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NORMALISED CONSTANT MODULUS ALGORITHM FOR BLIND CHANNEL EQUALISATION

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ABSTRACT

A new constant modulus algorithm and two of its variants are presented for blind equalisation of complex-valued communication channels. The proposed algorithm is obtained by solving a novel deterministic optimisation criterion which comprises the dispersion minimisation of ***a priori*** *as well as* ***a posteriori*** *quantities leading to an update equation having a particular zero-memory continuous nonlinearity. The convergence analysis of a variant of the proposed algorithm is presented.*

I. INTRODUCTION

Adaptive equalisation techniques are of great importance in modern high-efficiency communication systems. Conventionally, an adaptive equaliser is employed with the aid of a training sequence known to both the transmitting and receiving ends. This training session, however, can be rather costly or even unrealistic in certain applications such as asynchronous wireless network. To improve the overall throughput of a transmission system, the use of a training period is avoided by performing blind equalisation on the receiver side. Among all algorithms for blind equalisation, the constant modulus algorithm (CMA) [1], [2] plays a vital role. One of the most important features of CMA is that it can equalise constant modulus as well as non-constant modulus (like QAM) signals.

[1] The cost function CM$(q,2)$ can be written as [1]:

$$\boxed{\min_{\mathbf{w}_n} \mathbb{E}\left[(|y_n|^q - \gamma^q)^2\right]} \tag{1}$$

The criterion (1) minimises the dispersion of the modulus of *a priori* output y_n away from a statistical constant γ. The cost yields the following stochastic gradient algorithm:

$$\boxed{\text{CMA}(q,2)\text{: } \mathbf{w}_{n+1} = \mathbf{w}_n + \mu \mathbf{x}_n(\gamma^q - |y_n|^q)|y_n|^{q-2}y_n^*} \tag{2}$$

where the dispersion constant γ^q (which also serves as an equaliser gain) is obtained as:

$$\gamma^q = \frac{\mathbb{E}[|a|^{2q}]}{\mathbb{E}[|a|^q]} \tag{3}$$

[1]Assuming a time-invariant channel, the channel and equaliser outputs at the Baud rate are given by $x_n = \sum_{k=0}^{K-1} h_k a_{n-k} + \nu_n$ and $y_n = \mathbf{w}_n^H \mathbf{x}_n$, respectively, where $\{h_k\}$ is the (K-tap FIR) channel impulse response, $\mathbf{w}_n$ is the (N-tap FIR) equaliser vector at time instant n, a_n and ν_n are the channel input and additive noise sample, respectively, at time instant n, $\mathbf{x}_n$ is the regressor, superscripts T and H denote transpose and Hermitian transpose, respectively, and subscripts R and I denote the real and the imaginary parts of the complex entity, respectively.

The stochastic gradient algorithm (2) drops the expectation operator and minimises the resulting cost function by performing one iteration per sample period. It is interesting to note that only two members of this family, namely CMA$(1,2)$ and CMA$(2,2)$, have been widely and till recently discussed and studied (say [3], [4], [5], [6] and the references therein). The performance of CMA$(q,2)$ for $q > 2$ has been found pretty dissatisfactory especially for high-order non-constant modulus signals. In [7, page 35], describing the performance of CM$(q,2)$, Bellini states that

The value $q = 2$ provided faster convergence than $q = 1$. The performance of $q = 3$ was disappointing.

This behaviour is due to the mismatch between the signal constellation and CM cost function; as a result, the CMA update equation causes the adaptive weights to jitter (fluctuation noise) about their optimum settings even if the perfect equalisation is achieved [8]. This jitter becomes even more severe when q becomes greater than 2. Based on the jitter analysis carried out for CMA$(2,2)$ in [8], we obtain a generalised result for CMA$(q,2)$ that:

$$\text{Jitter Noise} \propto \mu^2 \mathbb{E}\left[(\gamma^q - |y_n|^q)^2 |y_n|^{2q}\right] \tag{4}$$

Notice from Eq. (4) that there is an increased jitter-noise for larger values of q. There is yet another point to mention. Refer to Fig. 1, where the error-function of CMA$(q,2)$, $y|y|^{q-2}(\gamma^q - |y|^q)$, has been plotted for a set of values of q for some real-valued signal. Observe in Fig. 1 that for larger values of q, the error-function gets aggregated at the larger values of $|y|$ and leaves a dead-zone for smaller values of $|y|$. Thus an equaliser implementing CMA$(q,2)$ for a large q virtually makes *no* adaptation for small values of $|y|$. We believe that the worse convergence behaviour of CMA$(q,2)|_{q>2}$ is also due to this phenomenon.

In the next section, we aim to obtain a newer form of CMA with reduced jitter-noise and even better error-function.

II. PROPOSED ALGORITHMS

After having the equaliser estimate $\mathbf{w}_n$, we want to adapt it by considering the following instantaneous optimisation problem:

$$\min_{\mathbf{w}_{n+1}} (|\mathbf{w}_{n+1}^H \mathbf{x}_n|^q - \gamma^q)^2 \tag{5}$$

where we denote $\mathbf{w}_{n+1}^H \mathbf{x}_n = s_n$ as the *a posteriori* output of the equaliser. It is obvious that we can minimise this cost function perfectly, leading to $|s_n| = \gamma$, $\forall q \geq 1$, while leaving $\mathbf{w}_{n+1}$ largely undetermined. To fix the degree of freedom in $\mathbf{w}_{n+1}$,

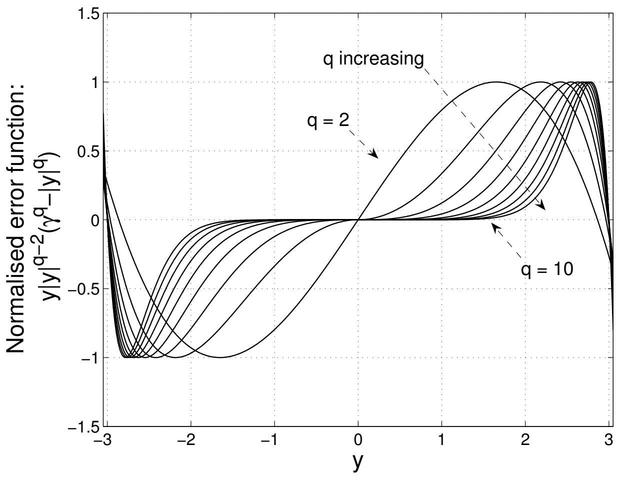

Fig. 1. Error-function of CMA$(q,2)$ for $q = 2, 3, \cdots, 10$.

we impose that $\mathbf{w}_{n+1}$ remains as close as possible to its prior estimate $\mathbf{w}_n$, while satisfying the constraints imposed by the new data, leading to $\min_{\mathbf{w}_{n+1}} \|\mathbf{w}_{n+1} - \mathbf{w}_n\|_2^2$ subject to $|s_n|^q = \gamma^q$. Using Lagrange multipliers, we may formulate our optimisation problem as follows:

$$\min_{\mathbf{w}_{n+1}} \left\{ \|\mathbf{w}_{n+1} - \mathbf{w}_n\|_2^2 + \lambda(|s_n|^q - \gamma^q) \right\} \quad (6)$$

Notice that, for $q = 2$, (6) becomes the same cost which had been used in [9] to obtain a sort of normalised constant modulus algorithm. Consider a normalised adaptive algorithm $\mathbf{w}_{n+1} = \mathbf{w}_n + \mu(\varphi[y_n] - y_n)^* \mathbf{x}_n \|\mathbf{x}_n\|_2^{-2}$, where $\varphi[y_n]$ is an appropriate blind estimate of the desired signal based on y_n. We obtain

$$s_n = \mathbf{w}_{n+1}^H \mathbf{x}_n = \mu\varphi[y_n] + (1-\mu)y_n. \quad (7)$$

This shows that the *a posteriori* output s_n is a convex combination of the *a priori* output y_n and the blind estimate $\varphi[y_n]$ (this observation has been borrowed from [10]). Hence, s_n will be closer to the $\varphi[y_n]$ than y_n (where the step-size $\mu \in (0,1]$ controls the extent to which s_n approaches $\varphi[y_n]$). It provides us a heuristic idea that better blind equalisation algorithms may be obtained if both *a posteriori* and *a priori* outputs are forced to come closer to the blind estimate. We can develop a cost function to minimise some joint dispersion of moduli of s_n and y_n, leading to a modified form of (6) as follows:

$$\min_{\mathbf{w}_{n+1}} \left\{ \|\mathbf{w}_{n+1} - \mathbf{w}_n\|_2^2 + \lambda\left(|s_n|^t |y_n|^p - \gamma^{t+p}\right) \right\} \quad (8)$$

$t, p \geq 1$. For tractable derivation, we use $t = 2$ and $p = 2q-2$ leading to

$$\boxed{\min_{\mathbf{w}_{n+1}} \left\{ \|\mathbf{w}_{n+1} - \mathbf{w}_n\|_2^2 + \lambda\left(|s_n|^2 |y_n|^{2q-2} - \gamma^{2q}\right) \right\}} \quad (9)$$

Now we differentiate (9) with respect to $\mathbf{w}_{n+1}$ and set the result equal to zero, we get

$$\mathbf{w}_{n+1}^* - \mathbf{w}_n^* + \lambda \mathbf{x}_n^* \mathbf{x}_n^T \mathbf{w}_{n+1}^* |\mathbf{w}_n^H \mathbf{x}_n|^{2q-2} = \mathbf{0} \quad (10)$$

Transposing (10) and post-multiplying it with $\mathbf{x}_n$ lead to

$$s_n - y_n + \lambda s_n |y_n|^{2q-2} \|\mathbf{x}_n\|_2^2 = 0 \quad (11)$$

Solving (11) yields into the optimum Lagrange multiplier, λ_*, as given by

$$\lambda_* = -\frac{1}{|y_n|^{2q-2}\|\mathbf{x}_n\|_2^2}\left(1 - \frac{y_n}{s_n}\right) \quad (12)$$

and the corresponding update equation is

$$\mathbf{w}_{n+1} = \mathbf{w}_n - \lambda_* s_n^* |y_n|^{2q-2} \mathbf{x}_n \quad (13)$$

At each n, the *hard constraint* in (9) enforces

$$|s_n|^2 = \left(\frac{\gamma}{|y_n|}\right)^{2q} |y_n|^2 \Leftrightarrow s_n = \left(\frac{\gamma}{|y_n|}\right)^q y_n$$

Therefore the optimum Lagrange multiplier in (12) is

$$\lambda_* = -\frac{1}{|y_n|^{2q-2}\|\mathbf{x}_n\|_2^2}\left(1 - \frac{|y_n|^q}{\gamma^q}\right) \quad (14)$$

At this stage, partly motivated by the work in [9], we introduce a factor of relaxation, η, in (14) to gain some control over the convergence speed. It implies that the constraint on s_n is now retained as a *soft constraint*. By introducing η, we have a *relaxed* Lagrange multiplier, $\lambda_{*,\eta}$, as given by

$$\lambda_{*,\eta} = -\frac{\eta}{|y_n|^{2q-2}\|\mathbf{x}_n\|_2^2}\left(1 - \frac{|y_n|^q}{\gamma^q}\right) \quad (15)$$

and the corresponding update equation is

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \eta \frac{\mathbf{x}_n}{\|\mathbf{x}_n\|_2^2}\left(1 - \frac{|y_n|^q}{\gamma^q}\right) s_n^* \quad (16)$$

By transposing and post-multiplying (16) with $\mathbf{x}_n$ we obtain

$$s_n = \frac{y_n}{1 - \eta\left(1 - \dfrac{|y_n|^q}{\gamma^q}\right)} \quad (17)$$

Substituting (17) in (16) we obtain

$$\boxed{\mathbf{w}_{n+1} = \mathbf{w}_n + \frac{\mathbf{x}_n}{\|\mathbf{x}_n\|_2^2} \frac{\eta\left(1 - \dfrac{|y_n|^q}{\gamma^q}\right)}{1 - \eta\left(1 - \dfrac{|y_n|^q}{\gamma^q}\right)} y_n^*.} \quad (18)$$

We denote (18) as *soft constraint satisfaction constant modulus algorithm of order* q (SCS-CMAq). For the given signal $\{a\}$, a small relaxation-factor η and the order q, the value of the dispersion constant γ^q for SCS-CMAq is obtained as follows:

$$\gamma^q = \left(\frac{1+2\eta}{1+\eta}\right) \frac{\mathbb{E}[|a|^{q+2}]}{\mathbb{E}[|a|^2]}. \quad (19)$$

The computational complexity of the proposed algorithm (18) is little higher than that of the conventional CMA. This complexity can be reduced by observing that, with a suitable choice of η and successful convergence, $\mathbb{E}[\eta|1 - |y_n|^q/\gamma^q|] \ll 1$, and can be removed from the denominator of (18). This observation leads to the following two simplified variants of (18):

$$\boxed{\text{SCS-CMA}q\text{-I: } \mathbf{w}_{n+1} = \mathbf{w}_n + \mu \frac{\mathbf{x}_n}{\|\mathbf{x}_n\|_2^2}\left(\gamma^q - |y_n|^q\right) y_n^*} \quad (20)$$

and

$$\boxed{\text{SCS-CMA}q\text{-II: } \mathbf{w}_{n+1} = \mathbf{w}_n + \mu\, \mathbf{x}_n \left(\gamma^q - |y_n|^q\right) y_n^*} \quad (21)$$

where μ is a suitable step-size and (21) is the unnormalised version of (20). *Also, comparing (21) and (2), we notice that* *they are similar only for* $q = 2$. The dispersion constant γ^q for (20) and (21) is obtained as

$$\gamma^q = \frac{\mathbb{E}[|a|^{q+2}]}{\mathbb{E}[|a|^2]}. \quad (22)$$

Notice that the jitter-noise exhibited by SCS-CMAq-II is given as

$$\text{Jitter Noise} \propto \mu^2 \mathbb{E}\left[(\gamma^q - |y_n|^q)^2 |y_n|^4\right] \quad (23)$$

which is much smaller than that of CMA$(q, 2)$ for $q > 2$. Also notice in Fig. 2 that the error-function of SCS-CMAq-II exhibits a highly admissible form for large values of q.

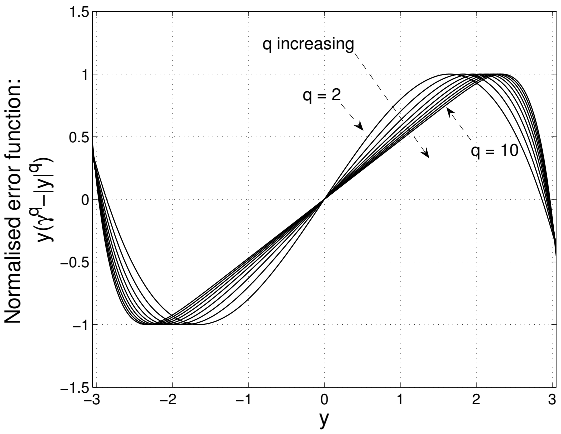

Fig. 2. Error-function of SCS-CMAq-II.

III. CONVERGENCE ANALYSIS OF SCS-CMAq-II

The convergence behaviour of stop-and-go[2] (selective) update CMA$(2, 2)$ has been studied by Rupp and Sayed [11]. They showed that for transmitted signals with constant modulus γ, the equaliser implementing CMA$(2, 2)$ is capable of making its outputs to lie within the circle of radius $\gamma\sqrt{c}$ infinitely often, for some value of c that is slightly larger than one. Due to the similarity which exist between CMA$(q, 2)$ and SCS-CMAq-II, we intend to carry out the similar analysis for SCS-CMAq-II to gain some insight into its convergence behaviour. At the end, we would be able to show that larger value of q has an advantageous effect of forcing c to come close to unity.

A. Stop-and-Go (Selective) Updating

The corresponding active update steps of SCS-CMAq-II have the form

$$\begin{cases} \text{if} \quad |y_n| \geq \gamma\sqrt{c} \\ \text{then } k = k + 1, \\ \qquad e_k = (\gamma^q - |y_k|^q)\, y_k, \\ \qquad \mathbf{w}_{k+1} = \mathbf{w}_k + \mu_k e_k^* \mathbf{x}_k. \end{cases} \quad (24)$$

Assume we run the above algorithm infinitely often (i.e., $n \to \infty$), and let K denote the maximum number of active updates that occurred in the process. We now prove that, by properly designing the step-size sequence, K can be made finite, which in turn means that the condition $|y_n| < \gamma\sqrt{c}$ will hold infinitely often. Let $\mathbf{w}$ denote the weight vector of the optimal equaliser and let $z_k = \mathbf{w}^H \mathbf{x}_k = a_{k-D}$ is the optimal output for some D [so that $|z_k| = \gamma$]. Define further the *a priori* and *a posteriori* estimation errors

$$\begin{aligned} e_k^a &= z_k - y_k = \widetilde{\mathbf{w}}_k^H \mathbf{x}_k \\ e_k^p &= z_k - s_k = \widetilde{\mathbf{w}}_{k+1}^H \mathbf{x}_k \end{aligned} \quad (25)$$

where $\widetilde{\mathbf{w}}_k = \mathbf{w} - \mathbf{w}_k$. We introduce a complex-valued function $h[z_1, z_2]$:

$$h[z_1, z_2] \triangleq \frac{z_1|z_1|^q - z_2|z_2|^q}{z_1 - z_2}, \quad (z_1 \neq z_2) \quad (26)$$

Using $h[\cdot, \cdot]$, we obtain (refer to [11])

$$e_k = (h[z_k, y_k] - \gamma^q)\, e_k^a \quad (27)$$

$$e_k^p = \left(1 - \frac{\mu_k}{\overline{\mu}_k}\left[h[z_k, y_k] - \gamma^q\right]\right) e_k^a \quad (28)$$

where $\overline{\mu}_k = 1/\|\mathbf{x}_k\|_2^2$ denotes the reciprocal of the input energy at the iteration k. In section II, we have pointed out that the *a posteriori* output s_n (or s_k) is closer to the blind estimate than *a priori* output y_n (or y_k), which requires that $|e_k^p| < |e_k^a|$. To ensure it, we need to select the step-size sequence μ_k so as to guarantee for all k

$$\left|1 - \frac{\mu_k}{\overline{\mu}_k}\left[h[z_k, y_k] - \gamma^q\right]\right| < d < 1 \quad (29)$$

for all possible combinations of z_k and y_k, the value of q and for some positive scalar d. Let $h_R[z_k, y_k]$ and $h_I[z_k, y_k]$ denote the real and imaginary parts of $h[z_k, y_k]$, respectively; from (29) we obtain

$$\frac{\mu_k^2}{\overline{\mu}_k^2}\left(h_I[z_k, y_k]\right)^2 + \left(1 - \frac{\mu_k}{\overline{\mu}_k}\left[h_R[z_k, y_k] - \gamma^q\right]\right)^2 < 1 \quad (30)$$

The values of μ_k for which (30) can be ensured, we have the following theorem:

Theorem 1 [Stop-and-Go SCS-CMAq-II]: Assume y_k stays uniformly bounded from above for all k, say

$$\gamma\sqrt{c} \leq |y_k| \leq P\gamma < \infty \quad (31)$$

for some $P \geq \sqrt{c} > 1$. Choose a positive number β^o in the interval

$$\frac{36P^{2q} - \epsilon^2 m_o^{4/q}}{36P^{2q} + \epsilon^2 m_o^{4/q}} < \beta^o < 1. \quad (32)$$

and compute an α^o via

$$\alpha^o = \frac{6(1 - \beta^o)P^q}{\epsilon\ m_o^{2/q}}. \quad (33)$$

Choose further the step-size μ_k for the active update from within the interval

$$(1-\beta^o)\frac{1}{\|\mathbf{x}\|_2^2}\frac{2}{q\epsilon\ m_o^{2/q}\gamma^q} < \mu_k < \alpha^o \frac{1}{\|\mathbf{x}\|_2^2}\frac{1}{(q+1)P^q\gamma^q} \quad (34)$$

It then holds that $K < \infty$. That is, $|y_n| < \gamma\sqrt{c}$ holds infinitely often.

Proof: The proof follows directly from [11] (where a stop-and-go CMA(2,2) is analysed) and is thus skipped. However, proofs of Equations (32)-(34) are provided in Appendix I.

[2]Notice that this stop-and-go principle is devised for the sake of convergence analysis in [11] and it has nothing to do with the conventional Bussgang-type stop-and-go adaptation strategy as appeared in [12] and [13].

Remark 1: From Theorem 1, we can say that, for suitably chosen step-sizes, the stop-and-go SCS-CMAq-II produces a sequence of estimates y_k that lies inside the circle of radius $\gamma\sqrt{c}$ with probability one.

Remark 2: We notice that the choice of μ_k is small and it becomes even smaller for large values of q. However at the same time, a larger q is beneficial in making c come close to unity. From Appendix I, we have

$$c \triangleq \left[\min_{r\in(0,1)}\left(\frac{1+r^{q+1}}{1+r}\right)\right]^{-\frac{2}{q}} + \epsilon \tag{35}$$

Notice in Table I, the corresponding values of c are decreasing monotonically and approaching unity with an increase in q.

TABLE I VALUES OF q AND c

q	1	2	3	4	5	6
$c-\epsilon$	1.4571	1.3333	1.2635	1.2185	1.1868	1.1634
q	7	8	9	10	11	∞
$c-\epsilon$	1.1452	1.1308	1.1190	1.1092	1.1009	1

IV. SIMULATIONS

In simulations, a complex-valued 7-tap transversal equaliser is used and it is initialised so that the center tap is set to one and other taps are set to zero. The propagation channel used in the simulation is a typical voice-band telephone channel and is taken from [12]. The signal to noise ratio (SNR) is taken as 30dB at the input of the equaliser. The residual ISI [14] is measured for (circular) 8-QAM signaling and compared as performance parameter. Signal alphabets are taken from the set $\left\{\pm 1 \pm \jmath, \frac{1}{\sqrt{2}}(1+\sqrt{3})(\pm 1 \pm \jmath)\right\}$. Each trace is the ensemble average of 500 independent runs with random initialisation of noise and data source. Notice that the values of the step-sizes are taken to be constant in all cases and are also mentioned in Fig. 3.

The performance of SCS-CMAq-II has been studied for $q = 1, 2, 4, 6$ and 8. As expected and demonstrated in Fig. 3, the performance gets better in terms of steady-state residual ISI when larger q is used. Fig. 3 also depicts convergence trace for the conventional normalised CMA (NCMA); notice that, except for $q = 1$, NCMA performed inferior to SCS-CMAq-II. In Fig. 4, we depict scatter diagrams for last 400 equalised symbols where it can be noticed that the clusters gets more aggregated when larger q is used. Moreover, with its capabilities to yield lower ISI floor and aggregated constellation, SCS-CMAq-II facilitates a reliable switch to decision-directed (tracking) mode from blind (acquisition) mode.

V. CONCLUSION

A new constant modulus algorithm and two of its variants have been presented for blind equalisation of complex-valued communication channels. The proposed algorithm has been obtained by solving a novel deterministic optimisation criterion, based on the dispersion minimisation of *a priori* as well as *a posteriori* quantities leading to an update equation having a particular zero-memory continuous nonlinearity. Furthermore, its convergence analysis and some simulation results have been presented.

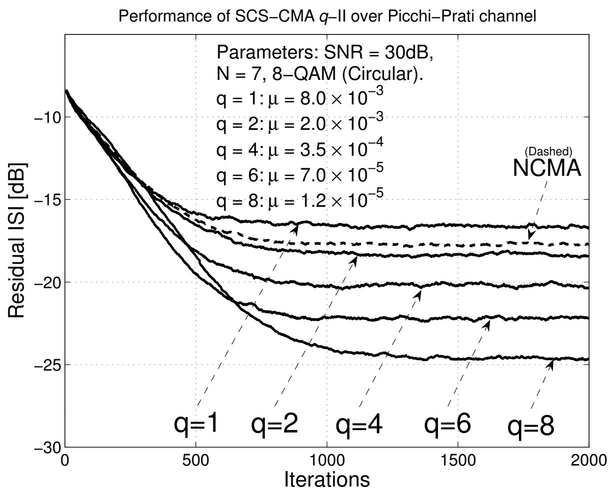

Fig. 3. Performance of SCS-CMAq-II: ISI traces.

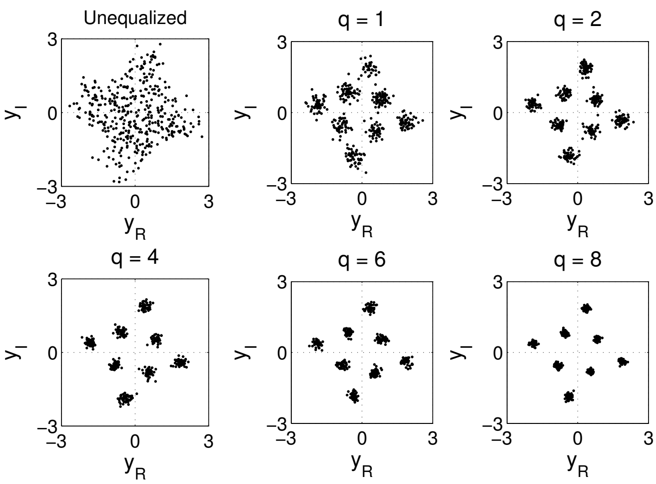

Fig. 4. Performance of SCS-CMAq-II: Scatter diagrams.

Future work will involve considering various equalisation applications (like block-processing, over-sampling, a DFE or MIMO-DFE in particular).

APPENDIX I

Let α and β be any two positive numbers satisfying

$$\alpha^2 + \beta^2 < 1 \tag{36}$$

We need to find a μ_k that satisfies

$$\left|\frac{\mu_k}{\overline{\mu}_k} h_I[z_k, y_k]\right| < \alpha \tag{37}$$

and

$$\left|1 - \frac{\mu_k}{\overline{\mu}_k}\left(h_R[z_k, y_k] - \gamma^q\right)\right| < \beta \tag{38}$$

From [11], it is straightforward to prove that[3]

$$h_R[z_k, y_k] \geq \gamma^q \left(1 + \epsilon\ m_o^{2/q}\right)^{q/2},$$
$$|h_I[z_k, y_k]| < \gamma^q (1+q) P^q.$$

We can satisfy (37) and (38) by selecting μ_k such that

$$\frac{1-\beta}{\|\mathbf{x}\|_2^2} \frac{\gamma^{-q}}{\left(1 + \epsilon\ m_o^{2/q}\right)^{q/2} - 1} < \mu_k < \frac{\alpha}{\|\mathbf{x}\|_2^2} \frac{\gamma^{-q}}{(q+1)P^q - 1}$$

Notice that for $0 < \epsilon \ll 1$ and $q \geq 1$, we can write

$$\left(1 + \epsilon\ m_o^{2/q}\right)^{q/2} \approx 1 + \frac{q}{2} \epsilon\ m_o^{2/q}$$

Also notice that $P \geq \sqrt{c} > 1$ and $q \geq 1$ give

$$\frac{1}{(q+1)P^q} < \frac{1}{(q+1)P^q - 1}$$

we found a simpler bound on μ_k as given by

$$\frac{1-\beta}{\|\mathbf{x}\|_2^2} \frac{2}{q\epsilon\ m_o^{2/q} \gamma^q} < \mu_k < \frac{\alpha}{\|\mathbf{x}\|_2^2} \frac{1}{(q+1)P^q \gamma^q} \quad (39)$$

which gives

$$\frac{1-\beta}{\alpha} < \frac{q\epsilon\ m_o^{2/q}}{2(q+1)P^q} < \frac{\epsilon\ m_o^{2/q}}{2P^q} \quad (40)$$

Let for some $\{\alpha^o, \beta^o\}$ we have

$$\frac{1-\beta^o}{\alpha^o} = \frac{1}{3} \cdot \frac{\epsilon\ m_o^{2/q}}{2P^q} = \frac{\epsilon\ m_o^{2/q}}{6P^q} \quad (41)$$

Then, $\{\alpha^o, \beta^o\}$ satisfy (40). Substituting into (36), we see that β^o must be such that

$$\left(\frac{6(1-\beta^o)P^q}{\epsilon\ m_o^{2/q}}\right)^2 + (\beta^o)^2 < 1$$

If we find a β^o that satisfies this inequality, then a pair of $\{\alpha^o, \beta^o\}$ satisfying (36) and (40) exists. So consider the following quadratic function

$$g(\beta) = \left(\frac{6(1-\beta)P^q}{\epsilon\ m_o^{2/q}}\right)^2 + (\beta)^2 - 1.$$

It has a negative minimum and it crosses the real axis at the positive roots

$$\beta^{(1)} = \frac{36P^{2q} - \epsilon^2 m_o^{4/q}}{36P^{2q} + \epsilon^2 m_o^{4/q}} < 1, \qquad \beta^{(2)} = 1.$$

Hence, β^o can be chosen as any value in the interval

$$\frac{36P^{2q} - \epsilon^2 m_o^{4/q}}{36P^{2q} + \epsilon^2 m_o^{4/q}} < \beta^o < 1. \quad (42)$$

For $q = 2$ and $m_o = 3/4$, the above result (42) can be found consistent with [11, Equation (51)]. The bounds on μ_k are thus justified. □

[3] Constants ϵ and m_o are related to c as given by $c \triangleq m_o^{-2/q} + \epsilon$; where $m_o = \min_{r \in (0,1)} \left(\frac{1 + r^{q+1}}{1+r}\right)$ and $0 < \epsilon \ll 1$.

APPENDIX II

To compute γ^q in (18), we need to solve the following [7]:

$$\mathbb{E}\left[\frac{|a|^2 \left(1 - \frac{|a|^q}{\gamma^q}\right)}{1 - \eta\left(1 - \frac{|a|^q}{\gamma^q}\right)}\right] = 0 \quad (43)$$

Since $\eta \ll 1$, we use the approximation $(1-x)^{-1} \approx 1 + x$ (where $|x| \ll 1$) to get

$$\mathbb{E}\left[|a|^2 \left(1 - \frac{|a|^q}{\gamma^q}\right)\left(1 + \eta\left(1 - \frac{|a|^q}{\gamma^q}\right)\right)\right] \approx 0 \quad (44)$$

Further we get

$$\mathbb{E}\left[|a|^2\right](1+\eta)\gamma^q - \mathbb{E}\left[|a|^{q+2}\right](1+2\eta) + \eta \frac{\mathbb{E}\left[|a|^{2q+2}\right]}{\gamma^q} = 0 \quad (45)$$

Again due to $\eta \ll 1$, we drop the last term in (45), and solve for the rest to get (19).

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