CRAMÉR-RAO TYPE LOWER BOUNDS FOR RELATIVE SENSOR REGISTRATION PROCESS

Stefano Fortunati[*], *Fulvio Gini*[*], *Maria Greco*[*], *Alfonso Farina*[#], *Antonio Graziano*[#], *Sofia Giompapa*[#]

[*]Department of Ingegneria dell'Informazione, University of Pisa
Via G. Caruso 16, 56122, Pisa, Italy
{stefano.fortunati, f.gini, m.greco}@iet.unipi.it
[#]SELEX Sistemi Integrati
Via Tiburtina, km 12.4, 00123 Roma, Italy
{afarina, agraziano, sgiompapa}@selex-si.com

ABSTRACT

This paper concerns the study of the Cramér-Rao type lower bounds for relative sensor registration (or grid-locking) problem. The theoretical performance bound is of fundamental importance both for algorithm performance assessment and for prediction of the best achievable performance given sensor locations, sensor number, and accuracy of sensor measurements. First, a general description of the relative grid-locking problem is given. Afterwards, the measurement model is analyzed. In particular, the nonlinearity of the measurement model and all the biases (attitude biases, measurement biases, and position biases) are taken into account. Finally, the Cramér-Rao lower bound (CRLB) is discussed and two different types of CRLB, the Hybrid CRLB (HCRLB) and the Modified CRLB (MCRLB), are calculated. Theoretical and simulated results are shown.

1. INTRODUCTION

An important prerequisite for successful multisensor integration is that the data from the reporting sensors are transformed to a common reference frame free of systematic or registration bias errors ([1], [2]). If not properly corrected, the registration errors can seriously degrade the global surveillance system performance by increasing tracking errors and even introducing ghost tracks. A first, basic distinction can be made between *relative* grid-locking and *absolute* grid-locking. The relative grid-locking process aligns remote data to local data making the assumption that the local data are bias free and that all biases reside with the remote sensor. The problem is that, actually, also the local sensor is affected by biases that cannot be corrected with this approach. The absolute grid-locking process assumes that all the sensors in the scenario are affected by errors that must be removed. One source of registration errors is sensor calibration (or offset) errors, also called measurement errors. Although the sensors are usually calibrated in an initial calibration procedure, the calibration may deteriorate over time. There are three measurement errors, one for each component of the measurement vector, i.e. range, azimuth, and elevation. Another kind of registration errors are attitude, or orientation errors. Attitude errors can be caused by bias errors in the gyros in the inertial measurement unit (IMU) of the sensor. There are three possible attitude errors, one for each body-fixed rotation axis. The last source of registration errors is represented by the location (or position) errors caused by bias errors in the navigation system associated with the sensors. There are three kinds of location errors, one for each component of the location vector defined in a three-dimensional coordinate system.

Various algorithms for sensor bias estimation have been proposed in the literature both for relative and absolute grid-locking process. In [3], both sensors are considered biased (i.e. affected by bias errors), but only the measurement errors in range and azimuth are taken into account; i.e. a 2-D scenario is considered and the elevation is neglected. A linearized measurement model is assumed and the CRLB is evaluated under this ideal assumption. In [4] both sensors are considered biased, but only two attitudes bias errors and two location bias errors are taken into account. A 2-D scenario is considered and the CRLB is not provided. In [5], two 3-D radars are considered. The location errors are neglected and a linearized measurement model is assumed. Also in this case, the CRLB is not provided. Finally, in [1] and [6], both sensors are considered biased and both the flat model and the ellipsoidal model for the Earth are considered. However, only the measurement bias errors are taken into account and the CRLB is evaluated under a linearized measurement model.

In this paper we derive two theoretical Cramér-Rao–like lower bounds, the Hybrid and the Modified CRLBs for relative grid-locking process. Unlike [3] and [6], no hypothesis of linearity of the model is made and all possible bias errors are taken into account. In our formulation, we need only the following assumptions: (1) one of the two radars is assumed as unbiased (relative grid-locking), i.e. free of registration errors; (2) the registration biases are time invariant during the observation interval; (3) K synchronous pairs of measures coming from a common target are available; (4) the Earth model is the flat model.

Section 2 provides a description of all the geometrical parameters involved in the relative grid-locking process. The measurement models for the measures coming from the two radars are discussed in Section 3. A brief overview on the CRLB is given in Section 4, where we also evaluate the HCRLB and the MCRLB. In Section 5 some numerical re-

 ISSN 2076-1465

sults are shown and discussed. Finally, the conclusions are collected in Section 6.

2. DESCRIPTION OF SCENARIO

The geometry of the scenario is shown in Fig. 1. The main parameters are:

- $\left(x_{S_1}, y_{S_1}, z_{S_1}\right)$: radar #1 reference system. Radar #1 is assumed to be ideal, then its reference system coincides with the absolute one.
- $\left(x_{S_2}, y_{S_2}, z_{S_2}\right)$: radar #2 reference system.
- $\mathbf{OP}^k$: true target position vector in the absolute reference system.
- $\mathbf{OS}_2$: true position vector of radar #2 in the absolute reference system.
- $\mathbf{S}_2\mathbf{P}^k$: true target position vector in radar #2 reference system.

From the geometry of the problem (Fig. 1), the following relation holds:

$$\mathbf{OP}^k = \mathbf{R}\left(\chi, \psi, \xi\right) \cdot \mathbf{S}_2\mathbf{P}^k + \mathbf{OS}_2 , \tag{1}$$

where k is the time index, $\mathbf{R}$ is the rotation matrix of angles χ, ψ and ξ that aligns the radar #2 reference frame to radar #1 reference frame. The angles χ, ψ and ξ are named roll, pitch and yaw and represent the rotation angles around x, y and z axes, respectively. As pointed out before, there are three different types of biases to take into account: attitude biases, measurement biases and location biases. In the rest of the paper we use the following notation:

- *Attitude biases*: we denote by $\mathbf{\Theta}_t=(\chi_t\ \psi_t\ \xi_t)^T$, $\mathbf{\Theta}_m=(\chi_m\ \psi_m\ \xi_m)^T$ and $d\mathbf{\Theta}=(d\chi\ d\psi\ d\xi\ )^T$ the true attitude angles, the measured attitude angles and the attitude bias errors, respectively.
- *Measurement biases*: we denote by $\mathbf{v}_t^k=(\rho_t^k\ \theta_t^k\ \varepsilon_t^k)^T$, $\mathbf{v}_m^k=(\rho_m^k\ \theta_m^k\ \varepsilon_m^k)^T$ and $d\mathbf{v}=(d\rho\ d\theta\ d\varepsilon)^T$ the true target position vector in spherical coordinates, the measured target position vector and the measurement bias errors, respectively.
- *Location biases*: we denote by $\mathbf{t}_t=(t_{x,t}\ t_{y,t}\ t_{z,t})^T$, $\mathbf{t}_m=(t_{x,m}\ t_{y,m}\ t_{z,m})^T$ and $d\mathbf{t}=(dt_x\ dt_y\ dt_z)^T$ the true relative position, the measured relative position and the location bias errors, respectively. It can be noted (see Fig. 1), that $\mathbf{t}_t=\mathbf{OS}_2$.

The assumption adopted in this paper is that the biases must be added to the measured value to obtain the true value of the specific parameter. According with this assumption, we have the following equations for the attitude angles (eq. (2)), for the relative position (eq. (3)) and for the measurement model (eq. (4)):

$$\mathbf{\Theta}_t = \mathbf{\Theta}_m + d\mathbf{\Theta} , \tag{2}$$

$$\mathbf{t}_t = \mathbf{t}_m + d\mathbf{t} , \tag{3}$$

$$\mathbf{v}_m^k = \mathbf{v}_t^k - d\mathbf{v} + \mathbf{n}^k , \tag{4}$$

where $\mathbf{n}^k$ is a zero-mean, Gaussian discrete random process with diagonal covariance matrix $\mathbf{C}_\mathbf{n}$. It must be noted that, without loss of generality, if the rotation around z is applied first, the azimuth measurement bias $d\theta$ and the attitude bias $d\xi$ cannot be distinguished and have to be merged into a single bias. Because of this geometrical coupling, we can define a single bias error as $d\xi' = d\xi + d\theta$.
In the rest of the paper we define the unknown parameters vector as:

$$\mathbf{\Phi} = \left(d\rho \quad d\varepsilon \quad d\chi \quad d\psi \quad d\xi' \quad dt_x \quad dt_y \quad dt_z\right)^T . \tag{5}$$

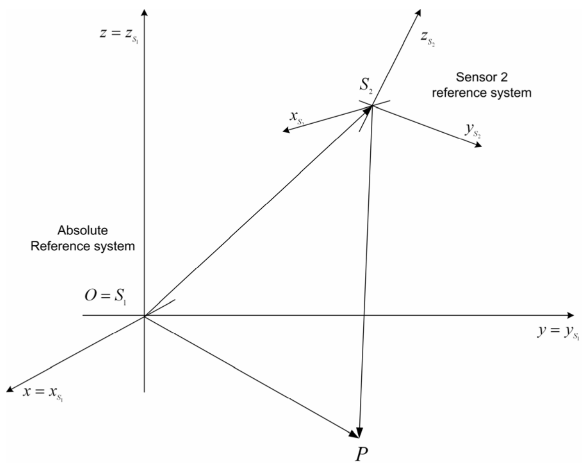

Figure 1 - Scenario's geometry.

Finally, in this paper, vectors indicated with $\mathbf{r}$ define the position vector in Cartesian coordinates, while the ones indicated with $\mathbf{v}$ define the same position vector in spherical coordinates. The spherical to Cartesian transformation is denoted with $\mathbf{h}(\cdot)$ and its inverse with $\mathbf{h}^{-1}(\cdot)$. In order to handle the non linear transformation of the measurement noise $\mathbf{n}^k$, we introduce the *unbiased conversion function* from spherical to Cartesian coordinates. In the following, we denote with $\mathbf{h}^u(\cdot)$ such conversion function, defined in [7], as:

$$\mathbf{h}^u\left(\rho_m, \theta_m, \varepsilon_m\right) = \begin{pmatrix} \lambda_\theta^{-1}\lambda_\varepsilon^{-1}\rho_m \cos\varepsilon_m \sin\theta_m \\ \lambda_\theta^{-1}\lambda_\varepsilon^{-1}\rho_m \cos\varepsilon_m \cos\theta_m \\ \lambda_\varepsilon^{-1}\rho_m \sin\varepsilon_m \end{pmatrix} , \tag{6}$$

where $\lambda_\theta^{-1} = e^{-\sigma_\theta^2/2}$ and $\lambda_\varepsilon^{-1} = e^{-\sigma_\varepsilon^2/2}$.

3. MEASUREMENT MODEL

In this Section, the measurements models coming from radars #1 and #2 are analyzed. In the following, the true target trajectory in the absolute reference frame, i.e. $\mathbf{OP}^k$ in Fig. 1, is assumed to be a discrete random process and is indicated with $\mathbf{r}_t^k$. If radar #1 is assumed to be unbiased, i.e. without bias errors, its reference system can be assumed as the absolute reference frame. Under this assumption, which characterizes the relative grid-locking problem, the radar #1 measurement model is:

$$\mathbf{v}_{1,m}^k = \mathbf{h}^{-1}\left(\mathbf{r}_t^k\right) + \mathbf{n}_1^k , \tag{7}$$

where the noise term $\mathbf{n}_1^k$ is zero-mean Gaussian distributed random vector with diagonal covariance matrix given by $\mathbf{C}_{\mathbf{n}_1} = diag\left(\sigma_{\rho,1}^2, \sigma_{\theta,1}^2, \sigma_{\varepsilon,1}^2\right)$. Now we have to derive the radar #2 measurement model. Eq. (1) can be rewritten as function of the bias errors as:

$$\mathbf{OP}^k = \mathbf{r}_t^k = \mathbf{R}\left(\mathbf{\Theta}_m + d\mathbf{\Theta}\right) \cdot \mathbf{S}_2\mathbf{P}^k + \left(\mathbf{t}_m + d\mathbf{t}\right) , \tag{8}$$

and solving eq. (8) for $\mathbf{S}_2\mathbf{P}^k$ yields to:

$$\mathbf{S}_2\mathbf{P}^k = \mathbf{R}^T\left(\boldsymbol{\Theta}_m + d\boldsymbol{\Theta}\right)\cdot\left[\mathbf{r}_t^k - \left(\mathbf{t}_m + d\mathbf{t}\right)\right], \tag{9}$$

where we have used the fact that if $\mathbf{R}$ is a rotation matrix, then $\mathbf{R}^{-1}=\mathbf{R}^T$. Finally, by applying the inverse coordinate transformation and by adding the measurement bias errors and the measurement noise, we get:

$$\begin{aligned}\mathbf{v}_{2,m}^k &= \mathbf{h}^{-1}\left(\mathbf{R}^T\left(\boldsymbol{\Theta}_m + d\boldsymbol{\Theta}\right)\left[\mathbf{r}_t^k - \left(\mathbf{t}_m + d\mathbf{t}\right)\right]\right) - d\mathbf{v} + \mathbf{n}_2^k \\ &= \boldsymbol{\mu}\left(\mathbf{r}_t^k;\boldsymbol{\Phi}\right) + \mathbf{n}_2^k,\end{aligned} \tag{10}$$

where $\mathbf{n}_2^k$ is a zero-mean, Gaussian distributed random vector with diagonal covariance matrix given by $\mathbf{C}_{\mathbf{n}_2} = diag\left(\sigma_{\rho,2}^2, \sigma_{\theta,2}^2, \sigma_{\varepsilon,2}^2\right)$.

4. CRAMÉR-RAO LOWER BOUND

The Cramér-Rao lower bound of the parameters vector $\boldsymbol{\Phi}$ is defined, for unbiased estimators, as [8]:

$$\mathrm{CRLB}\left(\hat{\Phi}_i\right) = \left[\mathbf{I}^{-1}\left(\boldsymbol{\Phi}\right)\right]_{ii} \tag{11}$$

where $\mathbf{I}(\boldsymbol{\Phi})$ is the Fisher Information Matrix (FIM) whose entries are defined as:

$$\begin{aligned}\left[\mathbf{I}\left(\boldsymbol{\Phi}\right)\right]_{ij} &= E\left\{\frac{\partial}{\partial\Phi_i}\ln\left(p\left(\mathbf{z};\boldsymbol{\Phi}\right)\right)\frac{\partial}{\partial\Phi_j}\ln\left(p\left(\mathbf{z};\boldsymbol{\Phi}\right)\right)\right\} \\ &= -E\left\{\frac{\partial^2}{\partial\Phi_i\partial\Phi_j}\ln\left(p\left(\mathbf{z};\boldsymbol{\Phi}\right)\right)\right\},\end{aligned} \tag{12}$$

where $\mathbf{z}$ is the observation vector. To evaluate the CRLB for the estimate of the grid-locking parameter vector defined in eq. (5), first we define three sets of K elements as follows:

$$V_1 = \left\{\mathbf{v}_{1,m}^k\right\}_{k=1}^K, V_2 = \left\{\mathbf{v}_{2,m}^k\right\}_{k=1}^K, R = \left\{\mathbf{r}_t^k\right\}_{k=1}^K, \tag{13}$$

where V_1 and V_2 are the sets of the K observations coming from radar #1 and #2, and R is the set of the K true targets positions. In order to evaluate the CRLB on $\boldsymbol{\Phi}$, we need to know the joint probability density function (pdf) of V_1 and V_2. From eqs. (7) and (10), we get:

$$\begin{aligned}p\left(V_1,V_2;\boldsymbol{\Phi}\right) &= \int p\left(V_1,V_2\left|R;\boldsymbol{\Phi}\right.\right)p\left(R\right)dR \\ &= E_R\left\{p\left(V_1,V_2\left|R;\boldsymbol{\Phi}\right.\right)\right\}.\end{aligned} \tag{14}$$

By using eq. (12), the FIM can be expressed as:

$$\begin{aligned}\left[\mathbf{I}\left(\boldsymbol{\Phi}\right)\right]_{ij} &= -E\left\{\frac{\partial^2}{\partial\Phi_i\partial\Phi_j}\ln E_R\left\{p\left(V_1,V_2\left|R;\boldsymbol{\Phi}\right.\right)\right\}\right\} \\ &= -\sum_{k=1}^K E\left\{\frac{\partial^2}{\partial\Phi_i\partial\Phi_j}\ln E_{\mathbf{r}^k}\left\{p\left(\mathbf{v}_{1,m}^k,\mathbf{v}_{2,m}^k\left|\mathbf{r}_t^k;\boldsymbol{\Phi}\right.\right)\right\}\right\}.\end{aligned} \tag{15}$$

Unfortunately, no further manipulation is possible because the logarithm of the expectation operator w.r.t. $\mathbf{r}_t^k$ cannot be evaluated analytically. To overcome this mathematical problem, we solve the problem differently.

As pointed out before, for the estimate of $\boldsymbol{\Phi}$ we have at disposal the measurement coming from radars #1 and #2 modelled as in eqs. (7) and (10). The classical way to arrange such measures is to assume as true target position the radar #1 measures and plug them into the radar #2 measurement model, so obtaining:

$$\mathbf{v}_{2,m}^k \simeq \boldsymbol{\mu}\left(\mathbf{h}^u\left(\mathbf{v}_{1,m}^k\right);\boldsymbol{\Phi}\right) + \mathbf{n}_2^k. \tag{16}$$

With this model, we have:

$$\mathbf{v}_{2,m}^k\left|\mathbf{v}_{1,m}^k\right. \sim \mathcal{N}\left(\boldsymbol{\mu}'\left(\mathbf{v}_{1,m}^k;\boldsymbol{\Phi}\right),\mathbf{C}_{\mathbf{n}_2}\right), \tag{17}$$

$$\mathbf{v}_{1,m}^k\left|\mathbf{r}_t^k\right. \sim \mathcal{N}\left(\mathbf{h}^{-1}\left(\mathbf{r}_t^k\right),\mathbf{C}_{\mathbf{n}_1}\right), \tag{18}$$

where, for ease of notation, in eq. (17) we have defined the function $\boldsymbol{\mu}'\left(\mathbf{v}_{1,m}^k;\boldsymbol{\Phi}\right) \triangleq \boldsymbol{\mu}\left(\mathbf{h}^u\left(\mathbf{v}_{1,m}^k\right);\boldsymbol{\Phi}\right)$. Such notation is used in the rest of the paper. It can be noted that the radar #2 measurement model implicitly depends on the realizations of the target position process $\mathbf{r}_t^k$ through $\mathbf{v}_1^k$.

4.1 Hybrid Cramér-Rao lower bound

For the specific case in which the parameters to be estimated are deterministic and the nuisance parameters, i.e. additional parameters whose estimation is not strictly required, are random, we can define the Hybrid CRLB (HCRLB) as the top-left $d_{\boldsymbol{\Phi}}\times d_{\boldsymbol{\Phi}}$ (where $d_{\boldsymbol{\Phi}}$ is the dimension of $\boldsymbol{\Phi}$) block matrix of the inverse of the FIM for the joint estimation of the hybrid vector $\boldsymbol{\Psi}$ ([9], [10]). In our case:

- hybrid vector: $\boldsymbol{\Psi} = \begin{pmatrix}\boldsymbol{\Phi}^T & \left(\mathbf{v}_{1,m}^1\right)^T & \cdots & \left(\mathbf{v}_{1,m}^K\right)^T\end{pmatrix}^T$,
- hybrid FIM: $\mathbf{I}\left(\boldsymbol{\Psi}\right) = \begin{pmatrix}\mathbf{I}_{\boldsymbol{\Phi}} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{I}_{V_1}\end{pmatrix}$, whose entries can be defined as: $\left[\mathbf{I}\left(\boldsymbol{\Psi}\right)\right]_{ij} = E_R\left\{\left[\mathbf{I}_R\left(\boldsymbol{\Psi}\right)\right]_{ij}\right\}$, where:

$$\left[\mathbf{I}_R\left(\boldsymbol{\Psi}\right)\right]_{ij} = -E_{V_1|R}\left\{E_{V_2|V_1}\left\{\frac{\partial^2\ln p\left(V_1,V_2\left|R;\boldsymbol{\Phi}\right.\right)}{\partial\Psi_i\partial\Psi_j}\right\}\right\}, \tag{19}$$

- finally, using the matrix inversion lemma, the HCRLB can be evaluated as:

$$\mathrm{HCRLB}\left(\Phi_i\right) = \left[\left(\mathbf{I}_{\boldsymbol{\Phi}} - \mathbf{A}\mathbf{I}_{V_1}^{-1}\mathbf{A}^T\right)^{-1}\right]_{ii}. \tag{20}$$

First, we start to evaluate each entry of $\mathbf{I}_R\left(\boldsymbol{\Psi}\right)$:

$$\begin{aligned}\left[\mathbf{I}_R\left(\boldsymbol{\Psi}\right)\right]_{ij} &= -E_{V_1|R}\left\{E_{V_2|V_1}\left\{\frac{\partial^2\ln p\left(V_1,V_2\left|R;\boldsymbol{\Phi}\right.\right)}{\partial\Psi_i\partial\Psi_j}\right\}\right\} \\ &= -\sum_{k=1}^K E_{\mathbf{v}_{1,m}^k|\mathbf{r}^k}\left\{E_{\mathbf{v}_{2,m}^k|\mathbf{v}_{1,m}^k}\left\{\frac{\partial^2\ln p\left(\mathbf{v}_{1,m}^k,\mathbf{v}_{2,m}^k\left|\mathbf{r}^k;\boldsymbol{\Phi}\right.\right)}{\partial\Psi_i\partial\Psi_j}\right\}\right\} \\ &= \sum_{k=1}^K E_{\mathbf{v}_{1,m}^k|\mathbf{r}^k}\left\{g_{ij}\left(\mathbf{v}_{1,m}^k;\boldsymbol{\Phi}\right) - \frac{\partial^2\ln p\left(\mathbf{v}_{1,m}^k\left|\mathbf{r}^k;\boldsymbol{\Phi}\right.\right)}{\partial\Psi_i\partial\Psi_j}\right\},\end{aligned} \tag{21}$$

where

$$g_{ij}\left(\mathbf{v}_{1,m}^k;\boldsymbol{\Phi}\right) = -E_{\mathbf{v}_{2,m}^k|\mathbf{v}_{1,m}^k}\left\{\frac{\partial^2\ln p\left(\mathbf{v}_{2,m}^k\left|\mathbf{v}_{1,m}^k;\boldsymbol{\Phi}\right.\right)}{\partial\Psi_i\partial\Psi_j}\right\}, \tag{22}$$

that can be easily evaluated, starting from the model in eq. (17) and following the approach in [8], as:

$$g_{ij}\left(\mathbf{v}_{1,m}^{k};\mathbf{\Phi}\right)=\left(\frac{\partial\boldsymbol{\mu}'\left(\mathbf{v}_{1,m}^{k};\mathbf{\Phi}\right)}{\partial\Psi_i}\right)^T\mathbf{C}_{\mathbf{n}_2}^{-1}\left(\frac{\partial\boldsymbol{\mu}'\left(\mathbf{v}_{1,m}^{k};\mathbf{\Phi}\right)}{\partial\Psi_j}\right). \quad (23)$$

It's easy to show that g_{ij} represent the entries of a block matrix $\mathbf{G}\left(\mathbf{v}_{1,m}^{k};\mathbf{\Phi}\right)$ that can be expressed as:

$$\mathbf{G}\left(\mathbf{v}_{1,m}^{k};\mathbf{\Phi}\right)=\begin{pmatrix}\mathbf{F}[k] & \mathbf{0}_{d_{\mathbf{\Phi}}\times 3(k-1)} & \mathbf{B}_k & \mathbf{0}_{d_{\mathbf{\Phi}}\times 3(K-k)}\\ \mathbf{0}_{3(k-1)\times d_{\mathbf{\Phi}}} & \cdots & \cdots & \cdots\\ \mathbf{B}_k^T & \cdots & \mathbf{N}_k & \cdots\\ \mathbf{0}_{3(K-k)\times d_{\mathbf{\Phi}}} & \cdots & \cdots & \mathbf{0}_{3(K-k)\times 3(K-k)}\end{pmatrix} \quad (24)$$

where $d_{\mathbf{\Phi}}$ = dim($\mathbf{\Phi}$), and the sub-matrix $\mathbf{F}[k]$, $\mathbf{B}_k$ and $\mathbf{N}_k$ are given by:

$$\left[\mathbf{F}[k]\right]_{ij}=g_{ij}\left(\mathbf{v}_{1,m}^{k};\mathbf{\Phi}\right),\ i,j=1,\ldots,d_{\mathbf{\Phi}} \quad (25)$$

$$\left[\mathbf{B}_k\right]_{ij}=g_{ij}\left(\mathbf{v}_{1,m}^{k};\mathbf{\Phi}\right),\quad i=1,\ldots,d_{\mathbf{\Phi}};\ j=d_{\mathbf{\Phi}}+(3k-2),\ldots,d_{\mathbf{\Phi}}+3k, \quad (26)$$

$$\left[\mathbf{N}_k\right]_{ij}=g_{ij}\left(\mathbf{v}_{1,m}^{k};\mathbf{\Phi}\right),\quad i,j=d_{\mathbf{\Phi}}+(3k-2),\ldots,d_{\mathbf{\Phi}}+3k. \quad (27)$$

Finally, the first part of the expectation operator in eq. (21) can be rewritten in matrix form as:

$$\sum_{k=1}^{K}E_{\mathbf{v}_{1,m}^{k}|\mathbf{r}^k}\left\{\mathbf{G}\left(\mathbf{v}_{1,m}^{k};\mathbf{\Phi}\right)\right\}=\mathbf{W}(R)=$$

$$=\begin{pmatrix}\sum_{k=1}^{K}E_{\mathbf{v}_{1,m}^{k}|\mathbf{r}^k}\left\{\mathbf{F}^k\right\} & E_{\mathbf{v}_{1,m}^{1}|\mathbf{r}^1}\left\{\mathbf{B}_1\right\} & \cdots & E_{\mathbf{v}_{1,m}^{K}|\mathbf{r}^K}\left\{\mathbf{B}_K\right\}\\ E_{\mathbf{v}_{1,m}^{1}|\mathbf{r}^1}\left\{\mathbf{B}_1^T\right\} & E_{\mathbf{v}_{1,m}^{1}|\mathbf{r}^1}\left\{\mathbf{N}_1\right\} & \cdots & \mathbf{0}_{3\times3}\\ \vdots & \vdots & \ddots & \vdots\\ E_{\mathbf{v}_{1,m}^{K}|\mathbf{r}^K}\left\{\mathbf{B}_K^T\right\} & \mathbf{0}_{3\times3} & \cdots & E_{\mathbf{v}_{1,m}^{K}|\mathbf{r}^K}\left\{\mathbf{N}_K\right\}\end{pmatrix}. \quad (28)$$

Now, we have to evaluate the second term of the sum in the expectation operator in eq. (21). By using the measurement model in eq. (18), through some algebraic manipulation, such term can be rewritten in matrix form as:

$$-\sum_{k=1}^{K}E_{\mathbf{v}_{1,m}^{k}|\mathbf{r}^k}\left\{\frac{\partial^2\ln p\left(\mathbf{v}_{1,m}^{k}\left|\mathbf{r}^k\right.;\mathbf{\Phi}\right)}{\partial\Psi_i\partial\Psi_j}\right\}_{i,j=1,\ldots,d_{\mathbf{\Phi}}+3K}=\mathbf{C}_K=$$

$$=\begin{pmatrix}\mathbf{0}_{d_{\mathbf{\Phi}}\times d_{\mathbf{\Phi}}} & \cdots & \cdots & \mathbf{0}_{d_{\mathbf{\Phi}}\times d_{\mathbf{\Phi}}}\\ \vdots & \mathbf{C}_{\mathbf{n}_1}^{-1} & \cdots & \mathbf{0}_{3\times3}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0}_{d_{\mathbf{\Phi}}\times d_{\mathbf{\Phi}}} & \mathbf{0}_{3\times3} & \cdots & \mathbf{C}_{\mathbf{n}_1}^{-1}\end{pmatrix}, \quad (29)$$

The expression in eq. (29) doesn't depend on the particular trajectory R. So, we get:

$$\mathbf{I}_R(\mathbf{\Psi})=\mathbf{W}(R)+\mathbf{C}_K, \quad (30)$$

and finally, calculating the expectation w.r.t. R, we obtain the hybrid FIM $\mathbf{I}(\mathbf{\Psi})$. Both expectation operators w.r.t. V_1 in eq. (28) and w.r.t. R are evaluated numerically through independent Monte Carlo trials.

4.2 Modified Cramér-Rao Lower Bound

We introduce the Modified CRLB (MCRLB) for an unbiased estimator of the parameters vector $\mathbf{\Phi}$, as described in [9]. The MCRLB is defined as the inverse of the top-left $d_{\mathbf{\Phi}}\times d_{\mathbf{\Phi}}$ block matrix of the hybrid FIM:

$$\mathrm{MCRLB}(\Phi_i)=\left[\mathbf{I}_{\mathbf{\Phi}}^{-1}\right]_{ii}, \quad (31)$$

where Φ_i is the ith component of the parameters vector $\mathbf{\Phi}$.

4.3 Relationships among the various bounds

In [11] it's shown that when the marginal pdf $p(V_2;\mathbf{\Phi})$ is used, the resulting lower bound on $\mathbf{\Phi}$ is tighter than the bound obtained for the joint estimation of the hybrid parameters vector $\mathbf{\Psi}$, i. e. CRLB($\mathbf{\Phi}$) ≥ HCRLB($\mathbf{\Phi}$). Moreover, taking into account that $[\mathbf{I}_{\mathbf{\Psi}}^{-1}]_{d\Phi\times d\Phi} \geq [\mathbf{I}_{\mathbf{\Psi}}]^{-1}_{d\Phi\times d\Phi}$ [12], the equality between HCRLB and MCRLB holds only when the estimation accuracy of $\mathbf{\Phi}$ doesn't depend on the nuisance parameters ($\mathbf{\Phi}$ and the nuisance parameters are decoupled). Collecting the previous results, we can write the following inequality chain:

$$\mathrm{CRLB}(\mathbf{\Phi})\geq\mathrm{HCRLB}(\mathbf{\Phi})\geq\mathrm{MCRLB}(\mathbf{\Phi}) \quad (32)$$

The distance between the HCRLB and the MCRLB gives us an idea about how much the estimate of $\mathbf{\Phi}$ depends on the nuisance parameters. If the HCRLB is much higher than the MCRLB, the estimate accuracy is strongly affected by the nuisance parameters, while if HCRLB and MCRLB are similar, the nuisance parameters don't affect the estimate.

5. NUMERICAL RESULTS

In this Section, the HCRLB and the MCRLB are calculated in a specific scenario. Radars #1 and #2 are characterized by the following parameters: $\sigma_{\rho,1}=\sigma_{\rho,2}$ = 50 m, $\sigma_{\theta,1}=\sigma_{\theta,2}$ = 0.3°, $\sigma_{\varepsilon,1}=\sigma_{\varepsilon,2}$ = 0.3°. The target locations are generated randomly, uniformly distributed in a given three-dimensional area defined in the absolute reference system of Fig.1 as $H=[-x_l,x_l]\times[-y_l,y_l]\times[0,z_l]$, then $\mathbf{r}_t^k\sim\mathcal{U}(H)$. This model is consistent with a real operational scenario, because we don't need to perform the plot-track association process. At every new acquisition, we only need to know that the measures collected by the two radars come from the same target, without the need to know which particular target generated them. The true target position is generated according to the following values: $x_l=y_l=5\cdot10^4$ m and $z_l=5\cdot10^3$ m. The actual bias errors values are: $d\rho$ = -10 m, $d\theta=d\varepsilon$ = = -0.0573°, $d\chi=d\psi$ = -0.0573°, $d\xi$ = -0.1146°, $dt_x=dt_y=dt_z$ = -30m. The radar #2 position vector is $\mathbf{OS}^2=\mathbf{t}_t$ = = $(1,1,1)\cdot10^3$ m. Finally, the number of the trajectories used to evaluate numerically the expectation operator of the FIM w.r.t. R is N_R = 100, while the number of noisy sequences $\mathbf{n}_1^k$ used to evaluate the expectation operator in eq. (28) is $N_{\mathbf{n}}$ = 500.

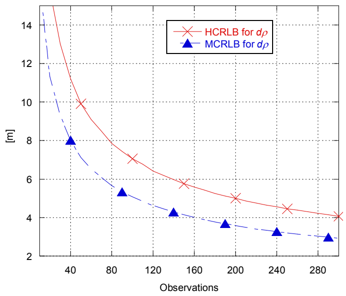

Figure 2 - $\sqrt{\text{HCRLB}}$ & $\sqrt{\text{MCRLB}}$ for $d\rho$.

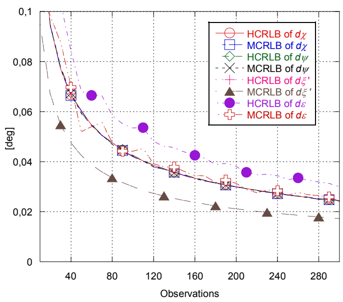

Figure 3 - $\sqrt{\text{HCRLB}}$ & $\sqrt{\text{MCRLB}}$ for $d\mathbf{\Theta}$ and $d\varepsilon$.

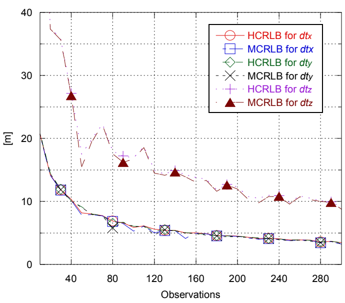

Figure 4 - $\sqrt{\text{HCRLB}}$ & $\sqrt{\text{MCRLB}}$ for position errors.

In Figs. 2, 3 and 4, the square root of HCRLB and MCRLB are shown. First, we can see that for the position and attitude bias errors, except for $d\xi'$, the hybrid and the modified CRLB are equal (Figs. 3, 4). This means that the estimate of such parameters is uncoupled with the nuisance parameters. A different behaviour is observed for the HCRLB and the MCRLB on the estimate of the measurement bias errors explained by noting that there exist a coupling between the radar #1 measurement noise $\mathbf{n}_1^k$ and the estimate of $d\rho$, $d\theta$ (through $d\xi'$) and $d\varepsilon$ since both $\mathbf{n}_1^k$ and $d\mathbf{v}$ are directly involved in the measure's inaccuracy of the target position. Finally, it can be noted that the bound on the estimate of the position error along z, dt_z, is higher than the ones on the estimate of dt_x and dt_y. This fact is due to the geometrical asymmetry of H with respect to the position of radar #1: the z component of the target position vector is forced to be always non-negative (i.e. there are no targets under the sea or ground level), while this geometrical constraint is not present for x and y components.

6. CONCLUDING REMARKS

In this paper, the HCRLB and the MCRLB have been discussed and explicitly derived for the estimate of the relative registration errors. The mathematical problems involved in the evaluation of the classical CRLB have been also shown. The HCRLB and MCRLB have been derived to overcome such problems, given a particular measurement model for the biased radar. We have found that the HCRLB and the MCRLB are generally identical, except for the estimate of the measurement errors that presents a coupling with the nuisance parameters. Future work will explore the performance bounds for the absolute grid-locking process.

REFERENCES

[1] B. Ristic and N. Okello, "Sensor registration in ECEF coordinates using the MLR algorithm," in *Proc. IF 2003*, Cairns, Australia, July 7-10, 2003, vol.1, pp. 135-142.

[2] M. P. Dana, "Registration: A prerequisite for multiple sensor tracking," in *Multitarget-Multisensor Tracking: Advanced Applications*, ch. 5, pp 155-185. Dedham, MA: Artech House, 1990.

[3] Y. Zhou, H. Leung and P. C. Yip, "An Exact Maximum Likelihood Registration Algorithm for Data Fusion," *IEEE Trans. on Signal Proc.*, vol. 45, no. 6, pp. 1560-1573, Jun 1997.

[4] J. J. Sudano, "A Least Square Algorithm with Covariance Weighting for Computing the Translational and Rotational Errors between Two Radar Sites," *Proc. IEEE NAECON 1993*, 24-28 May, 1993.

[5] R. E. Helmick and T. R. Rice, "Removal of Alignment Errors in an Integrated System of Two 3-D Sensors", *IEEE Trans. on Aerospace and Electronic Systems*, vol. 29, No. 4, pp. 1333-1343, October 1993.

[6] B. Ristic, N. Okello and H.-T. Ong, "Performance Bounds for Sensor Registration," *Proc. IF 2002,* Annapolis, USA, July 8-11, 2002, vol 1, pp 346-353.

[7] M. Lonqbin, S. Xiaoquan, Z. Yiyu, S. Z. Kanq and Y. Bar-Shalom, "Unbiased Converted Measurements for Tracking," *IEEE Trans. on Aerospace and Electronic Systems*, vol. 34, pp 1023-1027 July 1998.

[8] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory.* Englewood Cliffs, NJ: Prentice-Hall, 1993.

[9] F. Gini and R. Reggiannini, "On the use of Cramér-Rao-like bounds in the presence of random nuisance parameters," *IEEE Trans. Commun.,* vol. 46, no. 7, pp. 2120-2126, Dec. 2000.

[10] Y. Noam, H. Messer, "Notes on the Tightness of the Hybrid Cramér–Rao Lower Bound", *IEEE Trans. on Signal Proc.*, vol. 57, Issue 6, June 2009.

[11] I. Reuven, H. Messer, "A Barankin-type lower bound on the estimation error of a hybrid parameter vector," *IEEE Trans. Inform. Theory,* vol. 43, pp. 1084-1093, May 1997.

[12] L. L. Scharf, *Statistical Signal processing*, New Jork: Addison-Wesley, 1991.