CUBIC PHASE FUNCTION FOR TWO-DIMENSIONAL POLYNOMIAL-PHASE SIGNALS

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ABSTRACT

A cubic phase function for two-dimensional polynomial-phase signals of the third order (CPF 2-D) is proposed. The CPF 2-D based estimator is able to obtain all unknown parameters by using reduced number of phase differences, compared to the classical Francos-Friedlander (FF) approach. Statistical analysis shows that the proposed CPF 2-D based estimator is asymptotically unbiased and gives low mean squared error (MSE). Simulation results demonstrate that the proposed approach outperforms the FF approach.

1. INTRODUCTION

Two-dimensional (2-D) polynomial-phase signals (PPS) can be found in the radar signal processing and other important applications [1]. The most popular technique for parameter estimation of the 2-D PPS is based on a phase difference operator proposed by Friedlander and Francos [2], [3], which is referred to as the FF approach. The FF approach requires a fourth-order nonlinear transformation to estimate the third-order phase parameters of a 2-D cubic phase PPS (CP-PPS) [3]. Once the highest-order parameters are obtained, a dechirping procedure can be used for the lower-order parameters estimation. However, this estimation procedure suffers from the error propagation effect, i.e., spreading of the former estimation errors to the latter estimates.

In this paper, a generalization of the cubic phase function (CPF) [4], [5] is proposed for the 2-D signals. The generalized CPF is called the CPF 2-D, and it provides a simplified estimation of the CP-PPS with only a second-order nonlinearity. Numerical results show that the CPF 2-D technique outperforms the FF approach, with respect to the estimation threshold which is lower by about 7dB.

The manuscript is organized as follows. The signal model and the FF approach are described in Section II. The proposed technique is presented in Section III. Asymptotic accuracy study is summarized in Section IV. Simulation results are provided in Section V. Conclusions and discussions are given in Section VI, while Appendix provides a brief overview of the asymptotic accuracy study of the CPF 2-D based estimator.

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2. SIGNAL MODEL AND FF APPROACH

2.1 Signal model

Consider the following signal model:

$$y(n,m) = x(n,m) + \nu(n,m),$$

$$n \in [-N/2, N/2),\ m \in [-M/2, M/2), \quad (1)$$

where $x(n,m)$ is the 2-D CP-PPS,

$$x(n,m) = A\exp\left(j\phi(n,m)\right)$$

$$= A\exp\left(j\sum_{p=0}^{P}\sum_{q=0}^{Q} c(p,q)n^p m^q\right), \quad (2)$$

and $\nu(n,m)$ is a white complex Gaussian noise with zero-mean and variance σ^2. In (2), A is the constant amplitude, $\phi(n,m)$ is a polynomial phase with total order up to 3, and $c(p,q)$ is the $(p+q)$-layer parameter $(P+Q \leq 3)$.

The signal model described in (1) has numerous applications in radar, sonar, seismic signals, [6], etc. For example, reflected radar signal from moving targets can be represented by a sum of 2-D CP-PPSs [7]. The parameter estimation of this signal therefore is of a great interest for radar signal processing. Our approach is to estimate the second-order partial derivatives of the signal phase,

$$\begin{bmatrix} \frac{\partial^2\phi(n,m)}{\partial n^2} \\ \frac{\partial^2\phi(n,m)}{\partial n\partial m} \\ \frac{\partial^2\phi(n,m)}{\partial m^2} \end{bmatrix} = \begin{bmatrix} 2c(2,0)+2c(2,1)m+6c(3,0)n \\ c(1,1)+2c(2,1)n+2c(1,2)m \\ 2c(0,2)+2c(1,2)n+6c(0,3)m \end{bmatrix}, \quad (3)$$

and then, based on the above estimates, to estimate signal parameters $\{c(p,q) | p \in [0,P]$ and $q \in [0,Q],\ P+Q \leq 3\}$, as well as A, in a more accurate manner than the FF approach [2], [3], especially at low SNRs.

2.2 FF approach

To estimate the highest-layer parameters of the 2-D CP-PPS the FF approach uses three phase differences (PD), i.e.:

$$PD_{0,2}[y(n,m)] = y(n,m)y^{*2}(n,m+\tau_m)y(n,m+2\tau_m),$$

$$PD_{1,1}[y(n,m)] = y(n,m)y^{*}(n+\tau_n,m)\times$$

$$y^{*}(n,m+\tau_m)y(n+\tau_n,m+\tau_m),$$

$$PD_{2,0}[y(n,m)] = y(n,m)y^{*2}(n+\tau_n,m)y(n+2\tau_n,m), \quad (4)$$

where $*$ denotes the complex conjugation, and τ_n and τ_m are two lag coefficients in the n and m axes. For a noise-free signal $x(n,m)$, the phases of differences (4) are given by (terms not related to n or m are omitted):

$$\text{angle}\{PD_{0,2}[y(n,m)]\} = 2\tau_m^2 c(1,2)n + 6\tau_m^2 c(0,3)m$$

$$\text{angle}\{PD_{1,1}[y(n,m)]\} = 2\tau_n\tau_m c(2,1)n + 2\tau_n\tau_m c(1,2)m$$

$$\text{angle}\{PD_{2,0}[y(n,m)]\} = 6\tau_n^2 c(3,0)n + 2\tau_n^2 c(2,1)m.$$

As a result, it is seen that the PDs in (4) transform the 2-D CP-PPS into 2-D complex sinusoids with coefficients proportional to the third-order phase parameters. Therefore, by ignoring terms not related to n or m, $c(3,0)$, $c(2,1)$, $c(1,2)$ and $c(0,3)$ can be estimated by locating the positions of the peaks of the corresponding 2-D Fourier spectra. For example, point $(\hat{\omega}_n, \hat{\omega}_m)$ at which the 2-D Fourier transform of $PD_{0,2}[y(n,m)]$ reaches its maximal value,

$$(\hat{\omega}_n, \hat{\omega}_m) = \arg \max_{(\omega_n,\omega_m)} |FT_{2D}[PD_{0,2}[y(n,m)]]|,$$

is used for determination of $c(1,2)$ and $c(0,3)$, i.e.

$$(\hat{c}(2,1), \hat{c}(0,3)) = \left(\frac{\hat{\omega}_n}{2\tau_m^2}, \frac{\hat{\omega}_m}{6\tau_m^2}\right).$$

Phase parameters of the lower layer can be estimated by dechirping the original signal with the obtained highest-layer estimates, similarly to the 1-D case.

The fourth-order non-linearity of the PD operator limits the accuracy of the highest-order estimates, especially in the presence of noise or multicomponent signals, when the PD operator produces a great number of cross-terms. This fact can be easily noticed from (4) where the expansion of the each equation results in a sum of 12 elements. Only the one element from that sum is useful, while the other ones represent cross-terms. Furthermore, the estimate errors,

$$\delta c(p,q) = \hat{c}(p,q) - c(p,q), \; p+q = 3,$$

due to the dechirping procedure, propagate from higher to lower order coefficients, i.e.

$$y_d(n,m) = y(n,m)\exp\left(-j\sum_{p+q=3}\hat{c}(p,q)n^p m^q\right) = \nu_d(n,m)$$

$$+A\exp\left(j\sum_{p+q\le 2}c(p,q)n^p m^q - j\sum_{p+q=3}\delta c(p,q)n^p m^q\right).$$

So, $\delta c(p,q)$ has a great influence on the accuracy of lower-layer estimates. By reducing both the order of non-linearity and the number of dechirping, more accurate estimates can be obtained.

3. PROPOSED APPROACH

For the purpose of higher estimation accuracy we extend the CPF [4], [5] for parameter estimation of the 2-D PPS. In this paper, a new phase differencing operator, referred to as the chirp differencing, is introduced as a generalization of the CPF for the case of the 2-D CP-PPS. In distinction to phase differences in (4), the proposed chirp difference, defined as

$$r_y(n,m;\tau_n,\tau_m) = y(n+\tau_n, m+\tau_m)y(n-\tau_n, m-\tau_m), \quad (5)$$

has only a second-order non-linearity, which in the presence of noise reduces the number of cross-terms to 3. This property has a great benefit in improving the estimation accuracy, i.e., decreasing SNR threshold. Following evaluation of the chirp difference, the magnitude of the CPF 2-D is given as

$$f_y(n,m;\mathbf{\Psi}) = |g_y(n,m;\mathbf{\Psi})|^2 =$$

$$\Bigg|\sum_{\tau_n=-n_1}^{n_1}\sum_{\tau_m=-m_1}^{m_1} r_y(n,m;\tau_n,\tau_m)\times$$

$$\exp(-j\Omega_n\tau_n^2 - j\Omega_m\tau_m^2 - j2\Omega_{nm}\tau_n\tau_m)\Bigg|^2, \quad (6)$$

where $\mathbf{\Psi} = [\Omega_n, \Omega_{nm}, \Omega_m]$, $n_1 = \min(N/2-n-1, N/2+n)$ and $m_1 = \min(M/2-m-1, M/2+m)$.

Assuming that $y(n,m)$ is a noise free and expanding $\phi(n+\tau_n, m+\tau_m)$ and $\phi(n-\tau_n, m-\tau_m)$ in the Taylor series around (n,m) up to the 3rd order,

$$\phi(n\pm\tau_n, m\pm\tau_m) = \phi(n,m) \pm [\phi_n(n,m)\tau_n + \phi_m(n,m)\tau_m] +$$

$$\frac{1}{2}[\phi_{nn}(n,m)\tau_n^2 + 2\phi_{nm}(n,m)\tau_n\tau_m + \phi_{mm}(n,m)\tau_m^2]$$

$$\pm\frac{1}{6}[\phi_{nnn}(n,m)\tau_n^3 + 3\phi_{nnm}(n,m)\tau_n^2\tau_m$$

$$+3\phi_{nmm}(n,m)\tau_n\tau_m^2 + \phi_{mmm}(n,m)\tau_m^3],$$

where

$$\phi_{i_1 i_2 \ldots i_N}(n,m) = \frac{\partial^N \phi(n,m)}{\partial i_1 \partial i_2 \cdots \partial i_N},$$

it follows

$$r_y(n,m;\tau_n,\tau_m) =$$

$$A^2\exp(j\phi_{nn}(n,m)\tau_n^2 + j\phi_{mm}(n,m)\tau_m^2 + j2\phi_{nm}(n,m)\tau_n\tau_m). \quad (7)$$

From (7) it is clear that the first-order partial derivatives of the CPF 2-D is equal to zero at $[\phi_{nn}(n,m), \phi_{nm}(n,m), \phi_{mm}(n,m)]$, so, in absence of noise, the CPF 2-D reaches maxima at

$$\Omega_n(n,m) = 2c(2,0) + 2c(2,1)m + 6c(3,0)n,$$

$$\Omega_m(n,m) = 2c(0,2) + 2c(1,2)n + 6c(0,3)m,$$

$$\Omega_{mn}(n,m) = 2c(2,1)n + 2c(1,2)m + c(1,1), \quad (8)$$

which are the second-order partial derivatives of the 2-D PPS. Equations in (8) suggest that the proposed CPF 2-D can be used to estimate the second-order partial derivatives of the signal phase (see (3)), even in presence of a high noise.

Based on the estimates of $\{\Omega_n(n,m_i),\ \Omega_{nm}(n,m),\ \Omega_m(n,m)\}$, the relevant phase parameters in (3) can be estimated as follows:

1) Choose three instants points (n_i,m_i), $i=1,2,3$;

2) Estimate corresponding $\{\hat{\Omega}_n(n_i,m_i),\ \hat{\Omega}_{nm}(n_i,m_i),\ \hat{\Omega}_m(n_i,m_i)\}$ $i=1,2,3$, by searching for the maxima of (6);

3) Estimate seven phase parameters including four third-layer ones $\{c(3,0),\ c(2,1),\ c(1,2),\ c(0,3)\}$ and three second-layer ones $\{c(2,0),\ c(1,1),\ c(0,2)\}$ using:

$$\begin{bmatrix}\hat{c}(2,0)\\ \hat{c}(3,0)\\ \hat{c}'(2,1)\end{bmatrix} = \begin{bmatrix}2 & 6n_1 & 2m_1\\ 2 & 6n_2 & 2m_2\\ 2 & 6n_3 & 2m_3\end{bmatrix}^{-1}\begin{bmatrix}\hat{\Omega}_n(n_1,m_1)\\ \hat{\Omega}_n(n_2,m_2)\\ \hat{\Omega}_n(n_3,m_3)\end{bmatrix},$$

$$\begin{bmatrix}\hat{c}(0,2)\\ \hat{c}(0,3)\\ \hat{c}'(1,2)\end{bmatrix} = \begin{bmatrix}2 & 6m_1 & 2n_1\\ 2 & 6m_2 & 2n_2\\ 2 & 6m_3 & 2n_3\end{bmatrix}^{-1}\begin{bmatrix}\hat{\Omega}_m(n_1,m_1)\\ \hat{\Omega}_m(n_2,m_2)\\ \hat{\Omega}_m(n_3,m_3)\end{bmatrix},$$

$$\begin{bmatrix}\hat{c}(1,1)\\ \hat{c}''(2,1)\\ \hat{c}''(1,2)\end{bmatrix} = \begin{bmatrix}1 & 2n_1 & 2m_1\\ 1 & 2n_2 & 2m_2\\ 1 & 2n_3 & 2m_3\end{bmatrix}^{-1}\begin{bmatrix}\hat{\Omega}_{nm}(n_1,m_1)\\ \hat{\Omega}_{nm}(n_2,m_2)\\ \hat{\Omega}_{nm}(n_3,m_3)\end{bmatrix}. \quad (9)$$

It can be seen that (9) gives two estimates of $c(1,2)$ and $c(2,1)$, i.e., ($\{\hat{c}'(2,1),\ \hat{c}''(2,1)\}$ and $\{\hat{c}'(1,2),\ \hat{c}''(1,2)\}$). The final estimates of $c(1,2)$ and $c(2,1)$ can be therefore obtained by either choosing one of the two estimates or by averaging them.

After finding the above estimates, the lower-layer phase parameters and the amplitude can be estimated in a straightforward manner as in [8]. As such, for estimating $c(0,0)$, $c(0,1)$, $c(1,0)$ and A, the dechirping is required. Therefore, these estimates undergo the error-propagation effects from the third-layer and second-layer parameter estimation. Nevertheless, the second-layer parameter estimates using the above approach do not suffer from the error-propagation effects, as opposed to the FF based approach.

4. MSE PERFORMANCE

The statistical performance study of the proposed CPF 2-D is rather tedious, but in general it follows the idea from the similar analysis for the 1-D CP-PPS from [5]. Here, we only summarize the final results while the main steps of derivations are given in Appendix. The results show that the proposed estimators for the second- and third-order phase parameters are asymptotically unbiased. The mean-squared error (MSE) and corresponding CRLB for the second- and third-layer parameter estimates are given in Table 1.

Phase Parameters	MSE	CRLB
$c(2,0)$	$\frac{90\left(1+\frac{1}{2\mathrm{SNR}}\right)}{\mathrm{SNR}\,N^5 M}$	$\frac{90}{\mathrm{SNR}\,N^5 M}$
$c(0,2)$	$\frac{90\left(1+\frac{1}{2SNR}\right)}{\mathrm{SNR}\,M^5 N}$	$\frac{90}{\mathrm{SNR}\,M^5 N}$
$c(1,1)$	$\frac{72\left(1+\frac{1}{2SNR}\right)}{\mathrm{SNR}\,M^3 N^3}$	$\frac{72}{\mathrm{SNR}\,M^3 N^3}$
$c(3,0)$	$\frac{2036.03+\frac{1844.46}{\mathrm{SNR}}}{\mathrm{SNR}\,N^7 M}$	$\frac{1400}{\mathrm{SNR}\,N^7 M}$
$c(0,3)$	$\frac{2036.03+\frac{1844.46}{\mathrm{SNR}}}{\mathrm{SNR}\,N M^7}$	$\frac{1400}{\mathrm{SNR}\,N M^7}$
$c(2,1)$	$\frac{1440+\frac{2160}{\mathrm{SNR}}}{\mathrm{SNR}\,N^5 M^3}$	$\frac{1080}{\mathrm{SNR}\,N^5 M^3}$
$c(1,2)$	$\frac{1440+\frac{2160}{\mathrm{SNR}}}{\mathrm{SNR}\,N^3 M^5}$	$\frac{1080}{\mathrm{SNR}\,N^3 M^5}$

Table 1: Variance and CRLB for the second-layer and third-layer coefficients.

From Table 1 it can be concluded that the estimator of the second-layer coefficient is asymptotically efficient, i.e., the variance of these parameters estimate for high SNR approaches the CRLB (term SNR^{-2} can be neglected with respect to SNR^{-1}). In addition, it can be seen that in case of high SNR, the proposed estimator produces variance 1.63dB higher than the CRLB for parameters $c(3,0)$ and $c(0,3)$ and for only 1.25dB higher than the CRLB for the mixed parameters $c(2,1)$ and $c(1,2)$.

5. NUMERICAL EXAMPLE

In this section, numerical examples are provided to verify the proposed approach. A CP-PPS signal with parameters $A = 1$, $c(0,0) = 1$, $c(1,0) = 4.5 \cdot 10^{-1}$, $c(0,1) = 8.2 \cdot 10^{-2}$, $c(2,0) = -1.5 \cdot 10^{-3}$, $c(1,1) = 6 \cdot 10^{-3}$, $c(0,2) = -2.2 \cdot 10^{-3}$, $c(3,0) = 1.7 \cdot 10^{-5}$, $c(2,1) = 4 \cdot 10^{-5}$, $c(1,2) = 3.73 \cdot 10^{-5}$, $c(0,3) = -1.35 \cdot 10^{-5}$ is generated with $N = 100$ and $M = 100$.

The FF approach is used as a benchmark [3]. The relevant coefficients for the FF approach are chosen as $\tau_n = \tau_m = 33$ (the choice of values of τ_n and τ_m influences the estimation accuracy, so these parameters are chosen following the instructions from [3]), and corresponding search is performed over a 2-D space with 512×512 elements for all three functions (4). Additional interpolations are performed around initial estimates by a factor of 100. The CPF 2-D (6) is evaluated at instants $(50,50)$, $(50,40)$ and $(40,50)$. Numerical results are given in Figure 1, where the MSEs for four characteristic higher-order parameters of the 2-D CP-PPS are depicted. Results are obtained with 200 runs of the Monte-Carlo simulation. Thin solid lines represent the MSEs achieved by the FF approach, the thick dashed lines depict the MSEs of the CPF 2-D, while the thin dashed lines are for the corresponding CRLB.

It can be observed that the proposed approach outperforms the FF approach in terms of lower SNR threshold by about 7dB. This is significant advantage of the proposed approach. However, it is paid by increased calculation complexity.

6. CONCLUSION AND DISCUSSION

This paper has presented an algorithm for estimating the parameters of a noisy 2-D CP-PPS. The algorithm is based on the bilinear chirp difference operator which reduces the number of cross-terms in comparison to the FF approach. As a result of reducing the number of cross-terms, the proposed technique has considerably lower SNR threshold for estimation. The presented statistical analysis has shown that all the parameter estimates are asymptotically statistically (near) efficient at high SNR value.

If components of multicomponent 2-D CP-PPS do not overlap in the 2-D FT domain, the proposed CPF 2-D can be used for the parameter estimation of each component. After determination of components' regions in the 2-D FT, each component can be extracted by setting the 2-D FT values outside of the considered region to zero and performing the inverse 2D FT. The obtained signals are monocomponent 2-D CP-PPS and their parameters can be estimated by the proposed algorithm.

For a single point $[\Omega_n, \Omega_{nm}, \Omega_m]$, evaluation of the CPF 2-D requires $O(NM)$ operations and since it must be done for a large number of points, because the 3-D search is used, the proposed approach has a higher computational complexity in comparison to the FF technique, which overall complexity is of the order of magnitude $O(NM \log_2 NM)$. Therefore, our future research will consider the problem of reducing the computation complexity as well as generalization to higher-order PPS.

7. APPENDIX

Appendix provides statistical analysis of the proposed approach. Here, only the main steps of analysis are given, while the detailed derivations can be found in [9]. Note that in [9] we analyzed calculation complexity of the algorithm and compared it with the FF approach. In addition, efficient evaluation procedure based on the genetic algorithm has been proposed.

The 2D CPF of a signal (1) can be separated to two components,

- signal component

$$g_x(n,m;\mathbf{\Psi}) =$$

$$\sum_{\tau_n}\sum_{\tau_m} r_x(n,m;\tau_n,\tau_m) e^{-j\psi_n\tau_n^2 - j\psi_m\tau_m^2 - j2\psi_{nm}\tau_n\tau_m}$$

- and a component introduced by interferences

$$\delta g(n,m;\mathbf{\Psi}) =$$

$$\sum_{\tau_n}\sum_{\tau_m} z_{x\nu}(n,m,\tau_n,\tau_m) e^{-j\psi_n\tau_n^2 - j\psi_m\tau_m^2 - j2\psi_{nm}\tau_n\tau_m},$$

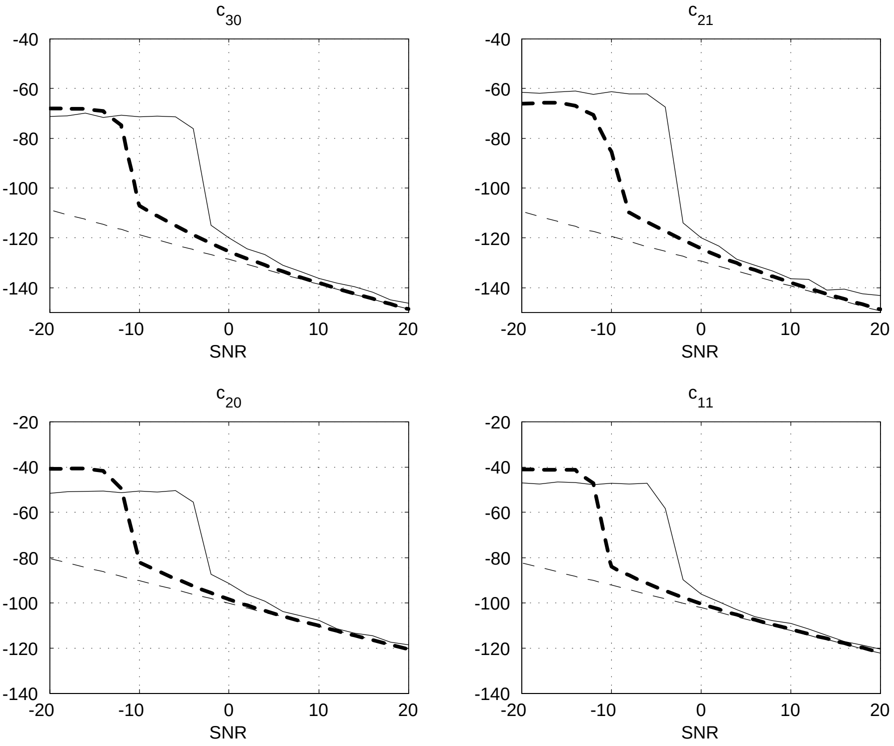

Figure 1: MSEs for $c(3,0)$, $c(2,1)$, $c(2,0)$ and $c(1,1)$ of the 2-D CP-PPS - MSEs achieved by the FF approach (solid lines); MSEs of the 2-D CPF (thick dashed lines) and the corresponding CRLBs (thin dashed lines).

where

$$z_{x\nu}(n,m,\tau_n,\tau_m) = x(n+\tau_n,m+\tau_m)\nu(n-\tau_n,m-\tau_m)+ \nu(n+\tau_n,m+\tau_m)x(n-\tau_n,m-\tau_m)+ \nu(n+\tau_n,m+\tau_m)\nu(n-\tau_n,m-\tau_m).$$

The maximum of $|g_y(n,m;\mathbf{\Psi})|^2$, due to presence of the noise, is dislocated from the real position, $\mathbf{\Psi} = \mathbf{\Omega}(n,m) = [\Omega_n(n,m), \Omega_{nm}(n,m), \Omega_m(n,m)] = \left[\frac{\partial^2\phi(n,m)}{\partial n^2}, \frac{\partial^2\phi(n,m)}{\partial n\partial m}, \frac{\partial^2\phi(n,m)}{\partial m^2}\right]$ for $\delta\mathbf{\Omega}(n,m) = [\delta\Omega_n(n,m), \delta\Omega_{nm}(n,m), \delta\Omega_m(n,m)]$. Therefore the first-order partial derivatives of $f_y(n,m;\mathbf{\Psi})$ are equal to 0 at $\mathbf{\Psi} = \mathbf{\Omega} + \boldsymbol{\delta\Omega}$:

$$\left[\frac{\partial f_x(n,m;\mathbf{\Psi})}{\partial\psi_i} + \frac{\partial\delta f(n,m;\mathbf{\Psi})}{\partial\psi_i}\right]|_{\mathbf{\Psi}=\mathbf{\Omega}+\delta\mathbf{\Omega}} = 0,\ i=1,2,3, \tag{10}$$

where ψ_i, $i=1,2,3$ are corresponding elements of the vector $\mathbf{\Psi}$, $\psi_1 = \psi_n$, $\psi_2 = \psi_{nm}$, $\psi_3 = \psi_m$ (for the sake of brevity, we removed the dependency of the second-order derivatives of the signal phase on position (n,m)).

By considering the fact that, at relatively large SNR, $\delta f(n,m;\mathbf{\Psi})$ can be approximated by

$$\delta f(n,m;\mathbf{\Psi}) \approx 2\,\mathrm{Re}\left\{g_x(n,m;\mathbf{\Psi})\delta g^*(n,m;\mathbf{\Psi})\right\},$$

the Taylor series expansion of (10), up to the second term, around $\mathbf{\Omega}$ gives the system of equations

$$\delta\mathbf{F}_1 + \mathbf{F}_2\delta\mathbf{\Psi} = \mathbf{0}, \tag{11}$$

where

$$[\delta\mathbf{F}_1]_i = 2\,\mathrm{Re}\left\{\frac{\partial g_x(n,m;\mathbf{\Psi})}{\partial\psi_i}\delta g^*(n,m;\mathbf{\Psi}) + g_x(n,m;\mathbf{\Psi})\frac{\partial\delta g^*(n,m;\mathbf{\Psi})}{\partial\psi_i}\right\},$$

$$\delta\mathbf{\Psi} = [\ \delta\psi_1 \quad \delta\psi_2 \quad \delta\psi_3\ ]^T$$

and

$$[\mathbf{F}_2]_{il} = \frac{\partial^2 f_x(n,m;\mathbf{\Psi})}{\partial\psi_i\partial\psi_l}|_{\mathbf{\Psi}=\mathbf{\Omega}} = 2\,\mathrm{Re}\left\{\frac{\partial^2 g_x(n,m;\mathbf{\Psi})}{\partial\psi_i\partial\psi_l}g_x^*(n,m;\mathbf{\Psi}) + \frac{\partial g_x(n,m;\mathbf{\Psi})}{\partial\psi_i}\frac{\partial g_x^*(n,m;\mathbf{\Psi})}{\partial\psi_l}\right\}, i=1,2,3, l=1,2,3.$$

The bias of the 2D CPF based estimator can be obtained from (11) by taking the expectation with respect to $\delta\mathbf{\Psi}$,

$$E\{\delta\mathbf{\Psi}\} = -\mathbf{F}_2^{-1}E\{\delta\mathbf{F}_1\}, \tag{12}$$

while the variances for the estimate errors are the diagonal of the covariance matrix of $\delta\mathbf{\Psi}$,

$$E\{(\delta\mathbf{\Psi})(\delta\mathbf{\Psi})^T\} = [\mathbf{F_2}]^{-1}C_{\delta\mathbf{F}_1}[\mathbf{F_2}]^{-1}, \tag{13}$$

where $\mathbf{C}_{\delta\mathbf{F}_1} = E\{(\delta\mathbf{F}_1)(\delta\mathbf{F}_1)^T\}$.

The detailed derivations of expressions for $\mathbf{F}_2, \mathbf{C}_{\delta\mathbf{F}_1}$ and $\delta\mathbf{F}_1$ are performed in [9]. Here, we will only present the obtained results:

$$\mathbf{F}_2 = -\frac{128}{9}A^4K^2L^2\begin{bmatrix} \frac{1}{5}K^4 & 0 & 0 \\ 0 & K^2L^2 & 0 \\ 0 & 0 & \frac{1}{5}L^4 \end{bmatrix},$$

$$\mathbf{C}_{\delta\mathbf{F}_1} = -8KL(2A^2\sigma^2 + \sigma^4)\mathbf{F}_2,$$

$$[\delta\mathbf{F}_1]_i = -8A^2KL\,\mathrm{Im}\left\{\Gamma(i,K,L)\right\},$$

where

$$\Gamma(i,K,L) = e^{j2\phi(n,m)}\sum_{\tau_n=-K}^{K}\sum_{\tau_m=-L}^{L}\lambda_i(\tau_n,\tau_m)\times$$

$$z^*_{x\nu}(n,m;\tau_n,\tau_m)e^{j\psi_n\tau_n^2+j\psi_m\tau_m^2+j2\psi_{nm}\tau_n\tau_m}$$

and

$$\lambda_i(\tau_n,\tau_m) = \begin{cases} \tau_n^2-\frac{K^2}{3} & i=1 \\ 2\tau_n\tau_m & i=2 \\ \tau_m^2-\frac{L^2}{3} & i=3. \end{cases}.$$

Substituting the above results into (12) and (13) gives

$$E\{\delta\mathbf{\Psi}\} = -\mathbf{F}_2^{-1}E\{\delta\mathbf{F}_1\} = \mathbf{0} \tag{14}$$

and

$$E\{(\delta\psi_1)^2\} = E\{(\delta\Omega_n)^2\} = \frac{45\left(2+\frac{1}{\mathrm{SNR}}\right)}{16\mathrm{SNR}K^5L} \tag{15}$$

$$E\{(\delta\psi_2)^2\} = E\{(\delta\Omega_{nm})^2\} = \frac{9\left(2+\frac{1}{\mathrm{SNR}}\right)}{16\mathrm{SNR}K^3L^3}$$

$$E\{(\delta\psi_3)^2\} = E\{(\delta\Omega_m)^2\} = \frac{45\left(2+\frac{1}{\mathrm{SNR}}\right)}{16\mathrm{SNR}KL^5}.$$

From (14) it is clear that the proposed estimator is unbiased in an asymptotic sense.

Assuming that the evaluation of the phase coefficients is performed for the central instant of the considered domain, MSEs of the second-layer phase parameters can be obtained as

$$E\{(\delta c(2,0))^2\} = \frac{E\{(\delta\Omega_n)^2\}|_{\substack{n=0\\m=0}}}{4} = \frac{90\left(1+\frac{1}{2\mathrm{SNR}}\right)}{\mathrm{SNR}\,N^5M}$$

$$E\{(\delta c(0,2))^2\} = \frac{E\{(\delta\Omega_m)^2\}|_{\substack{n=0\\m=0}}}{4} = \frac{90\left(1+\frac{1}{2\mathrm{SNR}}\right)}{\mathrm{SNR}\,M^5N}$$

$$E\{(\delta c(1,1))^2\} = E\{(\delta\Omega_{nm})^2\}|_{\substack{n=0\\m=0}} = \frac{72\left(1+\frac{1}{2\mathrm{SNR}}\right)}{\mathrm{SNR}\,M^3N^3}.$$

Since the estimation errors of the second phase derivatives are linearly related to the phase-parameters estimation errors (see (9)), MSEs of the third-layer phase-parameters can be easily evaluated. The relation between $\delta\Omega_n(n,m)$ and $\delta c(1,2)$ ($\delta c(0,3)$) could be established by estimating Ω_n at three points ($n=0$, $m=0$), ($n, m=0$) and ($n=0$, m) (see the first equation in (9)):

$$\mathbf{E} = \begin{bmatrix} \delta\hat{c}(2,0) \\ \delta\hat{c}(3,0) \\ \delta\hat{c}(2,1) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 6n & 0 \\ 2 & 0 & 6m \end{bmatrix}^{-1} \begin{bmatrix} \delta\Omega_n(0,0) \\ \delta\Omega_n(n,0) \\ \delta\Omega_n(0,m) \end{bmatrix}.$$

MSEs of the three estimates $c(2,0)$, $c(3,0)$ and $c(2,1)$ represent the diagonal elements of the covariance matrix of vector $\mathbf{E}$. From it follows

$$E\{(\delta c(3,0))^2\} = \frac{E\{(\delta\Omega_n(0,0))^2\}+E\{(\delta\Omega_n(n,0))^2\}}{36n^2} - \frac{2E\{\delta\Omega_n(0,0)\delta\Omega_n(n,0)\}}{36n^2},$$

$$E((\delta c(2,1))^2\} = \frac{E\{(\delta\Omega_n(0,0))^2\}+E\{(\delta\Omega_n(0,m))^2\}}{4m^2} - \frac{2E\{\delta\Omega_n(0,0)\delta\Omega_n(0,m)\}}{4m^2}.$$

Terms $E\{(\delta\Omega_n(0,0))^2\}$, $E\{(\delta\Omega_n(n,0))^2\}$ and $E\{(\delta\Omega_n(0,m))^2\}$ can easily be evaluated from (15), while $E\{\delta\Omega_n(0,0)\delta\Omega_n(n,0)\}$ and $E\{\delta\Omega_n(0,0)\delta\Omega_n(0,m)\}$ are equal to (see [9] for more information):

$$E\{\delta\Omega_n(0,0)\delta\Omega_n(n,0)\} =$$

$$= \frac{45^2E\left\{\Gamma(1,N/2,M/2)\Gamma^*(1,N/2-n,M/2)\right\}}{512A^4\left(\frac{N}{2}\right)^5\left(\frac{N}{2}-n\right)^5\left(\frac{M}{2}\right)^2},$$

$$E\{\delta\Omega_n(0,0)\delta\Omega_n(0,m)\} =$$

$$\frac{45^2E\left\{\Gamma(1,N/2,M/2)\Gamma^*(1,N/2,M/2-m)\right\}}{512A^4\left(\frac{N}{2}\right)^{10}\left(\frac{M}{2}\right)\left(\frac{M}{2}-m\right)}.$$

It is obvious that values of n and m have a great influence on MSEs. Numerical results show that $n\approx 0.11N$ and $m\approx 0.25M$ give minimum MSEs for a high SNR (e.g. $SNR=$ 20dB). So, substitution of $n=0.11N$ and $m=0.25M$ into the relations above results

$$E\left\{(\delta c(3,0))^2\right\} = \frac{2036.03+\frac{1844.46}{\mathrm{SNR}}}{\mathrm{SNR}N^7M},$$

$$E\{(\delta c(2,1))^2\} = \frac{1440+\frac{2160}{\mathrm{SNR}}}{\mathrm{SNR}N^5M^3}.$$

Similarly, the asymptotic accuracy for $c(0,3)$ and $c(1,2)$ can be derived as

$$E\left\{(\delta c(0,3))^2\right\} = \frac{2036.03+\frac{1844.46}{\mathrm{SNR}}}{\mathrm{SNR}NM^7},$$

$$E\{(\delta c(1,2))^2\} = \frac{1440+\frac{2160}{\mathrm{SNR}}}{\mathrm{SNR}N^3M^5},$$

respectively.

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