ANTENNA ARRAY CRAMÉR-RAO BOUND DESIGN BY ELEMENT RELOCATION

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ABSTRACT

We consider the use of an arbitrary planar array for direction-of-arrival (DOA) estimation, with the commonly used Cramér-Rao Bound (CRB) as a performance measure. By taking advantage of the cosine structure of the array Fisher information matrix (FIM), the FIM is reformulated to be completely represented by three parameters which depend on the array geometry, leading to a new FIM expression that reflects the impact of individual sensor relocations. The proposed expressions are particularly helpful for finding new sensor positions when given an objective FIM or CRB, and applies in certain more general scenarios. This usefulness is confirmed through the practical application of relocating a single sensor of an arbitrary planar array to achieve an FIM and CRB with given angular distribution.

1. INTRODUCTION

Direction of arrival (DOA) estimation using an antenna array has attracted much research interest, resulting in numerous estimation techniques [1], and several array design methods for common array geometries (see references in [2]). A popular performance measure is the Cramér-Rao Bound (CRB), because it describes the best achievable estimation performance for a general array, irrespective of the estimation algorithm [3]. The single source CRB is particularly interesting for its relative simplicity, and because it was shown in [4] to be attainable by the MUSIC algorithm of [5]. The CRB is further simplified when the array is planar, because for a fixed source elevation angle, it becomes a cosine function of the source azimuth [6]. The use of polar coordinates to define sensor locations can yield more compact expressions of the CRB [6], but cartesian coordinates are useful for indicating how the CRB is affected by specific array geometry characteristics, such as the array variance [7], or the moment of inertia of projected locations [2]. We wish to explore the dependance of the CRB on individual sensor locations, and consequently adopt the cartesian coordinate system.

The main contribution of this paper is to provide expressions of the Cramér-Rao bound which reflect the impact of individual sensor relocations. We develop these expressions by completely characterising the Fisher information matrix by three parameters, then by rewriting these to reflect the impact of individual sensor locations. These expressions are useful for finding the CRB of a given array, or more interestingly in solving for sensor locations given a CRB or constraints on the CRB. We apply the new expressions to solving a sensor relocation problem when the CRB is constrained to a given distribution over the azimuth range. We also explain how despite the planar array and coplanar source assumptions, our expressions apply to some more general scenarios.

This paper is organized as follows. Section 2 describes the assumptions, signal model, and existing expressions of the single source CRB and FIM. In Section 3, we reformulate the FIM to reflect its dependance on individual sensor locations. In Section 4, the FIM is expressed as a function of individual sensor relocations, and an expression is given for the special case where all but one sensor is relocated. We then apply those results in Section 5, where we consider the practical application of relocating a single sensor of an arbitrary array to achieve an FIM with given angular distribution. Conclusions follow in Section 6.

2. SYSTEM MODEL AND SINGLE SOURCE CRB

We consider a planar array of M isotropic sensors, where the mth sensor is located at $r_m = [x_m, y_m]^T$ and the location matrix of the entire array is given by:

$$\underline{\mathbf{r}} = [r_1, r_2, \ldots, r_M] = \begin{bmatrix} x_1 & x_2 & \ldots & x_M \\ y_1 & y_2 & \ldots & y_M \end{bmatrix}.$$

The centroid of this array is defined as

$$r_o = \begin{bmatrix} x_o \\ y_o \end{bmatrix} = \frac{1}{M}\sum_{m=1}^{M} r_m = \frac{1}{M}\sum_{m=1}^{M}\begin{bmatrix} x_m \\ y_m \end{bmatrix}.$$

A single coplanar wideband source impinges on the array with additive noise from direction θ in the far field, where θ is the azimuth angle measured counter-clockwise from the x-axis. We define the unit vector pointing towards the source as $u(\theta) = [\cos\theta, \sin\theta]^T$.

The source and noise signals are assumed to be zero mean Gaussian and mutually independent, and the noise collected at any two sensors is mutually independent as well. Over the observation period, N independent and identically distributed snapshots are taken, and within each snapshot we apply a J-point discrete Fourier transform to the wideband source.

For such a system model, a formulation of the CRB on the source DOA estimate is given in [8], based on the work in [3], [9], [10]. [2] further modified this to a more intuitive set of equations, which we use in a slightly modified form:

$$\begin{aligned}
\mathbf{CRB}(\theta) &= [\mathbf{FIM}(\theta)]^{-1} = [G(\underline{\mathbf{r}},\theta)\cdot P]^{-1}, \\
P &= \frac{2N}{c^2}\sum_{j=1}^{J}\frac{\omega_j^2}{n_j}p_j\left(1-\frac{n_j}{p_jM+n_j}\right), \\
G(\underline{\mathbf{r}},\theta) &= \frac{du(\theta)}{d\theta}^T\left(\sum_{m=1}^{M}(r_m-r_o)(r_m-r_o)^T\right)\frac{du(\theta)}{d\theta},
\end{aligned}$$

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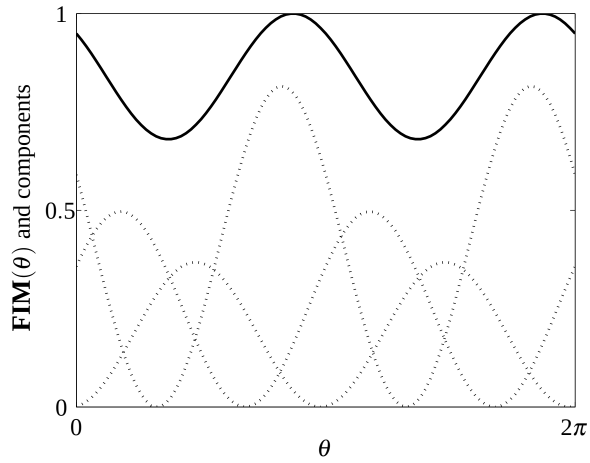

Figure 1: FIM of a 3 sensor array (—), and the 3 sine wave contributions from individual sensors $(\cdots)$.

where FIM is the array Fisher information matrix, c is the speed of propagation, and p_j and n_j are the signal power and noise power within the jth frequency interval, centred on ω_j.

In order to better reflect the contribution of individual sensors to the CRB, we use the definitions of $u(\theta)$, r_i and r_o to express the FIM in a format similar to the ones used in [7] and [11]:

$$\mathbf{FIM}(\theta) = P\sum_{m=1}^{M}\left((x_m - x_o)\sin\theta - (y_m - y_o)\cos\theta\right)^2. \quad (1)$$

3. FIM DEPENDENCE ON SENSOR LOCATIONS

3.1 Contribution of individual sensors to the FIM

Applying trigonometric identities, we can rewrite the individual contribution of the mth sensor to obtain the FIM as the following cosine function:

$$\mathbf{FIM}(\theta) = \frac{P}{2}\sum_{m=1}^{M}\left\{(x_m - x_o)^2 + (y_m - y_o)^2\right\} \times \left\{1 - \cos\left(2\theta - 2\arctan\left(y_m - y_o, x_m - x_o\right)\right)\right\}, \quad (2)$$

where $\arctan(y,x)$ is the generalization of $\arctan(y/x)$ over the entire circular range.

The FIM is therefore a summation of cosine functions, where each such function depends on a single set of sensor coordinates according to (2). An illustration of this fact is given in Fig.1 using an arbitrary three sensor array.

3.2 FIM as a cosine function

In [6], the CRB of an array was shown to always be a cosine function of the source azimuth, with double the azimuth frequency 2θ. The fact that the FIM is a summation of cosine functions with the same frequency agrees with this result. Due to this sine function structure, the FIM can be completely defined by three parameters and can be written as:

$$\mathbf{FIM}(\theta) = R\cos(2\theta + \Phi) + C, \quad (3)$$

where the parameters R, C, and Φ depend on P and on the array sensor locations. More specifically, the amplitude R, amplitude shift C, and phase shift Φ of the FIM depend on P and the amplitudes, amplitude shifts, and phase shifts of the sine waves produced by the sensor locations via (2). Expressions for these parameters are derived in Section 3.4.

3.3 FIM matching: a system of three equations

Because three parameters are sufficient to define any FIM, the problem of designing an unconstrained array to match any FIM sums up to a system of three equations that are independent of θ. A particular advantage of this is that the problem only involves variables with specific values, and omits θ which is defined over an angular range.

The problem of designing an array to match a specific FIM can be seen as an attempt to match two sine functions. The first sine function is given, while the second is defined by (3). Matching two sine waves only requires three matching points. Thus, selecting three angles with convenient cosine terms, this problem resumes to solving (4), where the left hand sides are given as objectives, and the right hand sides are dependant on the array FIM parameters R, C, Φ.

$$\begin{cases} \mathbf{FIM}\left(\frac{\pi}{2}\right) & = C - R\cos\Phi, \\ \mathbf{FIM}(0) & = C + R\cos\Phi, \\ \mathbf{FIM}\left(\frac{3\pi}{4}\right) & = C + R\sin\Phi. \end{cases} \quad (4)$$

3.4 FIM parameter definitions

Expressions for R, Φ, and C as functions of sensor locations shall now be derived.

The following system of equations is obtained by applying trigonometric identities to (2) and (3).

$$\begin{cases} R\cos\Phi & = \frac{P}{2}\sum_{m=1}^{M} -(x_m - x_o)^2 + (y_m - y_o)^2, \\ C & = \frac{P}{2}\sum_{m=1}^{M}(x_m - x_o)^2 + (y_m - y_o)^2, \\ R\sin\Phi & = P\sum_{m=1}^{M}(x_m - x_o)(y_m - y_o). \end{cases}$$

Expanding this and applying the definitions of x_o and y_o, the sensor coordinates can be isolated from the centroid coordinates:

$$\begin{cases} R\cos\Phi & = \frac{P}{2}\left(Mx_o^2 - My_o^2 + \sum_{m=1}^{M} -x_m^2 + y_m^2\right), \\ C & = \frac{P}{2}\left(-Mx_o^2 - My_o^2 + \sum_{m=1}^{M} x_m^2 + y_m^2\right), \\ R\sin\Phi & = P\left(-Mx_o y_o + \sum_{m=1}^{M} x_m y_m\right). \end{cases} \quad (5)$$

The FIM parameters can be determined from (5) through the simple application of $R = \sqrt{(R\cos(\Phi))^2 + (R\sin(\Phi))^2}$ and $\Phi = \arctan(R\sin(\Phi), R\cos(\Phi))$. Thus, (5) is useful in determining the FIM parameters for given sensor locations.

3.5 Simplifying the FIM expression

The system of equations (5) can be reformulated into (6). The advantage of the latter form is that specific parameter combinations are shown to depend on one of three isolated coordinate groupings: the x-coordinates, the y-coordinates, or the cross terms between x and y cooordinates. Thus, (6) should simplify the task of solving for sensor locations when given a set of FIM parameters.

$$\begin{cases} C - R\cos\Phi &= P\left(-Mx_o^2 + \sum_{m=1}^{M} x_m^2\right), \\ C + R\cos\Phi &= P\left(-My_o^2 + \sum_{m=1}^{M} y_m^2\right), \\ R\sin\Phi &= P\left(-Mx_o y_o + \sum_{m=1}^{M} x_m y_m\right). \end{cases} \quad (6)$$

4. FIM DEPENDENCE ON SENSOR MOVEMENT

4.1 Multiple sensor relocation

We will now consider the effect of sensor relocation on the FIM of an array. The relocation is represented by addition of a relocation array $\underline{\mathbf{R}}$ to the existing array $\underline{\mathbf{r}}$, with $\underline{\mathbf{R}}$ defined as

$$\underline{\mathbf{R}} = [R_1, R_2, ..., R_m] = \begin{bmatrix} X_1 & X_2 & \dots & X_m \\ Y_1 & Y_2 & \dots & Y_m \end{bmatrix}.$$

The relocation array's centroid is defined as

$$R_o = \begin{bmatrix} X_o \\ Y_o \end{bmatrix} = \frac{1}{M}\sum_{m=1}^{M} R_m = \frac{1}{M}\sum_{m=1}^{M} \begin{bmatrix} X_m \\ Y_m \end{bmatrix}.$$

As a result of adding the two array location matrices $\underline{\mathbf{r}}$ and $\underline{\mathbf{R}}$, the mth sensor is relocated to $(x_m + X_m, y_m + Y_m)$ and the new array centroid is $(x_o + X_o, y_o + Y_o)$. By substituting this information into (6), we obtain the relocated array's FIM given in (7).

$$\begin{cases} C - R\cos\Phi &= P\left(-M(x_o + X_o)^2 + \sum_{m=1}^{m} (x_m + X_m)^2\right), \\ C + R\cos\Phi &= P\left(-M(y_o + Y_o)^2 + \sum_{m=1}^{m} (y_m + Y_m)^2\right), \\ R\sin\Phi &= -PM(x_o + X_o)(y_o + Y_o) \\ & \quad + P\sum_{m=1}^{m} (x_m + X_m)(y_m + Y_m). \end{cases} \quad (7)$$

4.2 Single sensor relocation

If we only move a single sensor while maintaining all of the other sensors stationary, we can identify the impact that moving each sensor has on the overall array's FIM. By moving the kth sensor from (x_k, y_k) to $(x_k + X_k, y_k + Y_k)$, (7) can be rewritten as (8) or (9):

$$\begin{cases} R\cos\Phi &= a(Y_k^2 - X_k^2) - bX_k + cY_k - d + e, \\ C &= a(X_k^2 + Y_k^2) + bX_k + cY_k + d + e, \\ R\sin\Phi &= 2aX_kY_k + cX_k + bY_k + f. \end{cases} \quad (8)$$

$$\begin{cases} C - R\cos\Phi &= 2(aX_k^2 + bX_k + d), \\ C + R\cos\Phi &= 2(aY_k^2 + cY_k + e), \\ R\sin\Phi &= 2aX_kY_k + cX_k + bY_k + f. \end{cases} \quad (9)$$

where the constants a-f are defined as:

$$\begin{aligned} a &= \frac{P(M-1)}{2M} \quad , \quad d = \frac{P}{2}\sum_{m=1}^{M} x_m^2 - x_o^2 \quad , \\ b &= P(x_k - x_o) \quad , \quad e = \frac{P}{2}\sum_{m=1}^{M} y_m^2 - y_o^2 \quad , \\ c &= P(y_k - y_o) \quad , \quad f = P\sum_{m=1}^{M} x_m y_m - x_o y_o \quad . \end{aligned}$$

In the same way that (5) is helpful for calculating FIM parameters from a given array, (8) is useful for calculating the FIM parameters for a given displacement of the kth sensor. Just as (6) is useful for designing an array for a given objective FIM, (9) can help determine a new location for the kth sensor given FIM parameter values.

4.3 Relevance to more general scenarios

The expressions of Sections 3 and 4 were derived under the planar array and coplanar source assumptions. Nevertheless, the expressions are applicable to certain more general scenarios. Two such scenarios shall now be outlined, accompanied by the provisions for applying our expressions:

- In the case of 3-D source estimation with a planar array, the scenario can be completely characterised by a scalar-valued function of the source azimuth, where the scaling depends on source elevation [6]. Therefore, the problem can be reduced to coplanar source estimation with a planar array, provided that the objective CRB is scaled according to the source elevation, or that the objective CRB constraints are independent of the source elevation scaling. The design example of Section 5 demonstrates the latter case, since the CRB is constrained by a given direction and directivity.
- The second scenario to consider is the particular yet practical case of 3-D estimation with a 3-D array where physical limitations lead to a lack of control over the sensor elevations. This would occur, for instance, when an array of wireless sensors are scattered outdoors over an unknown uneven surface. Because the vertical coordinates are arbitrary, our array improvement must solely rely on planar coordinates. Therefore, from an array design perspective, the problem reduces to 3-D source estimation with a planar array, which as mentioned in the previous point can be further reduced to coplanar source estimation with a planar array.

5. APPLICATION: SINGLE SENSOR MOVEMENT FOR CRB CONSTRAINT MATCHING

5.1 Problem statement

In this section, the expressions derived in Sections 3 and 4 are applied. We consider the problem of moving one sensor from an arbitrary array to improve that array's potential estimation capability, using the CRB as a performance measure.

A scenario which from an array design perspective is reduced to a planar array for estimation of a coplanar source is considered (see Section 4.3). In such a case, the CRB is simply the inverse of the one-by-one Fisher information matrix, so any CRB requirements can be trivially translated into FIM parameters, and the results of Sections 3 and 4 are applicable.

5.2 Specifying the objective FIM parameters

If we were to aim for a specific FIM with specific values of C,R, and Φ, solving (9) would require the two quadratics in the first and second lines to share the same roots. Given that their leading coefficients are the same, this would mean that these quadratics are identical. This would only happen if $b = c$ and $e = d + 2R\cos\Phi$, which is so restrictive on the starting array and FIM parameter values that a solution is unlikely to exist. Thus, less restrictive constraints on the FIM are needed for a reasonable solution space.

The constraint of both Φ and R/C is meaningful because together these parameters define the relative distribution of the CRB over the azimuth range. The direction Φ corresponds to the bearing where the FIM is lowest, and the directivity R/C indicates how anisotropic the array is on a scale of 0 (isotropic array) to 1 (linear array). The distribution of the FIM over the bearing range is significant because it indicates which directions should see performance improvements and which directions can be compromised to allow such benefits.

We consider the design of the FIM angular distribution through Φ and R/C, with the understanding that for designing optimal arrays, these results would need to be balanced with additional objectives such as FIM maximization or application-specific array geometry restrictions.

In order to make the most of the given parameters Φ and R/C, (9) can be rewritten in the following form:

$$\begin{cases} C(1-\dfrac{R}{C}\cos\Phi) & = 2(aX_k^2 + bX_k + d), \\ C(1+\dfrac{R}{C}\cos\Phi) & = 2(aY_k^2 + cY_k + e), \\ C\dfrac{R}{C}\sin\Phi & = 2aX_kY_k + cX_k + bY_k + f. \end{cases} \tag{10}$$

5.3 Solutions

A method to solving (10) shall now be given, by working through the general and special cases that can arise.

5.3.1 *General case:* $\sin\Phi \neq 0$, $\cos\Phi \neq \pm C/R$

In the general case, (10) can be rewritten as the following:

$$\begin{cases} C & = \dfrac{2(aX_k^2 + bX_k + d)}{1-\frac{R}{C}\cos\Phi}, \\ C & = \dfrac{2(aY_k^2 + cY_k + e)}{1+\frac{R}{C}\cos\Phi}, \\ C & = \dfrac{2aX_kY_k + cX_k + bY_k + f}{\frac{R}{C}\sin\Phi}. \end{cases}$$

C is eliminated from this system by setting the right hand sides (RHS) of each equation equal to each other. By setting the RHS of the first equation to equal the RHS of the third, and the RHS of the second to equal the RHS of the third, than the RHS of the first equals the RHS of the second. Thus, this system of three equations can be reduced to the following system of two equations:

$$\begin{cases} \dfrac{2(aX_k^2 + bX_k + d)}{1-\frac{R}{C}\cos\Phi} & = \dfrac{2aX_kY_k + cX_k + bY_k + f}{\frac{R}{C}\sin\Phi}, \\ \dfrac{2(aY_k^2 + cY_k + e)}{1+\frac{R}{C}\cos\Phi} & = \dfrac{2aX_kY_k + cX_k + bY_k + f}{\frac{R}{C}\sin\Phi}. \end{cases} \tag{11}$$

The second line of (11) can be rearranged to yield X_k as a function of Y_k. Substituting this into the first line leads to a quartic equation in Y_k which equates 0. Closed form solutions for this equation yield its roots, of which the real ones are the new y-coordinates of the kth sensor. These coordinates Y_k are then substituted into the second line of (11) to obtain the complete (X_k, Y_k) coordinates of the kth sensor. A linear array cannot be obtained by relocating a single sensor if the other sensors are not aligned, but in any other case, there are exactly two real roots, so two solution locations.

5.3.2 *Particular cases*

Several particular cases exist where the general solution method cannot be applied as it would require division by zero:

- $\sin\Phi = 0$ and $R/C = \pm 1$
- $\sin\Phi = 0$ and $R/C \neq \pm 1$
- $\cos\Phi = \pm C/R$

The first case corresponds to a linear array either parallel or orthogonal to direction Φ, which is trivial to solve if all but the kth sensor are aligned already, and has no solution otherwise.

The last two cases are solved by first substituting the known parameters into (10), before reducing the system of three equations to a quartic polynomial as was done in the general case.

5.4 Example

As an example, the techniques of Section 5.3 are applied to an arbitrary 10-sensor array with the objective to achieve an FIM with $R/C = 0.7$ and $\Phi = \pi/3$.

Fig.2 indicates all the original sensor locations, as well as the solution locations for the single arbitrarily chosen sensor. Fig.3 presents the FIM and CRB of the array before and after the sensor relocation; they have been normalized to have peak values of 1 to ease identification of the directivity R/C. The amplitudes and phase shifts of the functions in these figures confirm the achievement of our objective directivity of 0.7 and direction of $\pi/3$ by relocating a single sensor.

6. CONCLUSIONS

The single source wideband CRB was used as a performance measure, which under the planar array and coplanar source assumptions is trivially found by inverting the FIM. We defined the FIM in terms of three parameters, and formulated it to reflect the impact of individual sensor locations. The existence of these three parameters simplifies the problem of designing an unconstrained array for a particular FIM to a system of three equations, thus avoiding optimisation over the entire range of θ.

We examined the effect of sensor movemement on the FIM by rewriting it as a function of sensor locations and their displacements. We also expressed the FIM for the particular case where only one of the sensors is relocated. These expressions allow for the introduction of constraints on the objective FIM, and are particularly useful in modifying existing arrays to achieve an FIM with specific characteristics.

Despite the planar array and coplanar source assumptions, our expressions are relevant to array geometry improvements in more general scenarios, namely 3-D source estimation with a coplanar array, or even 3-D source estimation with a 3-D array when the sensor elevation is arbitrary.

We applied our results to an array design problem where we wish to achieve a given FIM direction and directivity by moving a single sensor from a random array. We presented the general solution, which reduces to solving a quartic equation, then gave an example where we took an arbitrarily selected sensor from an arbitrary array, then found closed form solutions for its new locations. These results should be a useful contribution to multi-objective array design methods, particularly if complemented by other objectives such as CRB minimisation and array geometry constraints.

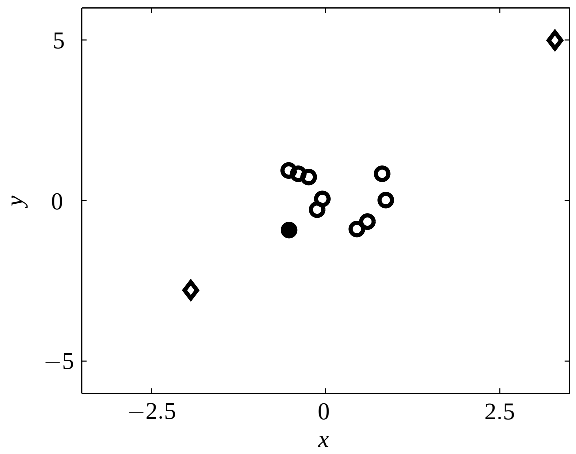

Figure 2: Sensor locations for an arbitrary array before and after relocating the mth sensor. mth sensor starting location (•) and two solution locations (◇), other sensor locations (**o**), for $M = 10, R/C = 0.7, \Phi = \pi/3$.

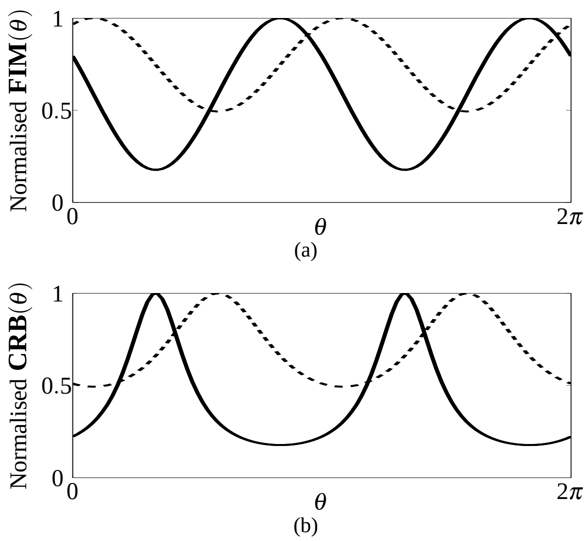

Figure 3: (a) Normalised FIM and (b) normalised CRB, with an arbitrary array before (- -) and after (—) relocating one sensor, for $M = 10, R/C = 0.7, \Phi = \pi/3$.

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