

A STUDY ON INTERLACED SAMPLING WITH UNKNOWN OFFSETS

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ABSTRACT

In this paper, interlaced sampling is discussed, for which signals/images are sampled several times by an identical sampling device like a CCD camera with slightly displaced locations. Since offset parameters are unknown, the problem becomes challenging. A typical example of this formulation is super-resolution image reconstruction from multiple low resolution images. A well-defined solution has already been devised for the case where the number of unknown parameters is less than or equal to the number of measurements. However, this condition easily fails in practical situations. Hence, this paper proposes a signal reconstruction method that provides stable solutions even if the condition does not hold. The key factors introduced here are a statistical assumption for target signals and the minimization of a cost function. Simulation results show that the proposed method performs much better than the conventional method.

1. INTRODUCTION

Sampling is the problem of reconstructing a target signal f from its sampled measurements $\{d_n\}$. By assuming that f belongs to a certain Hilbert space H , the measurements $\{d_n\}$ are modeled by the inner product as

$$d_n = \langle f, \psi_n \rangle, \quad (1)$$

where $\{\psi_n\}$ are sampling functions. The signal is reconstructed by a linear combination of reconstruction functions $\{\varphi_k\}$ as

$$\tilde{f}(x) = \sum_k c_k \varphi_k(x). \quad (2)$$

In the classical formulations, the sampling functions $\{\psi_n\}$ and reconstruction functions $\{\varphi_k\}$ were assumed to be fixed [1], [2], [3].

In contrast to them, recent discussions assume that $\{\psi_n\}$ and $\{\varphi_k\}$ involve unknown parameters, and thus are not fixed functions. For example, Vetterli *et al.* discuss problems in which locations of reconstruction functions are unknown [4],[5],[6]. They introduced the notion of rate of innovation, and provide perfect reconstruction formulas for signals with finite rate of innovation. A similar discussion is about sampling signals from union of subspaces [7].

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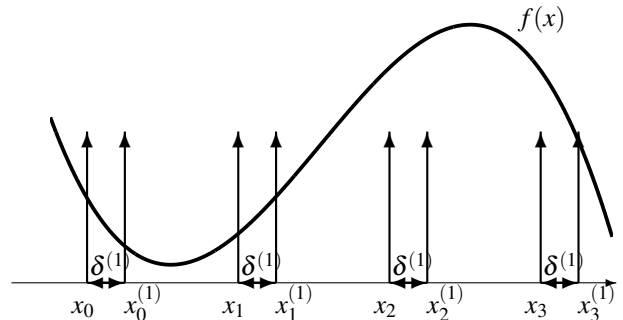


Figure 1: Interlaced sampling ($x_n^{(1)} = x_n + \delta^{(1)}$)

Table 1: Relation of the present paper to other works.

	Prior Information	
	Use	Not use
(Parameters) \leq (Samples)		[9]
(Parameters) $>$ (Samples)	This paper	[10]

On the other hand, the present paper discusses a sampling problem with unknown parameters in sampling functions. A signal is sampled by a sampling device several times with slightly displaced locations as shown in Fig. 1. This type of sampling is called interlaced sampling [8]. A typical example of this formulation is super-resolution from multiple low-resolution images.

Several solutions to this problem have been proposed so far. Vandewalle *et al.* devised a method for perfect reconstruction under a condition such that the number of unknown parameters is less than or equal to that of measurements [9]. In practical applications, however, there are many unknown parameters, and the condition does not hold in general. Browning proposed a method for signal reconstruction with unknown sample locations [10]. This method, however, does not provide good reconstruction results as shown in simulations later. Marziliano *et al.* discussed a problem of sampling discrete-time bandlimited signals with unknown locations [11]. However, this is not an interlaced sampling problem with unknown arbitrary real offsets.

Hence, in this paper, we propose a signal reconstruction method that provides a stable, high quality solution from interlaced samples with unknown offsets. To this end, we adopt a stochastic formulation, and introduce a generalized Wiener criterion. The proposed algorithm obtains a signal that min-

imizes the criterion among signals satisfying the given samples. For fixed offset parameters, the optimization problem has an analytical solution. After deriving the solution, we further minimize the criterion with respect to the offset parameters. Since this step can not be solved analytically, we use an algorithm based on an exhaustive search over a finite set of candidate parameters. Note that we use a mean signal as a priori knowledge. Even though it is not easy to know this information exactly, we do not need correlation operator nor probability density function.

This paper is organized as follows. Section 2 formulates the interlaced sampling problem. Section 3 briefly reviews conventional approaches to the problem. In Section 4, we propose a signal reconstruction algorithm based on the generalized Wiener criterion. Section 5 shows simulation results which illustrate the effectiveness of the proposed method. Section 6 concludes the paper.

2. INTERLACED SAMPLING

A signal f to be reconstructed is defined on a continuous domain \mathcal{D} . We assume that f belongs to a Hilbert space $H = H(\mathcal{D})$ of a finite dimension K . The inner product for f and g in H is denoted by $\langle f, g \rangle$, and the norm is induced as $\|f\| = \sqrt{\langle f, f \rangle}$. By using an arbitrarily fixed orthonormal basis $\{\varphi_k\}_{k=0}^{K-1}$, any f in H is expressed as

$$f = \sum_{k=0}^{K-1} a_k \varphi_k. \quad (3)$$

A K -dimensional vector with k -th element a_k is denoted by \mathbf{a} . A (conjugate) reconstruction operator A_r^* maps \mathbf{a} into f as

$$f = A_r^* \mathbf{a}. \quad (4)$$

A signal f is sampled J times by an observation device with displacements $\{\delta^{(j)}\}_{j=0}^{J-1}$, where $\delta^{(0)} = 0$. A J -dimensional vector with j -th element $\delta^{(j)}$ is denoted by δ . The observation device is characterized by sampling functions $\{\psi_n\}_{n=0}^{N-1}$. Then, the sampling function for the n -th sample in the j -th series is given by

$$\psi_n^{(j)}(x) = \psi_n(x - \delta^{(j)}), \quad (5)$$

and the sample is expressed as

$$d_n^{(j)} = \langle f, \psi_n^{(j)} \rangle. \quad (6)$$

A JN -dimensional vector with $n + jN$ -th element $d_n^{(j)}$ is \mathbf{d} . An operator that maps f into \mathbf{d} is $A_s(\delta)$:

$$A_s(\delta) f = \mathbf{d}. \quad (7)$$

A $JN \times K$ matrix with the $n + jN$, k -th element $\langle \varphi_k, \psi_n^{(j)} \rangle$ is denoted by B_δ . Since $A_s(\delta) A_r^* = B_\delta$, substituting Eq. (4) into Eq. (7) yields

$$B_\delta \mathbf{a} = \mathbf{d}. \quad (8)$$

Fig. 2 shows relations between spaces and operators.

Interlaced sampling is a problem of obtaining the perfect reconstruction or an optimum approximation under some criterion from samples $\{\{d_n^{(j)}\}_{n=0}^{N-1}\}_{j=0}^{J-1}$. If the offset parameters $\{\delta^{(j)}\}_{j=1}^{J-1}$ are known, the interlaced sampling is a linear

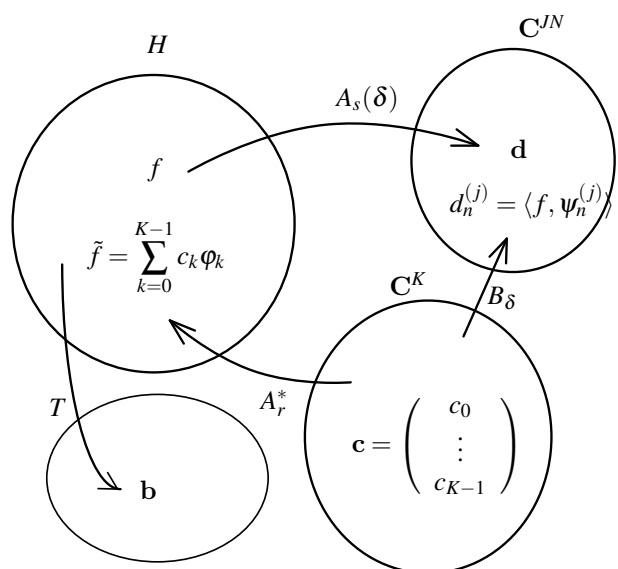


Figure 2: Spaces and operators

problem, and many solutions have already been derived [8]. If the offset parameters are unknown, the problem becomes nonlinear and challenging.

3. CONVENTIONAL APPROACHES

In order to reconstruct the signal f from interlaced samples with unknown offsets, we have to determine both $\{a_k\}_{k=0}^{K-1}$ and $\{\delta^{(j)}\}_{j=1}^{J-1}$. To this problem, two solutions have already been proposed.

Vandewalle *et al.* proposed a perfect reconstruction algorithm under a condition that the number of unknown parameters is less than or equal to the number of samples $\{\{d_n^{(j)}\}_{n=0}^{N-1}\}_{j=0}^{J-1}$, or

$$K + J - 1 \leq JN. \quad (9)$$

They proposed two algorithms, one of which is called the projection method. Let us briefly review the method.

Let $\{\varepsilon^{(j)}\}_{j=0}^{J-1}$ be parameters used for estimation of $\{\delta^{(j)}\}_{j=0}^{J-1}$, where $\varepsilon^{(0)} = 0$. A J -dimensional vector with j -th element $\varepsilon^{(j)}$ is ε . A matrix obtained by replacing δ in B_δ by ε is denoted by B_ε . The range and the Moore-Penrose generalized inverse of B_ε are denoted by $\mathcal{R}(B_\varepsilon)$ and B_ε^\dagger , respectively. The sample vector \mathbf{d} belongs to $\mathcal{R}(B_\varepsilon)$ if $\varepsilon = \delta$, but not otherwise. Hence, the algorithm obtains ε that minimizes the distance between \mathbf{d} and its projection onto $\mathcal{R}(B_\varepsilon)$, which is computed by $\|\mathbf{d} - B_\varepsilon B_\varepsilon^\dagger \mathbf{d}\|$. If the condition (9) does not hold, however, $\mathcal{R}(B_\varepsilon)$ can be the entire space \mathbf{C}^{JN} . Then, $\|\mathbf{d} - B_\varepsilon B_\varepsilon^\dagger \mathbf{d}\|$ becomes 0 irrespective of ε , and the minimization does not have any meaning.

Browning also proposed a method for signal approximation from samples at unknown locations [10]. It is interesting that the method is an extension of the algorithm proposed by Vandewalle *et al.* This method reconstructs a signal in a subspace $\{\varphi_k\}_{k=0}^{L-1}$, where $L < K$. Then, the distance $\|\mathbf{d} - B_\varepsilon B_\varepsilon^\dagger \mathbf{d}\|$ does not become zero generally. Hence, this method is applicable even if the condition (9) does not hold.

However, as shown in the simulations later, ε that minimizes the criterion can be far from the true value δ . A good approximation is not reconstructed, either. Hence, in this paper, we propose a method that provides a stable and high quality solution even if the condition (9) does not hold.

4. OPTIMUM APPROXIMATION RECONSTRUCTION

In order to solve the problem of interlaced sampling with unknown offset parameters, we introduce the following Wiener like criterion:

$$J_{org}[\tilde{f}] = E_f \|T(\tilde{f} - f)\|^2, \quad (10)$$

where E_f stands for expectation for $\{f\}$ and T is a pre-determined operator that maps f into a certain feature space. Typically, T is an identity operator or some derivative operator. It is well-known that Eq. (10) is expressed as the bias-variance decomposition:

$$J_{org}[\tilde{f}] = \|T(\tilde{f} - \bar{f})\|^2 + E_f \|T(\bar{f} - f)\|^2, \quad (11)$$

where \bar{f} is the mean of f :

$$\bar{f} = E_f f. \quad (12)$$

Note that the second term in Eq. (11) is a constant with respect to \tilde{f} . Hence, we can ignore the term. Further, since we assume noiseless samples in this paper, Eq. (7) should be satisfied. Thus, we consider the following problem.

Problem 1 Determine a vector ε that minimizes

$$J_{min}[\varepsilon] = J[\tilde{f}_\varepsilon] = \|T(\tilde{f}_\varepsilon - \bar{f})\|^2, \quad (13)$$

where \tilde{f}_ε stands for the signal f that minimizes

$$J[\tilde{f}] = \|T(\tilde{f} - \bar{f})\|^2 \quad (14)$$

subject to

$$A_s(\varepsilon)\tilde{f} = \mathbf{d} \quad (15)$$

for an arbitrarily fixed ε .

Let $\{c_k\}_{k=0}^{K-1}$ be coefficients of \tilde{f} with respect to the orthonormal basis $\{\varphi_k\}_{k=0}^{K-1}$: $c_k = \langle \tilde{f}, \varphi_k \rangle$. Then, \tilde{f} is expressed as

$$\tilde{f} = \sum_{k=0}^{K-1} c_k \varphi_k = A_r^* \mathbf{c}, \quad (16)$$

where \mathbf{c} is a K -dimensional vector with k -th element c_k . By using this expression, Problem 1 is restated in the following vector-matrix form.

Problem 2 Determine a vector ε that minimizes

$$J_{min}[\varepsilon] = J[\mathbf{c}_\varepsilon] = \|T(A_r^* \mathbf{c}_\varepsilon - \bar{f})\|^2, \quad (17)$$

where \mathbf{c}_ε stands for the vector \mathbf{c} that minimizes

$$J[\mathbf{c}] = \|T(A_r^* \mathbf{c} - \bar{f})\|^2 \quad (18)$$

subject to

$$B_\varepsilon \mathbf{c} = \mathbf{d} \quad (19)$$

for an arbitrarily fixed ε .

The *a priori* knowledge in Problems 1 and 2 is the mean signal \bar{f} . Having access to such an information is not an easy problem. But, we do not need any further information such as the correlation operator, the covariance operator, or the probability density function of f . From now on, we solve Problem 2.

Since the problem for a fixed ε is a constrained quadrature optimization problem, it is analytically solvable. We first express the solution. Let P_ε and Q_ε be matrices defined by

$$P_\varepsilon = I - B_\varepsilon^\dagger B_\varepsilon, \quad (20)$$

$$Q_\varepsilon = T A_r^* P_\varepsilon, \quad (21)$$

where I is the K -dimensional identity matrix. Further, \mathbf{w}_ε is a vector given by

$$\mathbf{w}_\varepsilon = T \bar{f} - T A_r^* B_\varepsilon^\dagger \mathbf{d}. \quad (22)$$

Theorem 1 For an arbitrarily fixed ε , if $\mathbf{d} \in \mathcal{R}(B_\varepsilon)$, \mathbf{c}_ε in Problem 2 is given by

$$\mathbf{c}_\varepsilon = B_\varepsilon^\dagger \mathbf{d} + Q_\varepsilon^\dagger \mathbf{w}_\varepsilon + (P_\varepsilon - Q_\varepsilon^\dagger Q_\varepsilon) \mathbf{u}, \quad (23)$$

where \mathbf{u} is an arbitrary K -dimensional vector. The minimum value $J_{min}[\varepsilon]$ is

$$J_{min}[\varepsilon] = \|(I - Q_\varepsilon Q_\varepsilon^\dagger) \mathbf{w}_\varepsilon\|^2. \quad (24)$$

(Proof) Since the condition $\mathbf{d} \in \mathcal{R}(B_\varepsilon)$ is true, \mathbf{c} that satisfies Eq. (19) is expressed as

$$\mathbf{c} = B_\varepsilon^\dagger \mathbf{d} + P_\varepsilon \mathbf{v}, \quad (25)$$

where \mathbf{v} is an arbitrary K -dimensional vector. Substituting Eq. (25) into Eq. (18) yields

$$\begin{aligned} J[\mathbf{c}] &= \|Q_\varepsilon \mathbf{v} - \mathbf{w}_\varepsilon\|^2 \\ &= \|Q_\varepsilon \mathbf{v} - Q_\varepsilon Q_\varepsilon^\dagger \mathbf{w}_\varepsilon\|^2 + \|(I - Q_\varepsilon Q_\varepsilon^\dagger) \mathbf{w}_\varepsilon\|^2 \\ &\geq \|(I - Q_\varepsilon Q_\varepsilon^\dagger) \mathbf{w}_\varepsilon\|^2, \end{aligned}$$

where we used Eqs. (21) and (22). The equality holds if and only if

$$Q_\varepsilon \mathbf{v} = Q_\varepsilon Q_\varepsilon^\dagger \mathbf{w}_\varepsilon. \quad (26)$$

This equation is true, for example, for $\mathbf{v} = Q_\varepsilon^\dagger \mathbf{w}_\varepsilon$. Hence, the minimum value is given by Eq. (24) indeed. All vectors \mathbf{v} that satisfies Eq. (26) is given by

$$\mathbf{v} = Q_\varepsilon^\dagger \mathbf{w}_\varepsilon + (I - Q_\varepsilon^\dagger Q_\varepsilon) \mathbf{u}. \quad (27)$$

Since $P_\varepsilon Q_\varepsilon^\dagger = Q_\varepsilon^\dagger$, substituting Eq. (27) into Eq. (25) yields Eq. (23). ■

Since an arbitrary vector \mathbf{u} exists in Eq. (23), there are infinitely many vector \mathbf{c}_ε . However, the minimum value $J_{min}[\varepsilon]$ is unique as shown in Eq. (24). Further, under the following condition, \mathbf{c}_ε also becomes unique.

Corollary 1 The vector \mathbf{c}_ε in Eq. (23) is unique if and only if

$$\mathcal{R}(T^*) + \mathcal{R}(A_s(\varepsilon)^*) = H. \quad (28)$$

In this case, \mathbf{c}_ε is given by

$$\mathbf{c}_\varepsilon = B_\varepsilon^\dagger \mathbf{d} + Q_\varepsilon^\dagger \mathbf{w}_\varepsilon. \quad (29)$$

(Proof) Assume that Eq. (28) holds. Applying A_r to both sides from the left yields

$$\mathcal{R}(A_r T^*) + \mathcal{R}(P_\varepsilon)^\perp = \mathbf{C}^K, \quad (30)$$

where we used $A_r A_s(\varepsilon)^* = B_\varepsilon^*$ and $\mathcal{R}(B_\varepsilon^*) = \mathcal{R}(P_\varepsilon)^\perp$. Further, applying P_ε to both sides from the left yields

$$\mathcal{R}(Q_\varepsilon^*) = \mathcal{R}(P_\varepsilon). \quad (31)$$

Hence, we have $Q_\varepsilon^\dagger Q_\varepsilon = P_\varepsilon$, which implies that \mathbf{c}_ε in Eq. (23) becomes unique and reduces to Eq. (29).

Conversely, assume that \mathbf{c}_ε in Eq. (23) is unique. Then, it follows that $Q_\varepsilon^\dagger Q_\varepsilon = P_\varepsilon$ and Eq. (31) holds. Hence, we have Eq. (30) and it follows that

$$\mathcal{R}(A_r T^*) + \mathcal{R}(A_r A_s(\varepsilon)^*) = \mathbf{C}^K. \quad (32)$$

Since $\mathcal{R}(A_r^\dagger) = H$ and $A_r^\dagger A_r$ is the identity operator, applying A_r^\dagger to both sides from the left to Eq. (32) yields Eq. (28). ■

For example, if T is the identity operator, then $\mathcal{R}(T^*) = H$ and Eq. (28) is true irrespective of $A_s(\varepsilon)$. Hence, \mathbf{c}_ε becomes unique as in Eq. (29).

Theorem 1 implies that the solution to Problem 2 is obtained by minimizing $J_{\min}[\varepsilon]$ in Eq. (24) with respect to ε . Since there are Moore-Penrose generalized inverses in Eq. (24), we can not analytically compute the gradient. Hence, we can not use optimization techniques such as the steepest descent method. Therefore, we solve the optimization problem by using the following algorithm based on an exhaustive search over a finite set of candidate parameters.

Algorithm 1 Before start, determine the candidate for each parameter $\varepsilon^{(j)}$ ($j = 1, \dots, J - 1$).

1. Initialize J_{\min} as infinity.
2. Repeat the following steps for all candidate vectors ε .
 - (a) If $\mathbf{d} \in \mathcal{R}(B_\varepsilon)$, calculate $J_{\min}[\varepsilon]$ by Eq. (24).
 - (b) Otherwise, set infinity to $J_{\min}[\varepsilon]$.
 - (c) If $J_{\min}[\varepsilon] < J_{\min}$, then update J_{\min} .

The condition $\mathbf{d} \in \mathcal{R}(B_\varepsilon)$ is verified by evaluating the value $\|\mathbf{d} - B_\varepsilon B_\varepsilon^\dagger \mathbf{d}\|$. If this is small enough, then we regard the condition true.

5. SIMULATION

Let H be a space spanned by functions

$$\varphi_k(x) = \begin{cases} 1 & (k = 0), \\ \sqrt{2} \cos \frac{k\pi x}{l} & (0 < k < K), \end{cases} \quad (33)$$

where l is a positive real number. The inner product is defined by

$$\langle f, g \rangle = \frac{1}{l} \int_0^l f(x) \overline{g(x)} dx. \quad (34)$$

Under this inner product, $\{\varphi_k\}_{k=0}^{K-1}$ is an orthonormal basis indeed. Hence, we can use the discussion in the previous sections. In this space H , the target signal f is generated by a normal distribution with a fixed signal \tilde{f} as mean and a variance of 0.5.

We adopted two operators for T . One is the identity operator: $T = I$. In this case, Eq. (10) reduces to the original Wiener criterion. The second one is the second derivative operator: $T = d^2/dx^2$. In this case, $T\varphi_k = (-k^2\pi^2/l^2)\varphi_k$, which implies that φ_k is the eigenfunction of T .

The sample points $\{x_n\}_{n=0}^{N-1}$ for the base sequence are given by

$$x_n = \frac{(2n+1)l}{2N} \quad (n = 0, 1, \dots, N-1). \quad (35)$$

For simplicity, let $l = N$. Then, the sampling interval reduces to one. The number of sampling sequences is $J = 3$, and the offset parameters are $\delta^{(1)} = -0.2$ and $\delta^{(2)} = 0.3$. The dimension of H is $K = 20$, and the number of sample points in each sequence is $N = 6$. Sampling is assumed to be ideal, i.e., $d_n^{(j)} = f(x_n + \delta^{(j)})$. It should be noted that the condition (9) does not hold.

Fig. 3 shows simulation results. Figures (a) ~ (c) show reconstructed signals by the proposed method with $T = I$, the proposed method with $T = d^2/dx^2$, and the conventional method [10], respectively. In the conventional method, we used $L = 16$, because it is the maximum number that satisfies the condition (9) in the case of $J = 3$ and $N = 6$. In all figures, the thick solid, the thin solid, and the dotted lines show the reconstructed signal \tilde{f} , the target signal f , and the mean signal \bar{f} , respectively. The estimated results for the parameters δ and the normalized error $\|\tilde{f} - f\|/\|f\|$ are shown in Table 2. The proposed methods outperform the conventional method in the sense of both the parameter estimation and the normalized error. It should be noted that the reconstructed signals in (a) or (b) are closer to the target signal than to the mean signal. Further, the proposed method with $T = I$ performs better than that with $T = d^2/dx^2$. This shows that, in this experiment, suppression for some special frequency components did not work effectively. Hence, we have to use appropriate T depending on a practical situation.

The computational complexity of Algorithm 1 exponentially increases in terms of J . If the candidate of each $\varepsilon^{(j)}$ is $-0.5, -0.4, \dots, 0.5$, then the main routine will be repeated $11^{(J-1)}$ times. With $J = 5$, the routine spend more than one day. This is a problem even though Algorithm 1 does not suffer from the issue of local minima. In order to solve this problem, we have to devise a more sophisticated optimization algorithm. One possibility is based on the coarse-to-fine strategy. Another one is to compute the gradient of $J_{\min}[\varepsilon]$ approximately. For example, let $J = 3$. We first set ε as a zero vector $(0, 0, 0)$ and compute $J_{\min}[\varepsilon]$ for this ε . Then, compute $J_{\min}[\varepsilon]$ for $\varepsilon = (0, 0.1, 0)$, $(0, -0.1, 0)$, $(0, 0, 0.1)$, and $(0, 0, -0.1)$. The ε which minimizes $J_{\min}[\varepsilon]$ within these five values is the next candidate. This algorithm will reduce computational complexity dramatically compared to Algorithm 1. However, this may be trapped in local minima.

Table 2: Estimation results of δ and normalized error of \tilde{f}

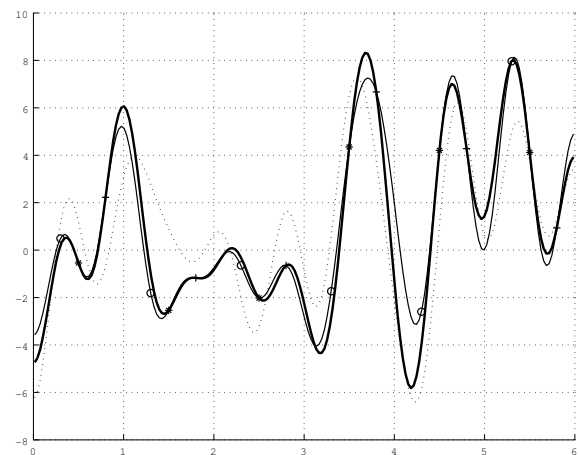
	δ	$\ \tilde{f} - f\ /\ f\ $
True value	(-0.20, 0.30)	-
Proposed with $T = I$	(-0.16, 0.30)	0.28
Proposed with $T = d^2/dx^2$	(-0.17, 0.40)	0.40
Conventional [10]	(0.30, -0.10)	1.74

6. CONCLUSION

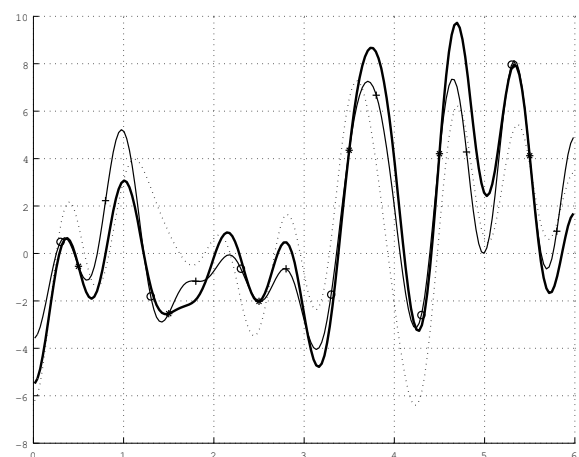
We proposed a stochastic signal reconstruction algorithm that provides a stable, high quality approximation from interleaved samples with unknown offset parameters. We introduced a generalized Wiener criterion. The proposed algorithm yields a signal that minimizes the criterion among signals consistent with the given samples. We derived an analytical solution for the optimization problem with fixed offset parameters. Then, the minimum value of the criterion for a fixed offset was further minimized with respect to all candidate offset parameters. Simulation results showed that the proposed method outperforms the conventional method in the sense of both the parameter estimation and the signal reconstruction. Our future work is to devise a more sophisticated optimization algorithm.

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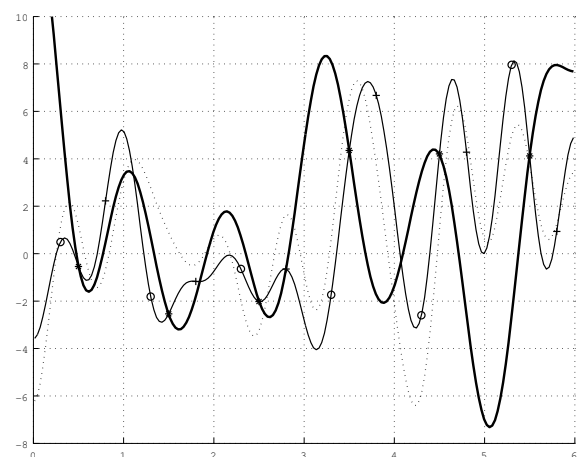
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(a) Proposed method with $T = I$



(b) Proposed method with $T = d^2/dx^2$



(c) Conventional method [10]

Figure 3: Simulation results