cannot simply sample the TFD in the time–frequency domain to obtain the DTFD; rather, we must form the DTFD from the discrete time-domain signal. Ideally, the DTFD should equal a sampled TFD and satisfy all the desirable properties of the continuous distribution. This situation is, unfortunately, not attainable [2–13]. (The desirable properties, which we shall discuss later, are a set of commonly presented properties [2–5, 7, 9, 10].)

There are two widely used DTFD definitions—namely, the generalised discrete TFD (GDTFD) [6, 8, 14] and the alias-free GDTFD (AF-GDTFD) [9, 10]. The AF-GDTFD, as the name suggests, claims to be an alias-free version of the GDTFD. But this is not completely correct [4, 10, 11], as the AF-GDTFD merely has the potential to suppress aliasing, similar to the way cross-terms are suppressed by defining an appropriate kernel [1] for the distribution. Thus, the amount of aliasing removed from the distribution depends on the structure of the kernel. Nonetheless, the AF-GDTFD conserves most of the desirable properties. The GDTFD, in contrast, is alias free; it does not, however, use all the available signal information and therefore does not satisfy all desirable properties.

In this article, we present a new method for defining DTFDs. This proposed DTFD remains alias free, regardless of the kernel structure, and satisfies more desirable properties than either the GDTFD or the AF-GDTFD satisfy. The disadvantage in using this DTFD is an increased computational load. For an N-point time-domain signal, the proposed DTFD generates a $2N$ by $2N$ matrix, whereas the GDTFD and AF-GDTFD generate N by N matrices.

2. REVIEW OF EXISTING METHODS

To define a DTFD, we require a discrete WVD (DWVD) and a discrete kernel—as shown for the continuous case in (2). Because the kernel is independent of the signal, we can sample the kernel in any of the four related domains, assuming a closed-form expression for the kernel exists in that domain. These four domains—namely, the time–frequency, time–lag, doppler–lag, and doppler–frequency domains—are related by discrete Fourier transforms. Also, if we assume that the kernel is time and frequency bandlimited, then the discrete kernel will be alias free. The difficulty arises, however, when forming the DWVD, as we cannot sample the WVD in the time–frequency domain. Instead, we must form the DWVD from the discrete-time signal.

Different methods for defining the DWVD exist, which we briefly review here. These definitions form the bases for the more general DTFD definitions.

2.1 Discrete Wigner–Ville Distributions

We assume that the signal we wish to analyse is the real-valued signal $s(t)$. The complex-valued signal $z(t)$ in (1) is the analytic signal, derived from the real-valued signal $s(t)$; we use this signal to avoid cross-terms between positive and negative frequencies in the distribution [1].

To form the WVD we follow two simple steps: construct the time–lag function $K_z(t,\tau) = z(t+\tau/2)z^*(t-\tau/2)$ from the signal $z(t)$, and then Fourier transform this function to the time–frequency domain. We use a similar process to form the

DWVD.

Two conditions are necessary to form an alias-free DWVD from the discrete signal $s(nT)$ [5, 13]. (We assume that the N-point signal $s(nT)$ is Nyquist sampled with sampling period T.) First, we obtain the discrete analytic signal $z(nT)$ from $s(nT)$ [13], and second, we zero-pad $z(nT)$ to length $2N$ and call this zero-padded signal $y(nT)$. We then use $y(nT)$ to form the DWVD [2, 4, 13].

The zero-padding procedure, however, introduces some energy into the negative frequency spectrum for $Y(k/2NT)$, where $Y(k/2NT)$ is the discrete Fourier transform (DFT) of $y(nT)$. Therefore, the DWVD of $y(nT)$ is not truly alias free [5,13]. In this article, however, we refer to the DWVD of $y(nT)$ as *alias free*, so we can distinguish between the aliasing caused by using the real-valued signal $s(nT)$, which is significant, and the aliasing caused by using the zero-padded analytic signal $y(nT)$, which is typically small in comparison. In [13, 15], we propose a simple procedure for reducing aliasing in the DWVD of $y(nT)$.

2.1.1 Ideal Sampling Approach

Ideally, we should sample the time–lag function $K(t,\tau)$, in both time and lag, with sampling period T to obtain the function $K^{\text{ideal}}(nT,mT)$. Fig. 1a illustrates this sampling grid. This discrete time–lag function, in terms of $y(nT)$, is

$$K_y^{\text{ideal}}(nT,mT) = y((n+\tfrac{m}{2})T)y^*((n-\tfrac{m}{2})T). \qquad (3)$$

Unfortunately, we cannot generate this function because we only have samples of $y(nT)$ when n is an integer.

2.1.2 Sampling Approach A

Claasen and Mecklenbräuker [2] proposed a simple alternative to the ideal sampling approach by just ignoring the lag samples in (3) when m is odd. This procedure uses the discrete grid illustrated in Fig. 1b to obtain the discrete function $K^{\text{A}}(nT,2mT)$. Unlike the ideal discrete time–lag function, we can generate $K_y^{\text{A}}(nT,2mT)$ from the discrete analytic signal:

$$K_y^{\text{A}}(nT,2mT) = y((n+m)T)y^*((n-m)T).$$

To compute the DWVD, we simply discrete Fourier transform (DFT) $K_y^{\text{A}}(nT,2mT)$:

$$W_y^{\text{A}}(nT,\tfrac{k}{2NT}) = \sum_{m=0}^{2N-1} K_y^{\text{A}}(nT,2mT)\mathrm{e}^{-\mathrm{j}\pi mk/N}. \qquad (4)$$

for $n,k = 0,1,\ldots,N-1$.

The full time–frequency region of support for the DWVD of $y(nT)$ is $0 \le t \le 2NT$ and $|f| \le 1/2T$. The DWVD in (4), however, is defined over the reduced region $0 \le t \le NT$ and $0 \le f \le 1/2T$—a quarter of the size of the full time–frequency region. Nonetheless, the DWVD W_y^{A} is periodic over this quarter-plane region:

$$W_y^{\text{A}}(nT,\tfrac{k}{2NT}) = W_y^{\text{A}}((n-pN)T,(k-qN)\tfrac{1}{2NT})$$

where p,q are integers. We call a distribution, that is periodic over the quarter time–frequency region, a *quarter-plane distribution;* likewise, we call a distribution, periodic over the full time–frequency region, a *full-plane distribution.* The DWVD W^{A} is, therefore, a quarter-plane distribution.

The continuous and discrete distributions are closely related—the DWVD W^{A} approximates samples of the WVD:

$$W_y^{\text{A}}(nT,\tfrac{k}{2NT}) \approx W_y(nT,\tfrac{k}{2NT})$$

for $n,k = 0,1,\ldots,N-1$. This equality is not exact because $y(nT)$ does not meet the two conditions for a completely alias-free DWVD [5, 13], as we discussed previously.

2.1.3 Sampling Approach B

Chan [3] proposed another approach for sampling the time–lag function. The method uses a nonuniform sampling grid, illustrated in Fig. 1c. The resultant time–lag function $K^{\text{B}}(nT/2,mT)$, for even–odd values of n, is

$$K_y^{\text{B}}(nT,2mT) = y((n+m)T)y^*((n-m)T)$$
$$K_y^{\text{B}}((n+\tfrac{1}{2})T,(2m+1)T) = y((n+m+1)T)y^*((n-m)T).$$

The DWVD is the DFT of time–lag function:

$$W_y^{\text{B}}(\tfrac{nT}{2},\tfrac{k}{4NT}) = \sum_{m=0}^{2N-1} K_y^{\text{B}}(\tfrac{nT}{2},mT)\mathrm{e}^{-\mathrm{j}\pi mk/(2N)} \qquad (5)$$

for $n,k = 0,1,\ldots,2N-1$. This distribution, like the DWVD W^{A}, is also a quarter-plane distribution.

Also, like the DWVD W^{A}, it approximates samples of the continuous WVD:

$$W_y^{\text{B}}(\tfrac{nT}{2},\tfrac{k}{4NT}) \approx W_y(\tfrac{nT}{2},\tfrac{k}{4NT})$$

for $n,k = 0,1,\ldots,2N-1$.

Sampling approach A and B differ; discrete grid B uses more sample points compared with grid A—Fig. 1a and Fig. 1c illustrate. The advantage of using the denser sampling grid, when forming W^{B}, is that all the available signal information is used. The benefit of this is that W^{B} satisfies more desirable properties, such as Moyal's formula and the frequency marginal, than W^{A} satisfies [3, 5].

Similar to W^{A}, W^{B} is periodic over the quarter-plane distribution, although an extra multiplicative factor is present:

$$W_y^{\text{B}}(\tfrac{nT}{2},\tfrac{k}{4NT}) = (-1)^{nq+kp}W_y^{\text{B}}((n-p2N)\tfrac{T}{2},(k-q2N)\tfrac{1}{4NT}). \qquad (6)$$

The two DWVDs are closely related [4, 5]: W^{A} is a decimated, in time and frequency, version of W^{B}—that is,

$$W_y^{\text{A}}(nT,\tfrac{k}{2NT}) = W_y^{\text{B}}(\tfrac{2nT}{2},\tfrac{2k}{4NT}). \qquad (7)$$

2.2 Discrete Time–Frequency Distributions

2.2.1 Generalised-DTFD method

The method to form the generalised DTFD (GDTFD) [6, 8, 14] is as follows. First, define the doppler–lag kernel $g(\nu,\tau)$ within the region $|\nu| \le 1/2T$ and $|\tau| \le NT$. Second, sample this kernel in the time–lag domain with discrete grid A, as discussed in Section 2.1.2. Third, map this discrete time–lag function to the time–frequency domain, using the DFT, to obtain the time–frequency kernel $\gamma^{\text{A}}(nT,k/2NT)$. Fourth, form the DWVD W_y^{A} defined in (4). Lastly, convolve the time–frequency kernel with the DWVD:

$$\rho_y^{\text{A}}(nT,\tfrac{k}{2NT}) = W_y^{\text{A}}(nT,\tfrac{k}{2NT}) \underset{n}{\circledast}\underset{k}{\circledast} \gamma^{\text{A}}(nT,\tfrac{k}{2NT}).$$

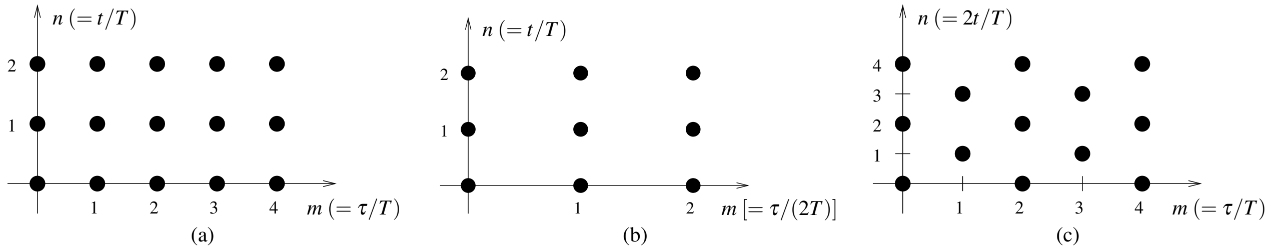

Figure 1: Different time–lag (t,τ) sampling grids with sampling period T: (a) ideal discrete grid (nT,mT), (b) discrete grid A $(nT,2mT)$, and (c) discrete grid B $(nT/2,mT)$.

The symbol $\circledast$ represents the circular convolution operation.

The relation between the GDTFD and the continuous WVD and kernel is

$$\rho_y^{\mathrm{A}}(nT,\tfrac{k}{2NT}) \approx W_y(nT,\tfrac{k}{2NT}) \underset{n}{\circledast}\underset{k}{\circledast} \gamma(nT,\tfrac{k}{2NT})$$

for $n,k = 0,1,\ldots,N-1$. The reason for the approximation was discussed in Section 2.1.2 in relation to the DWVD. In this article, we do not consider the potential artefacts introduced by the circular convolution [16] as aliasing; thus, we call the GDTFD alias free.

2.2.2 *Alias-free GDTFD method*

The method to form the alias-free GDTFD (AF-GDTFD) [9, 10] is as follows. First, define the doppler–lag kernel $g(\nu,\tau)$ within the region $|\nu| \leq 1/T$ and $|\tau| \leq NT$, which is a larger region in the doppler direction than that for the GDTFD. Second, sample the time–lag kernel with sampling grid B, as discussed in Section 2.1.3. Third, map this discrete time–lag function to the time–frequency domain, using the DFT, to obtain the time–frequency kernel $\gamma^{\mathrm{AF}}(nT/2,k/2NT)$. Fourth, form the DWD W_s^{B}, defined in (5), using the N-point real-valued signal $s(nT)$. Then, periodically extend the quarter-plane distribution W_s^{B} to a full-plane distribution so W_s^{B} extends from $0 \leq t \leq NT$ and $|f| \leq 1/2T$. Lastly, convolve the DWD with the time–frequency kernel:

$$\rho_s^{\mathrm{AF}}(nT,\tfrac{k}{NT}) = \left[W_s^{\mathrm{B}}(\tfrac{n'T}{2},\tfrac{k'}{2NT}) \underset{n'}{\circledast}\underset{k'}{\circledast} \gamma^{\mathrm{AF}}(\tfrac{n'T}{2},\tfrac{k'}{2NT}) \right]\Bigg|_{n=2n',k=2k'}$$

for $n,k = 0,1,\ldots,N-1$. We consider only the even values of n and k from the convolution, as the odd values are zero. Because we have defined the DWD W_s^{B} and kernel γ^{AF} as full-plane distributions, the DTFD ρ_s^{AF} is also a full-plane distribution.

We can relate the AF-GDTFD to the continuous WVD and the continuous kernel:

$$\rho_s^{\mathrm{AF}}(nT,\tfrac{k}{NT}) \approx \left[\sum_{m=0}^{1}\sum_{l=0}^{1} (-1)^{n'm+lk'-lmN} W_s((n'-lN)\tfrac{T}{2},(k'-mN)\tfrac{1}{2NT}) \underset{n'}{\circledast}\underset{k'}{\circledast} \gamma(\tfrac{nT}{2},\tfrac{k}{2NT}) \right]\Bigg|_{n=2n',k=2k'}$$

Thus, as the AF-GDTFD is based on an aliased DWVD, it will not be alias free. These aliasing components, however, may be suppressed by the convolution operation with the time–frequency kernel.

3. PROPOSED METHOD

We form the proposed DTFD as follows. First, define the doppler–lag kernel $g(\nu,\tau)$ within the region $|\nu| \leq 1/2T$ and $|\tau| \leq NT$. Second, sample the kernel in the time–lag domain with a uniform sample grid—sample in time and lag with period T. This uniform sampling grid is illustrated in Fig. 1a. Third, map this discrete time–lag function to the time–frequency domain, using the DFT, to obtain the time–frequency kernel $\gamma^{\mathrm{U}}(nT,k/2NT)$. Fourth, form the alias-free DWVD W^{B} of $y(nT)$, as discussed in Section 2.1.3. Lastly, convolve the DWVD with the time–frequency kernel:

$$\rho_y^{\mathrm{U}}(\tfrac{nT}{2},\tfrac{k}{4NT}) = W_y^{\mathrm{B}}(\tfrac{nT}{2},\tfrac{k}{4NT}) \underset{n}{\bar{\circledast}}\underset{k}{\bar{\circledast}} \gamma^{\mathrm{U}}(nT,\tfrac{k}{2NT}).$$

for $n,k = 0,1,\ldots,2N-1$. The symbol $\bar{\circledast}$ represents a *modified* circular convolution operation that compensates for the nonstandard periodic form of W^{B} in (6)—caused by the nonuniform discrete grid of K^{B}. The proposed DTFD has the same periodic relation as W^{B} in (6), and is, therefore, a quarter-plane distribution.

The proposed DTFD is related to the continuous WVD and continuous kernel as follows:

$$\rho_y^{\mathrm{U}}(\tfrac{nT}{2},\tfrac{k}{4NT}) \approx W_y(\tfrac{nT}{2},\tfrac{k}{4NT}) \underset{n}{\bar{\circledast}}\underset{k}{\bar{\circledast}} \gamma(nT,\tfrac{k}{2NT})$$

for $n,k = 0,1,\ldots,2N-1$. Thus, like the GDTFD, the proposed DTFD is alias free. (Again, we do not call the potential artefacts produced by the circular convolution as aliasing.)

3.1 Relation with AF-GDTFD and GDTFD

We can easily show that the proposed DTFD is closely related to the GDTFD:

$$\rho_y^{\mathrm{U}}(\tfrac{2nT}{2},\tfrac{2k}{4NT}) = \rho_y^{\mathrm{A}}(nT,\tfrac{k}{2NT}). \qquad (8)$$

Hence the GDTFD is contained within the proposed DTFD, just as the DWVD W^{A} is contained within the DWVD W^{B}, as described in (7).

Even though the AF-GDTFD and proposed DTFD both use the DWVD W^{B}, they are not directly related because of the following.

1. The AF-GDTFD uses the time-domain Nyquist sampled signal $s(nT)$. We have assumed that this N-point signal is real valued, although it could be a complex-valued, non-analytic signal. The proposed DTFD uses the $2N$-point zero-padded analytic signal $y(nT)$.
2. The kernel g^{AF} is defined, in the doppler–lag domain (ν,τ), over the region $|\nu| \leq 1/T$ and $|\tau| \leq NT$. The region for g^{U}, however, extends over the smaller region $|\nu| \leq 1/2T$ and $|\tau| \leq NT$. (Although both doppler–lag kernels have the same extent in the lag direction, this is a comparatively smaller region for the proposed DTFD because this distribution uses a $2N$-point signal, whereas the AF-GDTFD uses an N-point signal.)
3. The two time–lag kernels G^{AF} and G^{U} have different discrete grids: the AF-GDTFD kernel $G^{\mathrm{AF}}(nT/2,mT)$ has a nonuniform discrete grid, illustrated in Fig. 1c; the proposed kernel $G^{\mathrm{U}}(nT,mT)$ has a uniform discrete gird, illustrated in Fig. 1a.
4. The AF-GDTFD is a *full-plane distribution*—it is periodic over the time–frequency (t, f) region $0 \leq t \leq NT$ and $|f| \leq 1/2T$; the proposed DTFD is a *quarter-plane distribution*—it is periodic over the smaller region $0 \leq t \leq NT$ and $0 \leq f \leq 1/2T$.

3.2 Properties

Here we present a set of properties, inherent to the continuous TFD given a particular kernel constraint, which the discrete distribution should, ideally, satisfy [1–3, 5, 7, 9, 10]. Table 1 shows importance of the proposed DTFD: only this distribution satisfies all of these properties.

As discussed in the introduction, the WVD is an important distribution in the class of quadratic TFDs. Moyal's formula and signal recovery are two important properties of a DWVD [1–3, 5, 7]. The DWVD contained within the AF-GDTFD does not completely satisfy these properties.

The AF-GDTFD satisfies Moyal's formula under a certain kernel constraint [10]—but the DWVD satisfies this constraint only when N is odd [17]. The same is true for the signal recovery property—that is, the DWVD from the AF-GDTFD satisfies the signal property only when N is odd.

4. EXAMPLE

We present an example to illustrate the difference between the AF-GDTFD and the proposed DTFD. Each distribution uses three different doppler–lag kernels: a lag-independent kernel, which uses a Hamming window; a doppler-independent kernel, which uses a Hanning window; and a separable kernel, which combines the Hamming and Hanning windows [1].

The test signal we use is a linear frequency modulated (LFM) signal:

$$s_1(nT) = \cos\left(2\pi(0.1n + 0.3n^2/128)\right)$$

for $n = 0,1,\ldots,N-1$, where $N = 64$ and $T = 1$. The AF-GDTFDs of $s_1(nT)$, using the three different kernels, are plotted in Fig. 2. The plots indicate that the amount of aliasing suppressed is dependent on the kernel. Others have noted this behaviour [10, 11].

The proposed DTFD, which first transforms the real-valued signal $s_1(nT)$ to the $2N$-point zero-padded analytic signal, is also plotted in Fig. 2 for the three different kernels.

Table 1: Discrete properties for the three DTFDs.

	GDTFD	AF-GDTFD	Proposed DTFD
real	•	•	•
nonnegative	•	•	•
TF covariance	•	•	•
time marginal	•	•	•
freq. marginal		•	•
time support	•		•
freq. support	•		•
IF	•	•	•
group delay		•	•
Moyal's formula		•[1]	•
signal recovery		•[1]	•

Legend: IF: instantaneous frequency; TF: time–frequency; •: property satisfied;
[1] DWVD satisfies this property for N odd only.

The distribution is clearly different from the AF-GDTFD: it is alias-free for all kernels. We do not plot the GDTFD here as it is merely decimated version of the proposed DTFD, as detailed in (8).

5. CONCLUSION

We proposed a new DTFD definition which—unlike either the GDTFD or the AF-GDTFD—satisfies all presented properties and is alias free. We call the distributions alias free, as we did not consider the potential artefacts arising from either the circular convolution operations or the signal's approximation of the finite-time, finite-bandwidth constraint as aliasing. The proposed DTFD requires a larger time–frequency array than the other definitions require: for an N-point signal, the proposed DTFD contains $2N$ by $2N$ samples points, whereas the GDTFD and AF-GDTFD contain N by N sample points.

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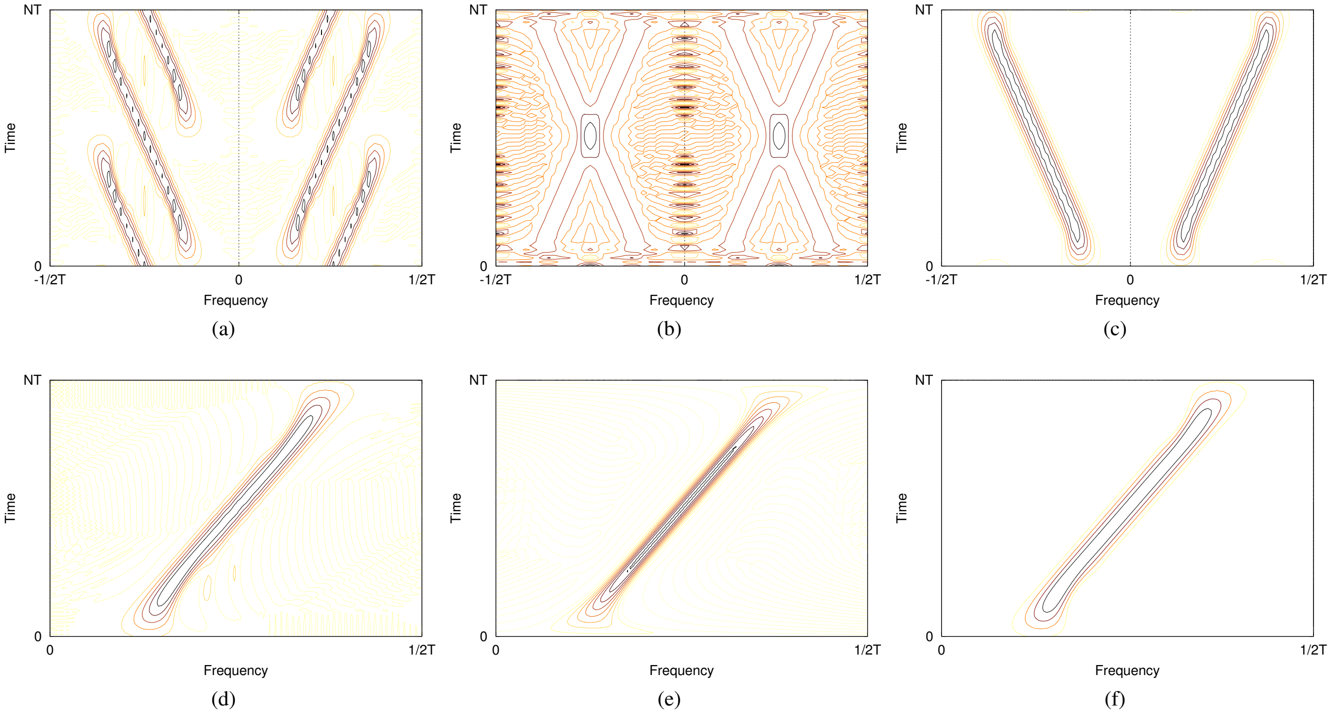

Figure 2: Comparison of AF-GDTFD and proposed DTFD of a LFM signal $s_1(nT)$: AF-GDTFD with (a) lag-independent kernel, (b) doppler-independent kernel, and (c) separable kernel; the proposed DTFD with (d) lag-independent kernel, (e) doppler-independent kernel, and (f) separable kernel.

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