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A NEW BAYESIAN LOWER BOUND ON THE MEAN SQUARE ERROR OF ESTIMATORS

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ABSTRACT

In this paper, the Weiss-Weinstein family of Bayesian lower bounds on the mean-square-error of estimators is extended to an integral form. A new class of Bayesian lower bounds is derived from this integral form by approximating each entry of the vector of estimation error in a closed Hilbert subspace of $\mathcal{L}_2$. This Hilbert subspace is spanned by a set of linear transformations of elements in the domain of an integral transform of a particular function, which is orthogonal to any function of the observations. It is shown that new Bayesian bounds can be derived from this class by selecting the particular function from a known set and modifying the kernel of the integral transform. A new computationally manageable lower bound is derived from the proposed class using the kernel of the Fourier transform. The bound is computationally manageable and provides better prediction of the signal-to-noise ratio threshold region, exhibited by the maximum *a-posteriori* probability estimator. The proposed bound is compared with other known bounds in terms of threshold SNR prediction in the problem of frequency estimation.

1. INTRODUCTION

Bayesian lower bounds are used as a benchmark for performance evaluation of Bayesian estimators, such as the minimum-mean-square error (MMSE) and the maximum a *a-posteriori* probability (MAP) estimators, as well as for global performance evaluation of non-Bayesian estimators, such as the maximum likelihood (ML) estimator. Bayesian bounds can be partitioned into two categories: the Ziv-Zakai [1] family, derived from a binary hypothesis testing problem and the Weiss-Weinstein family [2], derived from the covariance inequality. The Ziv-Zakai class contains the Ziv-Zakai [1], Bellini-Tartara [3], Chazan-Zakai-Ziv [4], Weinstein [5], extended-Ziv-Zakai [6] and Bell [7] bounds. The Weiss-Weinstein class contains the Bayesian Cramér-Rao (BCR) [8], Bayesian Bhattacharyya (BBH) [2], Bobrovsky-Zakai (BZ) [9], Reuven-Messer (RM) [10], Weiss-Weinstein (WW) [11], Bayesian Abel (BA) bound [12] and the combined Cramér-Rao/Weiss-Weinstein (CCRWW) [13] bounds. In this paper, we concern the Weiss-Weinstein class.

The Weiss-Weinstein class of bounds was derived via projection of the estimation error on a particular function, denoted by ψ, which is orthogonal to any function of the observations. The bounds contained in this family are derived by different selections of ψ. To this day, only a small set of ψ's, which satisfy the orthogonality property and in some cases yield tight and computationally manageable bounds, has been introduced. Moreover, in derivation of bounds, such as the RM, WW, BA and CCRWW, these ψ's should be evaluated, using the sampling operator, at large amount of test points of the parameter space in order to obtain tight bounds. Selection of these test points is carried out via numerical search methods, which might become computationally cumbersome as the number of test points and the dimensionality of the parameters increase.

In this paper, a new class of Bayesian lower bounds on the mean-square-error (MSE) of estimators is derived from an integral form of the Weiss-Weinstein family by applying an integral transform on some well known ψ's. By modifying the kernel of the integral transform, new ψ's for which the orthogonality property is satisfied, can be easily derived from existing ψ's. Moreover, the integral transform generalizes the sampling or derivative operators, used in some well known bounds in the Weiss-Weinstein family. Therefore, by applying an integral transform on ψ, such that the elements in the field of the transform are "compressed" into few elements in the domain of the transform, tight and computationally manageable bounds can be obtained by evaluating the domain of the integral transform in a smaller amount of test points. In similar to [14] and [15], the proposed class is derived by approximating each entry of the vector of estimation error in a closed Hilbert subspace of $\mathcal{L}_2$, which contains linear transformations of elements in the domain of an integral transform of ψ. It is shown that new Bayesian lower bounds can be derived from this class by selecting ψ from a set of ψ's, which has already been introduced and by modifying the kernel of the integral transform. A new lower bound is derived from this class using ψ applied in [13] and the kernel of the Fourier transform. The motivation of using the Fourier transform is as follows. First, the Fourier transform is easy to compute. Second, in cases where the power spectrum of a given ψ is concentrated in a small subset of the frequency domain, most of the information on ψ in the parameter space can be "compressed" into a few frequency components using the Fourier transform. Hence, it is shown that the proposed bound is computationally manageable and provides better prediction of the signal-to-noise ratio (SNR) threshold region, exhibited by the MAP estimator.

The paper is organized as follows: In Section 2, an integral form of the Weiss-Weinstein class of Bayesian lower bounds is derived. In Section 3, a new class of Bayesian lower bounds on the MSE of estimators is derived from the integral form of the Weiss-Weinstein family. In Section 4, a new bound is derived from the proposed class using ψ applied in [13] and the kernel of the Fourier transform. In Section 5, the proposed bound is compared with some other known bounds in terms of threshold SNR prediction in the problem of frequency estimation. Section 6, summarizes the main points of this contribution.

2. INTEGRAL FORM OF THE WEISS-WEINSTEIN CLASS: GEOMETRIC INTERPRETATION

In this section, an integral form of the Weiss-Weinstein class of lower bounds [2] is derived by approximating each entry of the vector of estimation error in a Hilbert subspace of $\mathcal{L}_2$. This Hilbert subspace contains linear transformations of a particular function, which is orthogonal to any function of the observations.

Derivation the integral form of the Weiss-Weinstein family is preceded by the following definitions and assumptions:

1. All the functions used in this paper, are assumed to be measurable [16].
2. Let $(\boldsymbol{\Theta}, \mathcal{M})$ denote a measurable space with finite Lebesgue measure, where $\boldsymbol{\Theta} \subset \mathbb{R}^M$ denotes a parameter space and $\mathcal{M}$ is a σ-algebra on $\boldsymbol{\Theta}$. In this paper, we consider the estimation of $\mathbf{g}(\boldsymbol{\theta})$, where $\mathbf{g} : \boldsymbol{\Theta} \to \mathbb{R}^L$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ is a random unknown multivariate parameter.
3. Let $(\mathcal{X}, \mathcal{F})$ denote a measurable space, where $\mathcal{X}$ and $\mathcal{F}$ denote an observation space of points $\mathbf{x}$ and σ-algebra on $\mathcal{X}$, respectively. The space $(\mathcal{X} \times \boldsymbol{\Theta}, \mathcal{F} \times \mathcal{M}, \mathcal{P})$ is a complete probability space, where $\mathcal{P}$ denotes a probability measure on the measurable product space $(\mathcal{X} \times \boldsymbol{\Theta}, \mathcal{F} \times \mathcal{M})$ and $\mathcal{F} \times \mathcal{M}$ is the σ-algebra on the Cartesian product $\mathcal{X} \times \boldsymbol{\Theta}$.
4. Let μ denote a σ-finite measure on $(\mathcal{X}, \mathcal{F})$ and let λ denote the Lebesgue measure on $(\boldsymbol{\Theta}, \mathcal{M})$. The measure $\mathcal{P}$ is assumed to have density $f(\mathbf{x}, \boldsymbol{\theta})$ relative to the product measure $\mu \times \lambda$ such that the joint probability of observing $S \in \mathcal{F}$ and $E \in \mathcal{M}$ is $\mathcal{P}(S, E) = \int_S \int_E f(\mathbf{x}, \boldsymbol{\theta})\, d\mu(\mathbf{x})\, d\boldsymbol{\theta}$.
5. The space $\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta}, \mathcal{F} \times \mathcal{M}, \mathcal{P})$ is the Hilbert space of functions $\zeta : \mathcal{X} \times \boldsymbol{\Theta} \to \mathbb{C}$ with finite second-order moments w.r.t. $\mathcal{P}$. The inner product of two elements in this space is given by $\langle \zeta, \zeta' \rangle_{\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta},\mathcal{F}\times\mathcal{M},\mathcal{P})} = \int_{\mathcal{X}} \int_{\boldsymbol{\Theta}} \zeta(\mathbf{x}, \boldsymbol{\theta})\, \zeta'^*(\mathbf{x}, \boldsymbol{\theta}) \cdot f(\mathbf{x}, \boldsymbol{\theta})\, d\mu(\mathbf{x})\, d\boldsymbol{\theta} = \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\zeta(\mathbf{x}, \boldsymbol{\theta})\, \zeta'^*(\mathbf{x}, \boldsymbol{\theta})]$, where $(\cdot)^*$ and $\mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}[\cdot]$ denote the complex-conjugate and the expectation operator w.r.t. $f(\mathbf{x}, \boldsymbol{\theta})$, respectively.
6. Let $\hat{\mathbf{g}}(\mathbf{x})$ and $\mathbf{e}(\mathbf{x}, \boldsymbol{\theta}) = \hat{\mathbf{g}}(\mathbf{x}) - \mathbf{g}(\boldsymbol{\theta})$ denote the estimator of $\mathbf{g}(\boldsymbol{\theta})$ and the vector of estimation error, respectively. The MSE matrix is given by $\mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{e}(\mathbf{x}, \boldsymbol{\theta})\, \mathbf{e}^T(\mathbf{x}, \boldsymbol{\theta})\right]$, where it is assumed that $[\mathbf{e}(\mathbf{x}, \boldsymbol{\theta})]_l \in \mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta}, \mathcal{F} \times \mathcal{M}, \mathcal{P})\ \forall l = 1, \ldots, L$.
7. Let $\boldsymbol{\Lambda} \subset \mathbb{R}^M$ denote a measurable space with finite Lebesgue measure. According to [2], a particular function $\boldsymbol{\psi} : \mathcal{X} \times \boldsymbol{\Theta} \times \boldsymbol{\Lambda} \to \mathbb{R}^P$ is orthogonal to any function of the observations iff $\forall \boldsymbol{\tau} \in \boldsymbol{\Lambda}$

$$\int_{\boldsymbol{\Theta}} \boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau})\, f(\mathbf{x}, \boldsymbol{\theta})\, d\boldsymbol{\theta} = \mathbf{0} \quad \text{for a.e. } \mathbf{x} \in \mathcal{X}. \tag{1}$$

We assume that $[\boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau})]_p \in \mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta}, \mathcal{F} \times \mathcal{M}, \mathcal{P})$ $\forall \boldsymbol{\tau} \in \boldsymbol{\Lambda}$ and $\forall p = 1, \ldots, P$. The orthogonality of $\boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau})$ to any function of the observations is a necessary condition for obtaining valid bounds, which are independent of the estimator.

Using the definitions and assumptions above, an integral form of the Weiss-Weinstein family is derived by the following steps.

First, the following Hilbert subspace of $\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta}, \mathcal{F} \times \mathcal{M}, \mathcal{P})$, which contains linear transformations of $\boldsymbol{\psi}$ is constructed.

$$\mathcal{H}_\phi^{(\boldsymbol{\psi})} = \left\{ \phi_{\mathbf{q}}(\mathbf{x}, \boldsymbol{\theta}) = \int_{\boldsymbol{\Lambda}} \mathbf{q}^H(\boldsymbol{\tau})\, \boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau})\, d\boldsymbol{\tau} \right. \\ \left. : \phi_{\mathbf{q}}(\mathbf{x}, \boldsymbol{\theta}) \in \mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta}, \mathcal{F} \times \mathcal{M}, \mathcal{P}) \right\}, \tag{2}$$

where $\mathbf{q} : \boldsymbol{\Lambda} \to \mathbb{R}^P$. It is assumed that $\mathcal{H}_\phi^{(\boldsymbol{\psi})}$ is closed, i.e. any Cauchy sequence in $\mathcal{H}_\phi^{(\boldsymbol{\psi})}$ converges to a limit in $\mathcal{H}_\phi^{(\boldsymbol{\psi})}$.

Second, according to (1), (2) and using the Hilbert projection theorem [16], it is shown in [17] that the best approximation of $[\mathbf{e}(\mathbf{x}, \boldsymbol{\theta})]_l$, $l = 1, \ldots, L$ in $\mathcal{H}_\phi^{(\boldsymbol{\psi})}$, in the sense of minimum norm of approximation error in $\mathcal{L}_2(\mathcal{X} \times \boldsymbol{\Theta}, \mathcal{F} \times \mathcal{M}, \mathcal{P})$ yields

$$\mathbf{e}_{\boldsymbol{\psi}}(\mathbf{x}, \boldsymbol{\theta}) = \int_{\boldsymbol{\Lambda}} \int_{\boldsymbol{\Lambda}} \mathbf{Z}_{\boldsymbol{\psi}}(\boldsymbol{\tau}')\, \mathbf{G}_{\boldsymbol{\psi}}(\boldsymbol{\tau}', \boldsymbol{\tau})\, \boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau})\, d\boldsymbol{\tau}' d\boldsymbol{\tau}, \tag{3}$$

where

$$\begin{aligned} \mathbf{Z}_{\boldsymbol{\psi}}(\boldsymbol{\tau}) &= \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{e}(\mathbf{x}, \boldsymbol{\theta})\, \boldsymbol{\psi}^T(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau})\right] \\ &= \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{g}(\boldsymbol{\theta})\, \boldsymbol{\psi}^T(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau})\right]. \end{aligned} \tag{4}$$

The last equality in (4) stems from equation (1) and the definition of $\mathbf{e}(\mathbf{x}, \boldsymbol{\theta})$. The matrix function $\mathbf{G}_{\boldsymbol{\psi}}(\cdot, \cdot)$ is defined in the following manner. Let

$$\mathbf{K}_{\boldsymbol{\psi}}(\boldsymbol{\tau}, \boldsymbol{\tau}') = \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau})\, \boldsymbol{\psi}^T(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau}')\right], \tag{5}$$

then $\mathbf{G}_{\boldsymbol{\psi}}(\cdot, \cdot)$ is the inverse of $\mathbf{K}_{\boldsymbol{\psi}}(\cdot, \cdot)$, such that

$$\int_{\boldsymbol{\Lambda}} \mathbf{K}_{\boldsymbol{\psi}}(\boldsymbol{\tau}, \boldsymbol{\tau}')\, \mathbf{G}_{\boldsymbol{\psi}}(\boldsymbol{\tau}', \boldsymbol{\tau}'')\, d\boldsymbol{\tau}' = \delta(\boldsymbol{\tau} - \boldsymbol{\tau}'')\, \mathbf{I}_P, \tag{6}$$

where $\delta(\cdot)$ denotes the Dirac's delta function, such that $\delta(\boldsymbol{\tau}) = \prod_{m=1}^{M} \delta([\boldsymbol{\tau}]_m)$ and $\mathbf{I}_P$ is the P-dimensional identity matrix.

Finally, let $\mathbf{u}(\mathbf{x}, \boldsymbol{\theta}) = \mathbf{e}(\mathbf{x}, \boldsymbol{\theta}) - \mathbf{e}_{\boldsymbol{\psi}}(\mathbf{x}, \boldsymbol{\theta})$, denote the vector of approximation error. According to the Hilbert projection theorem [16], each entry of $\mathbf{u}(\mathbf{x}, \boldsymbol{\theta})$ is orthogonal to each entry of $\mathbf{e}_{\boldsymbol{\psi}}(\mathbf{x}, \boldsymbol{\theta})$. In addition, the autocorrelation matrix of $\mathbf{u}(\mathbf{x}, \boldsymbol{\theta})$ is Hermitian positive semidefinite. Therefore,

$$\begin{aligned} &\mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{u}(\mathbf{x}, \boldsymbol{\theta})\, \mathbf{u}^H(\mathbf{x}, \boldsymbol{\theta})\right] = \\ &\mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{e}(\mathbf{x}, \boldsymbol{\theta})\, \mathbf{e}^T(\mathbf{x}, \boldsymbol{\theta})\right] - \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{e}_{\boldsymbol{\psi}}(\mathbf{x}, \boldsymbol{\theta})\, \mathbf{e}^T(\mathbf{x}, \boldsymbol{\theta})\right] \succeq \mathbf{0}. \end{aligned} \tag{7}$$

Hence, by substituting (3) into (7) and using (4), an integral form of the Weiss-Weinstein class of lower bounds on the MSE matrix is given by

$$\begin{aligned} \mathbf{C}(\boldsymbol{\psi}) &= \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{e}_{\boldsymbol{\psi}}(\mathbf{x}, \boldsymbol{\theta})\, \mathbf{e}^T(\mathbf{x}, \boldsymbol{\theta})\right] \\ &= \int_{\boldsymbol{\Lambda}} \int_{\boldsymbol{\Lambda}} \mathbf{Z}_{\boldsymbol{\psi}}(\boldsymbol{\tau}')\, \mathbf{G}_{\boldsymbol{\psi}}(\boldsymbol{\tau}', \boldsymbol{\tau})\, \mathbf{Z}_{\boldsymbol{\psi}}^T(\boldsymbol{\tau})\, d\boldsymbol{\tau}' d\boldsymbol{\tau}. \end{aligned} \tag{8}$$

Geometric interpretation of the class of bounds in (8) for the one-dimensional case i.e. $L = 1$ is depicted in Fig. 1. We note that the integral form in (8) is a new representation of the Weiss-Weinstein class. The commonly known discrete form [2] is a special case of (8).

Observing (8), one can notice that by selecting $\boldsymbol{\psi}$ such that (1) is satisfied, numerous bounds can be derived from (8). However, to this day only a small set of $\boldsymbol{\psi}$'s, which satisfy (1) and in some cases yield tight and computationally manageable bounds has been introduced. Moreover, in derivation of bounds, such as the RM, WW, BA and CCRWW these $\boldsymbol{\psi}$'s should be evaluated at many of test points of the parameter space in order to obtain tight bounds. Selection of these test points is usually carried out via numerical search methods, which become computationally cumbersome as the number of test points and the dimensionality of the parameters increase.

3. A NEW CLASS OF BAYESIAN LOWER BOUNDS

In this section, a new class of Bayesian lower bounds is derived from (8) by applying an integral transform on $\boldsymbol{\psi}$. The motivation for using the integral transform is discussed in Section 1. The proposed class is derived by the following steps.

First, a Hilbert subspace of $\mathcal{H}_\phi^{(\boldsymbol{\psi})}$, which contains linear transformations of elements in the domain of an integral transform defined on the set $\mathcal{S}_{\boldsymbol{\psi}} = \{\boldsymbol{\psi}(\mathbf{x},\boldsymbol{\theta};\boldsymbol{\tau}), \boldsymbol{\tau} \in \boldsymbol{\Lambda}\}$, is constructed in the following manner. Let $\mathbf{V} \times \boldsymbol{\Lambda}$ denote a product measurable space, where $\mathbf{V} \subset \mathbb{R}^M$ is a measurable space with finite Lebesgue measure and let $\mathbf{H} : \mathbf{V} \times \boldsymbol{\Lambda} \to \mathbb{C}^{Q\times P}$. An integral transform on $\mathcal{S}_{\boldsymbol{\psi}}$ is given by

$$(T_{\mathbf{H}}\boldsymbol{\psi})(\boldsymbol{\alpha}) = \int_{\boldsymbol{\Lambda}} \mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})\,\boldsymbol{\psi}(\mathbf{x},\boldsymbol{\theta};\boldsymbol{\tau})\,d\boldsymbol{\tau} = \boldsymbol{\varphi}_{\mathbf{H},\boldsymbol{\psi}}(\mathbf{x},\boldsymbol{\theta};\boldsymbol{\alpha}), \tag{9}$$

where $\mathbf{H}(\cdot,\cdot)$ and $S_{\boldsymbol{\varphi}}^{(\mathbf{H},\boldsymbol{\psi})} = \left\{\boldsymbol{\varphi}_{\mathbf{H},\boldsymbol{\psi}}(\mathbf{x},\boldsymbol{\theta};\boldsymbol{\alpha})\right\}$ denote the kernel and the domain of $(T_{\mathbf{H}}\boldsymbol{\psi})$, respectively and $\boldsymbol{\alpha} \in \mathbf{V}$. Hence, by fixing $\mathbf{H}$, a Hilbert subspace of $\mathcal{H}_\phi^{(\boldsymbol{\psi})}$ is given by

$$\mathcal{H}_\mu^{(\mathbf{H},\boldsymbol{\psi})} = \Bigg\{\mu_{\mathbf{a}}^{(\mathbf{H},\boldsymbol{\psi})}(\mathbf{x},\boldsymbol{\theta}) = \int_{\boldsymbol{V}} \mathbf{a}^H(\boldsymbol{\alpha})\,\boldsymbol{\varphi}_{\mathbf{H},\boldsymbol{\psi}}(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\alpha})\,d\boldsymbol{\alpha} \;:\; \mu_{\mathbf{a}}^{(\mathbf{H},\boldsymbol{\psi})}(\mathbf{x},\boldsymbol{\theta}) \in \mathcal{H}_\phi^{(\boldsymbol{\psi})}\Bigg\}, \tag{10}$$

where it is assumed that $\mathcal{H}_\mu^{(\mathbf{H},\boldsymbol{\psi})}$ is closed, i.e. any Cauchy sequence in $\mathcal{H}_\mu^{(\mathbf{H},\boldsymbol{\psi})}$ converges to a limit in $\mathcal{H}_\mu^{(\mathbf{H},\boldsymbol{\psi})}$ and $\mathbf{a} : \mathbf{V} \to \mathbb{R}^Q$. In [17] it is shown that a sufficient condition for $\mu_{\mathbf{a}}^{(\mathbf{H},\boldsymbol{\psi})}(\mathbf{x},\boldsymbol{\theta}) \in \mathcal{H}_\phi^{(\boldsymbol{\psi})}$ is absolute integrability of $\mathbf{a}^H(\boldsymbol{\alpha})\,\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})\,\boldsymbol{\psi}(\mathbf{x},\boldsymbol{\theta};\boldsymbol{\tau})$ on $\mathbf{V}\times\boldsymbol{\Lambda}$ for a.e. $\mathbf{x} \in \mathcal{X}$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.

Second, each entry of $\mathbf{e}(\mathbf{x},\boldsymbol{\theta})$ is approximated in $\mathcal{H}_\mu^{(\mathbf{H},\boldsymbol{\psi})}$. Using (1), (9), (10) and the Hilbert projection theorem [16], it is shown in [17] that the best approximation of $[\mathbf{e}(\mathbf{x},\boldsymbol{\theta})]_l$, $l = 1,\ldots,L$ in $\mathcal{H}_\mu^{(\mathbf{H},\boldsymbol{\psi})}$, in the sense of minimum norm of approximation error in $\mathcal{L}_2(\mathcal{X}\times\boldsymbol{\Theta},\mathcal{F}\times\mathcal{M},\mathcal{P})$ yields

$$\mathbf{e}_{\mathbf{H},\boldsymbol{\psi}}(\mathbf{x},\boldsymbol{\theta}) = \int_{\boldsymbol{V}}\int_{\boldsymbol{V}} \boldsymbol{\Gamma}_{\mathbf{H},\boldsymbol{\psi}}^H(\boldsymbol{\alpha}')\,\mathbf{G}_{\mathbf{H},\boldsymbol{\psi}}(\boldsymbol{\alpha}',\boldsymbol{\alpha})\,\boldsymbol{\varphi}_{\mathbf{H},\boldsymbol{\psi}}(\mathbf{x},\boldsymbol{\theta};\boldsymbol{\alpha})\,d\boldsymbol{\alpha}'d\boldsymbol{\alpha}, \tag{11}$$

where

$$\boldsymbol{\Gamma}_{\mathbf{H},\boldsymbol{\psi}}(\boldsymbol{\alpha}) = \int_{\boldsymbol{\Lambda}} \mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})\,\mathbf{Z}_{\boldsymbol{\psi}}^T(\boldsymbol{\tau})\,d\boldsymbol{\tau}. \tag{12}$$

The matrix function $\mathbf{G}_{\mathbf{H},\boldsymbol{\psi}}(\cdot,\cdot)$ is defined in the following manner. Let

$$\mathbf{K}_{\mathbf{H},\boldsymbol{\psi}}(\boldsymbol{\alpha},\boldsymbol{\alpha}') = \int_{\boldsymbol{\Lambda}}\int_{\boldsymbol{\Lambda}} \mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})\,\mathbf{K}_{\boldsymbol{\psi}}(\boldsymbol{\tau},\boldsymbol{\tau}')\,\mathbf{H}^H(\boldsymbol{\alpha}',\boldsymbol{\tau}')\,d\boldsymbol{\tau}d\boldsymbol{\tau}', \tag{13}$$

then $\mathbf{G}_{\mathbf{H},\boldsymbol{\psi}}(\cdot,\cdot)$ is the inverse of $\mathbf{K}_{\mathbf{H},\boldsymbol{\psi}}(\cdot,\cdot)$ such that

$$\int_{\boldsymbol{V}} \mathbf{K}_{\mathbf{H},\boldsymbol{\psi}}(\boldsymbol{\alpha},\boldsymbol{\alpha}')\,\mathbf{G}_{\mathbf{H},\boldsymbol{\psi}}(\boldsymbol{\alpha}',\boldsymbol{\alpha}'')\,d\boldsymbol{\alpha}' = \delta(\boldsymbol{\alpha}-\boldsymbol{\alpha}'')\,\mathbf{I}_Q. \tag{14}$$

Finally, in similar to the derivation of (8) from (7), the proposed class of lower bounds on the MSE matrix is given by

$$\begin{aligned}\mathbf{C}'(\mathbf{H},\boldsymbol{\psi}) &= \mathrm{E}_{\mathbf{x},\boldsymbol{\theta}}\left[\mathbf{e}_{\boldsymbol{\psi},\mathbf{H}}(\mathbf{x},\boldsymbol{\theta})\,\mathbf{e}^T(\mathbf{x},\boldsymbol{\theta})\right] \\ &= \int_{\mathbf{V}}\int_{\mathbf{V}} \boldsymbol{\Gamma}_{\mathbf{H},\boldsymbol{\psi}}^H(\boldsymbol{\alpha}')\,\mathbf{G}_{\mathbf{H},\boldsymbol{\psi}}(\boldsymbol{\alpha}',\boldsymbol{\alpha})\,\boldsymbol{\Gamma}_{\mathbf{H},\boldsymbol{\psi}}(\boldsymbol{\alpha})\,d\boldsymbol{\alpha}'d\boldsymbol{\alpha} \\ &= \int_{\boldsymbol{\Lambda}}\int_{\boldsymbol{\Lambda}} \mathbf{Z}_{\boldsymbol{\psi}}(\boldsymbol{\tau}')\,\tilde{\mathbf{G}}_{\mathbf{H},\boldsymbol{\psi}}(\boldsymbol{\tau}',\boldsymbol{\tau})\,\mathbf{Z}_{\boldsymbol{\psi}}^T(\boldsymbol{\tau})\,d\boldsymbol{\tau}'d\boldsymbol{\tau}.\end{aligned} \tag{15}$$

The second equality of (15) is obtained by substitution of (11) into the first equality of (15) and using (4), (9) and (12). The last equality of (15) is obtained by substitution of (12) into the second equality of (15), where

$$\tilde{\mathbf{G}}_{\mathbf{H},\boldsymbol{\psi}}(\boldsymbol{\tau}',\boldsymbol{\tau}) \triangleq \int_{\boldsymbol{V}}\int_{\boldsymbol{V}} \mathbf{H}^H(\boldsymbol{\alpha}',\boldsymbol{\tau}')\,\mathbf{G}_{\mathbf{H},\boldsymbol{\psi}}(\boldsymbol{\alpha}',\boldsymbol{\alpha})\,\mathbf{H}(\boldsymbol{\alpha},\boldsymbol{\tau})\,d\boldsymbol{\alpha}'d\boldsymbol{\alpha}. \tag{16}$$

Geometric interpretation of the proposed class in (15) for the one-dimensional case, i.e. $L = 1$, is depicted in Fig. 1. Observing (15), one can notice that numerous bounds can be derived by fixing $\boldsymbol{\psi}$ and modifying $\mathbf{H}(\cdot,\cdot)$.

In the following, the relation of the proposed class of bounds in (15) to the integral form of the Weiss-Weinstein class in (8), in terms of tightness and computational manageability, is discussed. Since $\mathcal{H}_\phi^{(\boldsymbol{\psi})} \supseteq \mathcal{H}_\mu^{(\mathbf{H},\boldsymbol{\psi})}$, it is obvious that $\mathbf{C}(\boldsymbol{\psi}) \succeq \mathbf{C}'(\mathbf{H},\boldsymbol{\psi})$ $\forall\mathbf{H}$. In [17], it is also shown that for any invertible integral transform, $(T_{\mathbf{H}}\boldsymbol{\psi})$, $\mathbf{C}(\boldsymbol{\psi}) = \mathbf{C}'(\mathbf{H},\boldsymbol{\psi})$. If $(T_{\mathbf{H}}\boldsymbol{\psi})$ is non-invertible, then $\mathbf{G}_{\boldsymbol{\psi}}(\boldsymbol{\tau}',\boldsymbol{\tau})$ is approximated by $\tilde{\mathbf{G}}_{\mathbf{H},\boldsymbol{\psi}}(\boldsymbol{\tau}',\boldsymbol{\tau})$ and a less tighter bounds than $\mathbf{C}(\boldsymbol{\psi})$ is derived. However, calculation of $\mathbf{C}(\boldsymbol{\psi})$ involves the computation of $\mathbf{G}_{\boldsymbol{\psi}}(\boldsymbol{\tau}',\boldsymbol{\tau})$, according to (6). Hence, in cases where this task is analytically impossible, $\mathbf{C}(\boldsymbol{\psi})$ is practically incomputable and non-invertible integral transforms should be used in order to obtain computationally manageable bounds.

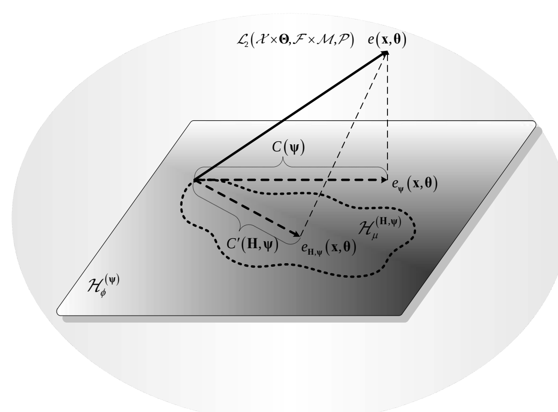

Fig. 1. Geometric interpretation of the integral form of the Weiss-Weinstein family in (8) and the proposed class of bounds in (15), for the one-dimensional case, i.e. $L = 1$. The scalars $e_{\boldsymbol{\psi}}(\mathbf{x},\boldsymbol{\theta})$ and $e_{\mathbf{H},\boldsymbol{\psi}}(\mathbf{x},\boldsymbol{\theta})$ denote the projections of $e(\mathbf{x},\boldsymbol{\theta})$ on $\mathcal{H}_\phi^{(\boldsymbol{\psi})}$ and $\mathcal{H}_\mu^{(\mathbf{H},\boldsymbol{\psi})}$, respectively.

4. A NEW BAYESIAN BOUND BASED ON THE FOURIER TRANSFORM

The Fourier transform is easy to compute and in some cases has a strong "energy compaction" property. Therefore, in cases where $\forall p = 1, \ldots, P$ the power spectrum of $[\boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau})]_p$ is concentrated in a small subset of the frequency domain, most of the information in each entry of $\boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau})$ can be concentrated in a few frequency components via the Fourier transform. Motivated by this fact, a new bound is derived from (15) using the kernel of the Fourier transform.

The proposed bound is derived using the following $\boldsymbol{\psi}$ and $\mathbf{H}$:

$$\boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau}) = [\boldsymbol{\psi}_{\text{BCR}}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau}), \psi_{\text{WW}}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau})]^T, \quad (17)$$

where

$$\boldsymbol{\psi}_{\text{BCR}}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau}) = \frac{\partial \log f(\mathbf{x}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \nu_{\boldsymbol{\Theta}'}(\boldsymbol{\theta}), \quad (18)$$

and

$$\psi_{\text{WW}}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau}) = \left(\rho^t(\mathbf{x}; \boldsymbol{\theta} + \boldsymbol{\tau}, \boldsymbol{\theta}) - \rho^{1-t}(\mathbf{x}; \boldsymbol{\theta} - \boldsymbol{\tau}, \boldsymbol{\theta})\right) \nu_{\boldsymbol{\Theta}'}(\boldsymbol{\theta}), \quad (19)$$

such that $\boldsymbol{\Theta}' = \{\boldsymbol{\theta} : f(\mathbf{x}, \boldsymbol{\theta}) > 0 \text{ for a.e. } \mathbf{x} \in \mathcal{X}\}$, $\nu_{\boldsymbol{\Theta}'}(\boldsymbol{\theta}) = \begin{cases} 1 & \boldsymbol{\theta} \in \boldsymbol{\Theta}' \\ 0 & \boldsymbol{\theta} \notin \boldsymbol{\Theta}' \end{cases}$, $\rho(\mathbf{x}; \boldsymbol{\theta} + \boldsymbol{\tau}, \boldsymbol{\theta}) = \frac{f(\mathbf{x}, \boldsymbol{\theta} + \boldsymbol{\tau})}{f(\mathbf{x}, \boldsymbol{\theta})}$ and $0 < t < 1$. We note that the function $\boldsymbol{\psi}$ in (17) was used for computation of the CCRW, where $\boldsymbol{\psi}_{\text{BCR}}$ and ψ_{WW} were used for computation of the BCR and the WW bounds, respectively.

$$\mathbf{H}(\boldsymbol{\alpha}, \boldsymbol{\tau}) = \begin{bmatrix} \mathbf{H}_{\text{BCR}}(\boldsymbol{\alpha}, \boldsymbol{\tau}) & \mathbf{0} \\ \mathbf{0}^T & H_{\text{Fourier}}(\boldsymbol{\alpha}, \boldsymbol{\tau}) \end{bmatrix}, \quad (20)$$

where

$$\mathbf{H}_{\text{BCR}}(\boldsymbol{\alpha}, \boldsymbol{\tau}) = \delta(\boldsymbol{\alpha}) \delta(\boldsymbol{\tau}) \mathbf{I}_M \quad (21)$$

and

$$H_{\text{Fourier}}(\boldsymbol{\alpha}, \boldsymbol{\tau}) = \sum_{j=0}^{J-1} \sum_{n=0}^{N-1} \delta(\boldsymbol{\alpha} - \boldsymbol{\Omega}_j) \delta(\boldsymbol{\tau} - \boldsymbol{\tau}_n) e^{-i\boldsymbol{\alpha}^T \boldsymbol{\tau}}, \quad (22)$$

such that $\boldsymbol{\Omega}_j$, $j = 1, \ldots, J$ denote frequency test bins and $\boldsymbol{\tau}_n \in \boldsymbol{\Lambda}, n = 1, \ldots, N$ denote test points on the parameter space. We note that $\{\boldsymbol{\tau}_n\}_{n=1}^N$ is obtained by uniform sampling of $\boldsymbol{\Lambda}$.

It can be shown that substitution of (17) and (20) into (15) yields the following bound [17]:

$$\mathbf{C}_{\text{Fourier}} = \boldsymbol{\Gamma} \mathbf{I}_{\text{BFIM}}^{-1} \boldsymbol{\Gamma}^T + \mathbf{Q} \mathbf{W}^H \left(\mathbf{W} \mathbf{R} \mathbf{W}^H\right)^{-1} \mathbf{W} \mathbf{Q}^T, \quad (23)$$

where the matrix elements composing the bound are given by

1. $\mathbf{I}_{\text{BFIM}} = -\text{E}_{\mathbf{x}, \boldsymbol{\theta}} \left[\frac{\partial^2 \log f(\mathbf{x}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T}\right]$ is the Bayesian Fisher information matrix.
2. $\boldsymbol{\Gamma} = -\text{E}_{\boldsymbol{\theta}} \left[\frac{d\mathbf{g}(\boldsymbol{\theta})}{d\boldsymbol{\theta}}\right]$.
3. $\mathbf{Q} = \boldsymbol{\Gamma} \mathbf{I}_{\text{BFIM}}^{-1} \mathbf{D} - \boldsymbol{\Phi}$.
4. $\boldsymbol{\Phi} = [\boldsymbol{\phi}(\boldsymbol{\tau}_1), \ldots, \boldsymbol{\phi}(\boldsymbol{\tau}_N)]$, where $\boldsymbol{\phi}(\boldsymbol{\tau}_n) = \text{E}_{\mathbf{x}, \boldsymbol{\theta}} \left[(\mathbf{g}(\boldsymbol{\theta}) - \mathbf{g}(\boldsymbol{\theta} + \boldsymbol{\tau}_n)) \rho^t(\mathbf{x}; \boldsymbol{\theta} + \boldsymbol{\tau}_n, \boldsymbol{\theta})\right]$.
5. $\mathbf{D} = [\mathbf{d}(\boldsymbol{\tau}_1), \ldots, \mathbf{d}(\boldsymbol{\tau}_N)]$, where $\mathbf{d}(\boldsymbol{\tau}_n) = \text{E}_{\mathbf{x}, \boldsymbol{\theta}} \left[\left(\frac{\partial \log f(\mathbf{x}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right)^T \psi_{\text{WW}}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau}_n)\right]$.
6. The matrix $\mathbf{R} = \boldsymbol{\Psi} - \mathbf{D}^T \mathbf{I}_{\text{BFIM}}^{-1} \mathbf{D}$, where $[\boldsymbol{\Psi}]_{m,n} = K_{\psi_{\text{WW}}}(\boldsymbol{\tau}_m, \boldsymbol{\tau}_n)$, $m, n = 0, \ldots, N-1$.
7. The elements of the Fourier matrix are given by $[\mathbf{W}]_{j,n} = \exp\left(-i\boldsymbol{\Omega}_j^T \boldsymbol{\tau}_n\right)$, $j = 0, \ldots, J-1$, $n = 0, \ldots, N-1$.

We note that the bound in (23) is composed of the BCR bound, supplemented by a positive semidefinite term. Therefore, the regularity conditions used in derivation of the BCR bound, are required also here. In cases where these conditions are not satisfied, (17) and (20) can be modified such that $\boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau}) = \psi_{\text{WW}}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau})$ and $\mathbf{H}(\boldsymbol{\alpha}, \boldsymbol{\tau}) = H_{\text{Fourier}}(\boldsymbol{\alpha}, \boldsymbol{\tau})$. Hence, (23) becomes $\mathbf{C}'_{\text{Fourier}} = \boldsymbol{\Phi} \mathbf{W}^H \left(\mathbf{W} \mathbf{R} \mathbf{W}^H\right)^{-1} \mathbf{W} \boldsymbol{\Phi}^T$.

Observing (23), one can notice that $\left[\mathbf{W} \mathbf{Q}^T\right]_{j,l}$, $j = 1, \ldots, J$, $l = 1, \ldots, L$ is the discrete Fourier transform (DFT) of the l^{th} column of $\mathbf{Q}^T$ evaluated at $\boldsymbol{\Omega}_j$ and that $\left[\mathbf{W} \mathbf{R} \mathbf{W}^H\right]_{j,k}$, $j, k = 1, \ldots, J$ is the two-dimensional DFT of $\mathbf{R}$, evaluated at $(\boldsymbol{\Omega}_j, -\boldsymbol{\Omega}_k)$. Hence, implementation of the bound can be easily performed using the fast Fourier transform (FFT).

The bound in (23) is computed using N equally spaced test points in $\boldsymbol{\Lambda}$ and J frequency test bins in $\mathbf{V}$. For each SNR, the frequency test bins are selected, such that the bound is maximized. In many cases, most of the information in $\psi_{\text{WW}}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau})$ is "compressed" into few frequency components, i.e. $J \ll N$. Therefore, in these cases the computational complexity of the proposed bound is significantly lower in comparison to known bounds, such as the RM, WW, BA and CCRWW bounds, in which maximization w.r.t. $N > J$ test points in $\boldsymbol{\Lambda}$ is required in order to obtain tight bounds.

5. EXAMPLE

In this section, the proposed bound is compared with the BCR [8], BA [12] and CCRWW [13], in the problem of frequency estimation with zero-mean additive white circular complex Gaussian noise. The comparison criterion is prediction of the threshold SNR region exhibited by the MAP estimator. The observation model is given by

$$\mathbf{x} = s\mathbf{a}(\theta) + \mathbf{n}, \quad (24)$$

where $\mathbf{x}$ denotes an $L' \times 1$ observation vector, $s \in \mathbb{C}$ is a known complex amplitude, $\mathbf{a}(\theta) = [1, \exp(i\theta), \ldots, \exp(i(L'-1)\theta)]^T$ is the normalized sinusoid signal, $\mathbf{n}$ denotes an $L' \times 1$ complex circular Gaussian noise vector, with zero-mean and known covariance $\mathbf{C}_{\mathbf{n}} = \sigma_n^2 \mathbf{I}_{L'}$ and $\theta \in \boldsymbol{\Theta} = (-\pi, \pi]$ is the parameter of interest. The *a-priori* probability density function (PDF) of θ zero-mean Gaussian with variance $\sigma_\theta^2 = \frac{1}{2}$, such that the tails of the PDF for $|\theta| > \pi$ are negligible.

Hence, by choosing $t = \frac{1}{2}$ in (19) it is shown in [17] that the terms composing (23) are given by

1. $\mathbf{I}_{\text{BFIM}} = \frac{1}{3}\text{SNR} \cdot L'(L'-1)(2L'-1) + \frac{1}{\sigma_\theta^2}$, where $\text{SNR} = \frac{|s|^2}{\sigma_n^2}$.
2. $\boldsymbol{\Gamma} = -1$.
3. $\boldsymbol{\phi}(\tau_n) = -\tau_n \exp(-\mu(\tau_n, 0))$, $m, n = 0, \ldots, N-1$, where $\mu(\tau_n, \tau_m) = \frac{(\tau_n - \tau_n)^2}{8\sigma_\theta^2} - \frac{1}{2}\text{SNR}\left(\frac{\sin\left(\frac{(\tau_n - \tau_m)L'}{2}\right) \cdot \cos\left(\frac{(\tau_n - \tau_m)(L'-1)}{2}\right)}{\sin\left(\frac{\tau_n - \tau_m}{2}\right)} - L'\right)$.
4. $\mathbf{d}(\tau_n) = 2\exp(-\mu(\tau_n, 0)) \cdot \eta(\tau_n)$, where $\eta(\tau_n) = \frac{\tau_n}{2\sigma_\theta^2} + \frac{\text{SNR}}{4\sin\left(\frac{\tau_n}{2}\right)}\left(\frac{\sin\left(\tau_n\left(L' - \frac{1}{2}\right)\right)}{\tan\left(\frac{\tau_n}{2}\right)} - (2L'-1)\cos\left(\tau_n\left(L' - \frac{1}{2}\right)\right)\right)$.
5. $K_{\psi_{\text{WW}}}(\boldsymbol{\tau}_m, \boldsymbol{\tau}_n) = 2\left(e^{-\mu(\tau_n, \tau_m)} - e^{-\mu(\tau_n, -\tau_m)}\right)$.

6. $[\mathbf{W}]_{j,n} = \exp(-i\omega_j n)$, where $\omega_j = \mathbf{\Omega}_j \Delta\tau \in (-\pi, \pi]$.

The comparison was carried out under the following conditions. The number of observations was set to $L' = 2^7$. The proposed bound was computed using a set of $N = 2^8$ equally spaced test points in $\mathbf{\Lambda}$, given by $\left\{\tau_n = \frac{2\pi n}{N} - \pi\right\}_{n=1}^{N}$ and $J = 1$ frequency test bin, denoted by ω. For each SNR, the proposed bound was maximized w.r.t. $\omega \in \left\{\frac{2\pi k}{N} - \pi\right\}_{k=1}^{N}$. All other compared bounds, except the BCR, were computed using a single test point in $\mathbf{\Lambda}$, denoted by τ. For each SNR, these bounds were maximized w.r.t. $\tau \in \left\{\frac{2\pi n}{N} - \pi\right\}_{n=1}^{N}$.

Fig. 2 depicts the power spectrum of $\{\psi_{\mathrm{WW}}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau}_n)\}_{n=1}^{N}$ for SNR of -15 dB, obtained by applying the two dimensional DFT on $\{K_{\psi_{\mathrm{WW}}}(\boldsymbol{\tau}_m, \boldsymbol{\tau}_n)\}_{m,n=1}^{N}$. One can notice that most of the power is concentrated in low frequencies. Therefore, a major part of the information in $\psi_{\mathrm{WW}}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau})$ can be "compressed" into a few frequency components and the use of the proposed bound is suitable for this scenario. Fig. 3 depicts the compared bounds on the root MSE (RMSE) as a function of SNR. The RMSE of the MAP estimator is depicted as well in order to compare the SNR threshold values predicted by the compared bounds. According to Fig. 3, the proposed bound in (23) is the tightest and allows better prediction of the SNR threshold region.

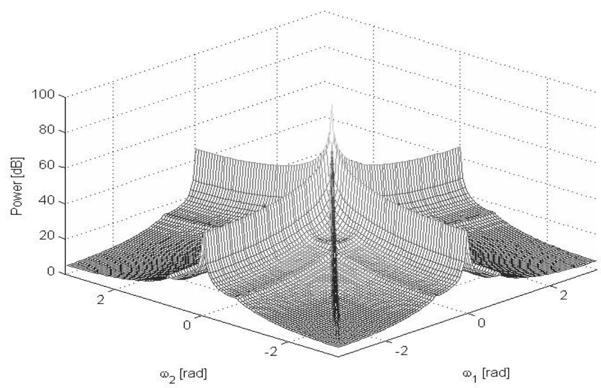

Fig. 2. The power spectrum of the random series $\{\psi_{\mathrm{WW}}(\mathbf{x}, \boldsymbol{\theta}; \boldsymbol{\tau}_n)\}_{n=1}^{N}$ in the scenario of Bayesian frequency estimation, where the number of observations, $L' = 2^7$, $N = 2^{10}$ and SNR=-15 dB.

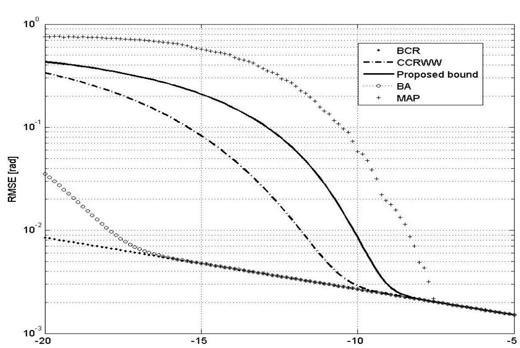

Fig. 3. Comparison of RMSE lower bounds versus SNR.

6. CONCLUSIONS

In this paper, the Weiss-Weinstein class of lower bounds was extended to an integral form and a new class of Bayesian lower bounds on the MSE of estimators was derived from this integral from. It was shown that new Bayesian lower bounds can be derived from this class by modifying the kernel of an integral transform applied on a particular function, which is orthogonal to any function of the observations. Using the kernel of the Fourier transform, a new lower bound was derived from the proposed class. It was shown by simulations that the proposed bound is computationally manageable and provides better prediction of the SNR threshold region, exhibited by the MAP estimator.

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