INTERPOLATION AND DECIMATION OF SPECTRALLY CORRELATED STOCHASTIC PROCESSES

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ABSTRACT

The problem of interpolation and decimation of jointly spectrally correlated (SC) discrete-time stochastic processes is addressed. Jointly SC processes have Loève bifrequency cross-spectrum with spectral masses concentrated on a countable set of support curves of the bifrequency plane. Jointly almost-cyclostationary (ACS) processes are obtained as special case when the support curves are lines with unit slope. It is shown that two jointly wide-sense stationary or jointly ACS processes expanded or decimated with different rates give rise to jointly SC processes. In addition, interpolation filtering of jointly SC processes is considered and sufficient conditions for alias-free filtering are provided.

1. INTRODUCTION

In multirate digital signal processing, systems constituted by cascade interconnections of interpolators, decimators, and linear time-invariant (LTI) filters acts on input stochastic processes [2], [13]. Since interpolators and decimators are linear time-variant systems, the stationarity or nonstationarity properties of the input processes turn out to be modified at the output. For example, the expanded version of a wide-sense stationary (WSS) process is cyclostationary with period of cyclostationarity equal to the expansion factor [11].

The effects of multirate systems on second-order WSS and cyclostationary processes are analyzed in [11]. In [1], conditions that preserve at the output of an interpolation filter the wide-sense stationarity of the input sequence are derived. Moreover, conditions for the joint wide-sense stationarity of the outputs of interpolation filters are also obtained. Almost-cyclostationary (ACS) processes and higher-order statistics are considered in [5].

In [1], [5], and [11], multirate operations on a single process are addressed and no cross-statistics between a process and its expanded or decimated version are considered. Moreover, no cross-statistics are considered between processes obtained by multirate elaborations of the same process with different rates.

In the present paper, results of [1], [5], and [11], are extended to treat cross-statistics of stochastic processes elaborated with different expansion or decimation factors. Elaborations of one process with different rates or scales is encountered, for example, in tree-structured filter banks. It is shown that in several cases of interest the (joint) nonstationary behavior of such processes can be modeled by using the (jointly) spectrally correlated (SC) processes. SC processes are finite-power processes whose distinct spectral component are correlated and are characterized by Loève bifrequency spectrum with spectral masses concentrated on a countable set of support curves in the bifrequency plane [8]. The case of support lines with non necessarily unit slope is treated in [6]. ACS processes are obtained as special case of SC processes when the support curves are lines with unit slope [3]. In such a case, spectral components of the process are correlated when the frequency separation belongs to a countable set which is the set of the cycle frequencies, that is, the frequencies of the Fourier series expansion of the almost-periodically time-variant autocorrelation function. WSS processes are obtained as further specialization when the unique support line of the Loève bifrequency spectrum is the main diagonal of the principal frequency domain. In such a case, distinct spectral components are uncorrelated.

It shown that expansions with different rates transforms jointly WSS, jointly ACS, and jointly SC processes into jointly SC processes. An analogous result is obtained by decimation with different rates. The effects of interpolation filters on jointly SC processes are also considered. Sufficient conditions to assure that the Loève bifrequency cross-spectrum of the interpolated processes is a frequency-scaled alias-free version of that of the original processes are derived. The case of jointly ACS input processes is treated in detail. Furthermore, some known results for a single ACS input process are obtained as special cases.

2. SPECTRALLY CORRELATED PROCESSES

In this section, the second-order characterization of discrete-time SC processes is briefly reviewed. For the sake of generality, a joint characterization of two processes \( x_1(n) \) and \( x_2(n) \) in terms of cross-statistics is provided.

The discrete-time processes \( x_1(n) \) and \( x_2(n) \) are said to be second-order jointly harmonizable if their cross-correlation function can be expressed by a Fourier-Stieltjes integral

\[
E \{ x_1(n_1) \ast x_2^*(n_2) \} = \int_{[-1/2,1/2]^2} e^{j2\pi [\nu_1n_1 + (-)\nu_2n_2]} d\mathcal{Y}(\nu_1, \nu_2) \tag{1}
\]

where \( \mathcal{Y}(\nu_1, \nu_2) \) is a (spectral) cross-correlation function of bounded variation [7]

\[
\int_{[-1/2,1/2]^2} |d\mathcal{Y}(\nu_1, \nu_2)| < \infty \tag{2}
\]

In (1), superscript \( * \) denotes optional complex conjugation,
Let \( x_1(n) \) and \( x_2(n) \) be discrete-time complex-valued second-order jointly harmonizable stochastic processes. Their Loève bifrequency cross-spectrum is defined as [7]

\[
S_x(v_1, v_2) \triangleq E \left\{ X_1(v_1) X_2^{*}(v_2) \right\}
\]

where

\[
X_i(v) \triangleq \sum_{n \in \mathbb{Z}} x_i(n) e^{-j2\pi vn}
\]

is the Fourier transform of \( x_i(n) \), \( i = 1, 2 \), and is assumed to exist (at least) in the sense of distributions [4]. It results that \( d\gamma_x(v_1, v_2) = S_x(v_1, v_2) dv_1 dv_2 \).

Let us assume that \( x_1(n) \) and \( x_2(n) \) do not contain any additive finite-strength sinewave component. The processes are said to be \textit{jointly spectrally correlated} if their Loève bifrequency cross-spectrum can be expressed as [8]

\[
S_x(v_1, v_2) = \sum_{k \in \mathbb{Z}} S_x^{(k)}(v_1) \tilde{\delta} \left( v_2 - \Psi_x^{(k)}(v_1) \right) \quad (5a)
\]

\[
= \sum_{k \in \mathbb{Z}} G_x^{(k)}(v_2) \tilde{\delta} \left( v_1 - \Phi_x^{(k)}(v_2) \right). \quad (5b)
\]

In (5a) and (5b), \( I \) is a countable set and \( \tilde{\delta}(v) \triangleq \sum_{p \in \mathbb{Z}} \delta(v - p) \) is the periodic Dirac delta train with period 1. In addition, the complex valued functions \( S_x^{(k)}(v) \) and \( G_x^{(k)}(v) \), referred to as spectral cross-correlation densities, and the real-valued functions \( \Psi_x^{(k)}(v) \) and \( \Phi_x^{(k)}(v) \), referred to as support functions, are periodic functions of \( v \) with period 1. Furthermore, each \( \Psi_x^{(k)}(\cdot) \) is assumed to be differentiable and locally invertible in every interval of width 1, \( \Phi_x^{(k)}(\cdot) \) assumed differentiable is the periodic replication with period 1 of one of the local inverses and, accounting for the variable change property in the argument of the Dirac delta [14, Sec. 1.7], it results

\[
S_x^{(k)}(v_1) = \left| \Psi_x^{(k)}(v_1) \right| G_x^{(k)} \left( \Psi_x^{(k)}(v_1) \right) \quad (6a)
\]

\[
G_x^{(k)}(v_2) = \left| \Phi_x^{(k)}(v_2) \right| S_x^{(k)} \left( \Phi_x^{(k)}(v_2) \right). \quad (6b)
\]

with superscript \(^t\) denoting first-order derivative.

From (5a) it follows that discrete-time jointly SC processes have Loève bifrequency cross-spectrum with spectral masses concentrated on the countable set of support curves

\[
v_2 = \Psi_x^{(k)}(v_1) \quad (5a)
\]

\[
v_2 = \Phi_x^{(k)}(v_1) \quad (5b)
\]

where \( k \in \mathbb{I} \), where \( \text{mod} \) is the modular 1 operation with values in \([-1/2, 1/2]\). Moreover, the spectral mass distribution is periodic with period 1 in both frequency variables \( v_1 \) and \( v_2 \). Without lack of generality, it can be assumed that two support curves \( v_2 = \Psi_x^{(k)}(v_1) \mod 1 \) and \( v_2 = \Psi_x^{(k')} (v_1) \mod 1 \), with \( k \neq k' \), intersect at most in a finite or countable set of points \( (v_1, v_2) \).

Note that in the general case of nonlinear support functions \( \Psi_x^{(k)}(\cdot) \), for every \( k \), two spectral cross-correlation density functions (6a) and (6b) should be considered, depending on which one of \( v_1 \) and \( v_2 \) is considered as independent variable in the argument of the Dirac delta in (5a) and (5b).

SC processes are an appropriate model for signals occurring in mobile wide-band communications [8] and in the analysis of fractional Brownian motion [10].

Almost all modulated signals encountered in communications, radar, sonar, and telemetry can be modeled as almost-cyclostationary. That is, their statistical functions such as distribution functions, moments, and cumulants are almost-periodic functions of time [3]. Second-order jointly ACS signals in the wide-sense are characterized by an almost-periodic cross-correlation function. That is [3]

\[
E \left\{ x_1(n + m) x_2^{*}(n) \right\} = \sum_{m \in \mathcal{A}} R_x^a(m) e^{j2\pi am} \quad (7)
\]

where the Fourier coefficients

\[
R_x^a(m) \triangleq \lim_{N \to +\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} E \left\{ x_1(n + m) x_2^{*}(n) \right\} e^{-j2\pi am} \quad (8)
\]

are referred to as cyclic cross-correlation functions and

\[
\mathcal{A} \triangleq \{ \alpha \in [-1/2, 1/2] : R_x^a(m) \neq 0 \} \quad (9)
\]

is the countable set of cycle frequencies \( \alpha \) in the principal domain \([-1/2, 1/2]\). By double Fourier transforming both sides of (7), the following expression for the Loève bifrequency cross-spectrum is obtained

\[
S_x(v_1, v_2) = \sum_{\alpha \in \mathcal{A}} S_x^a(v_1) \tilde{\delta}(v_2 - (\alpha - \gamma_x(v_1))) \quad (10)
\]

where

\[
S_x^a(v) = \sum_{m \in \mathbb{Z}} R_x^a(m) e^{-j2\pi vm} \quad (11)
\]

are the cyclic spectra. From (10) it follows that discrete-time jointly ACS processes are obtained as special case of jointly SC processes when the spectral support curves are lines with slope \( \pm 1 \) in the principal frequency domain \( (v_1, v_2) \in [-1/2, 1/2]^2 \). When \( (\cdot) \) is present, if the set \( \mathcal{A} \) contains the only element \( \alpha = 0 \), then the cross-correlation function does not depend on \( n \) and the processes \( x_1 \) and \( x_2 \) are jointly WSS. In such a case the Loève bifrequency cross-spectrum has support contained in the main diagonal of the principal frequency domain.

More generally, the processes \( x_1(n) \) and \( x_2^{*}(n) \) are said to exhibit joint almost-cyclostationarity at cycle frequency \( \alpha_0 \) if the cross-correlation function is not necessarily an almost-periodic function of \( n \) but contains a finite-strength additive sinewave component at frequency \( \alpha_0 \). In such a case, the cyclic cross-correlation function (8) is nonzero for \( \alpha = \alpha_0 \). Second-order ACS processes are those processes with almost-periodic autocorrelation function that are also SC [8], [9].

3. EXPANSION AND DECIMATION

In this section, the effects of expansion and decimation operations on jointly SC processes are analyzed. The special case of jointly ACS processes is treated in detail.
3.1 Expansion

Let \( x_i(n) \) be the \( L_i \)-fold expanded version of \( x(n) \), \( i = 1, 2 \). That is

\[
x_i(n) = \begin{cases}
x_i \left( \frac{n}{L_i} \right) & n = kL_i, k \in \mathbb{Z} \\
0 & \text{otherwise}
\end{cases}
\]  

(12)

whose Fourier transform is

\[
X_i(v) = X_i(vL_i).
\]  

(13)

If \( x_1(n) \) and \( x_2(n) \) are jointly SC with Loève bifrequency cross-spectrum (5a), the Loève bifrequency cross-spectrum of the expanded processes \( x_1(n) \) and \( x_2(n) \) is given by

\[
\begin{align*}
E \left\{ X_{1}(v_1) X_{2}^{(*)}(v_2) \right\} &= E \left\{ X_{1}(v_1L_1) X_{2}^{(*)}(v_2L_2) \right\} \\
&= \sum_{k \in \mathbb{Z}} S_{X}^{(k)}(v_1L_1) \delta(v_2L_2 - \Psi_{X}^{(k)}(v_1L_1)) \\
&= \sum_{k \in \mathbb{Z}} S_{X}^{(k)}(v_1L_1) \frac{1}{L_2} \sum_{h \in \mathbb{Z}} \delta \left( v_2 - \frac{h}{L_2} - \frac{1}{L_2} \Psi_{X}^{(k)}(v_1L_1) \right)
\end{align*}
\]  

(14)

where, in the third equality, the scaling property of the Dirac delta is used [14, Sec. 1.7]. From (14) it follows that the expanded processes are jointly SC. In addition, their Loève bifrequency cross-spectrum is periodic in \( v_1 \) with period \( 1/L_1 \) and in \( v_2 \) with period \( 1/L_2 \).

In the special case of \( x_1(n) \) and \( x_2(n) \) jointly ACS, accounting for (10), the Loève bifrequency cross-spectrum (14) of the expanded processes \( x_1(n) \) and \( x_2(n) \) specializes into

\[
\begin{align*}
E \left\{ X_{1}(v_1) X_{2}^{(*)}(v_2) \right\} &= \sum_{\alpha \in \mathbb{A}} S_{X}^{\alpha}(v_1L_1) \delta(v_2L_2 - (-)(\alpha - v_1L_1)) \\
&= \sum_{\alpha \in \mathbb{A}} S_{X}^{\alpha}(v_1L_1) \frac{1}{L_2} \sum_{h \in \mathbb{Z}} \delta \left( v_2 - \frac{h}{L_2} - (-) \left( \frac{\alpha}{L_2} - \frac{L_1}{L_2} v_1 \right) \right)
\end{align*}
\]  

(15)

Thus, \( E \{ X_{1}(v_1) X_{2}^{(*)}(v_2) \} \) is periodic in \( v_1 \) with period \( 1/L_1 \) and in \( v_2 \) with period \( 1/L_2 \). In addition, from (15) it follows that the Loève bifrequency cross-spectrum of the expanded processes \( x_1(n) \) and \( x_2(n) \) in the principal domain \( (v_1, v_2) \in [-1/2, 1/2]^2 \) has support curves \((-\alpha)\) present

\[
v_2 = (L_1/L_2)v_1 - \alpha/L_2 \quad \alpha \in \mathbb{A}
\]  

(16)

that is, lines with slope \( L_1/L_2 \). Consequently, \( x_1(n) \) and \( x_2(n) \) are jointly SC. In particular, considering a single process \( x_1(n) \equiv x_2(n) \equiv x(n) \), by taking \( L_1 = 1 \), that is \( x_1(n) \equiv x(n) \), it follows that the ACS process \( x(n) \) and its expanded version \( x_2(n) \) are jointly SC with support lines with slope \( 1/L_2 \). In the special case of \( x(n) \) WSS, \( x(n) \) and \( x_2(n) \) are jointly SC with support constituted by a unique line. These results constitute a strong motivation to treat the problem of multirate processing within the framework of the SC processes. In fact, even in the case of WSS input process, the joint characterization of input and output processes involves jointly SC processes.

By double inverse Fourier transforming both sides of (15), it can be shown that when \( x_1(n) \) and \( x_2(n) \) are jointly ACS, for \( L_1 \neq L_2 \) the cross-correlation function of \( x_1(n) \) and \( x_2(n) \) does not contain any finite-strength additive sinewave component. That is, \( x_1(n) \) and \( x_2^{(*)}(n) \) do not exhibit joint almost-cyclostationarity at any cycle frequency. In particular, their time-averaged cross-correlation is identically zero. Moreover, \( x_1(n) \) and \( x_2(n) \) are jointly ACS if and only if \( L_1 = L_2 = L \). In such a case, the cyclic cross-spectra of \( x_1(n) \) and \( x_2(n) \) are given by

\[
S_{x_{1}x_{2}^{(*)}}^{\alpha}(v) = \frac{1}{L} S_{xL}^{\alpha}(vL) \quad \alpha L \in \mathbb{A}.
\]  

(17)

In the special case of \( x_1 \equiv x_2 \), (17) reduces to [5, eq. (34)] specialized to second order.

3.2 Decimation

Let \( x_{Di}(n) \) be the decimated version of \( x_i(n) \), with decimation factor \( M_i \) \( (i = 1, 2) \). That is,

\[
x_{Di}(n) = x_i(nM_i).
\]  

(18)

Its Fourier transform is

\[
X_{Di}(v) = \frac{1}{M_i} \sum_{p=0}^{M_i-1} X_i \left( \frac{v-P}{M_i} \right).
\]  

(19)

Therefore, if \( x_1(n) \) and \( x_2(n) \) are jointly SC with Loève bifrequency cross-spectrum (5a), by using (19), the Loève bifrequency cross-spectrum of \( x_{D1}(n) \) and \( x_{D2}(n) \) can be expressed as

\[
\begin{align*}
E \left\{ X_{D1}(v_1) X_{D2}^{(*)}(v_2) \right\} &= \frac{1}{M_1M_2} \sum_{p_1=0}^{M_1-1} \sum_{p_2=0}^{M_2-1} S_{X}^{(k)} \left( \frac{v_1-p_1}{M_1} \right) \\
&\quad \delta\left( \frac{v_2-p_2}{M_2} - \Psi_{X}^{(k)} \left( \frac{v_1-p_1}{M_1} \right) \right) \\
&= \frac{1}{M_1} \sum_{p_1=0}^{M_1-1} \sum_{k \in \mathbb{A}} S_{X}^{(k)} \left( \frac{v_1-p_1}{M_1} \right) \\
&\quad \sum_{h \in \mathbb{Z}} \delta \left( v_2 - \frac{h}{M_2} - \Psi_{X}^{(k)} \left( \frac{v_1-p_1}{M_1} \right) \right)
\end{align*}
\]  

(20)

where, in the second equality, the scaling property of the Dirac delta [14, Sec. 1.7] is used. From (20) it follows that the decimated versions of jointly SC processes are jointly SC. In addition, their Loève bifrequency cross-spectrum is periodic with period \( 1 \) in both variables \( v_1 \) and \( v_2 \).

In the special case where \( x_1(n) \) and \( x_2(n) \) are jointly ACS, then

\[
\begin{align*}
E \left\{ X_{D1}(v_1) X_{D2}^{(*)}(v_2) \right\} &= \frac{1}{M_1} \sum_{p_1=0}^{M_1-1} \sum_{k \in \mathbb{A}} S_{X}^{(k)} \left( \frac{v_1-p_1}{M_1} \right) \\
&\quad \sum_{h \in \mathbb{Z}} \delta \left( v_2 - \frac{h}{M_2} - \left( \alpha M_2 - (v_1-p_1) \frac{M_2}{M_1} \right) \right)
\end{align*}
\]  

(21)
The delta train is periodic in both \( t_1 \) and \( t_2 \) with period 1. In addition, from (21) it follows that the Loe\`e birefringence cross-spectrum of the decimated processes \( x_1(n) \) and \( x_2(n) \) in the principal domain \( (v_1, v_2) \in [-1/2, 1/2]^2 \) has support curves (\( -k \))

\[
v_2 = (M_2/M_1)v_1 - \alpha M_2 - (M_2/M_1)p_1
\]

\[\alpha \in \mathbb{C}, \ p_1 \in \{0, 1, \ldots, M_1 - 1\}\]

(22)

that is, lines with slope \( M_2/M_1 \). Consequently, the decimated processes \( x_1(n) \) and \( x_2(n) \) are jointly SC. In particular, considering a single process \( x_1(n) \equiv x_2(n) = x(n) \), by taking \( M_1 = 1 \), that is \( x_1(n) \equiv x(n) \), it follows that the ACS process \( x(n) \) and its decimated version \( x_1(n) \) are jointly SC with support lines with slope \( M_2 \). In the special case of \( x(n) \) WSS, the support line is unique.

From (21) it follows that \( x_1(n) \) and \( x_2(n) \) are jointly ACS if and only if \( M_1 = M = 1 \). In such a case, their cyclic cross-spectrum can be expressed as

\[
S^\alpha_{x_1x_2}(v) = \frac{1}{M} \sum_{p=0}^{M-1} \sum_{q=0}^{M-1} S^{(\alpha-q)/M}(\frac{v-p}{M}).
\]

(23)

In the special case of \( x_1 \equiv x_2 \), (23) reduces to [5, eq. (24)] specialized to second order.

### 4. Interpolation Filters

In this section, the effects of interpolation filters on jointly SC processes are analyzed and sufficient conditions are established to assure that the Loe\`e birefringence cross-spectrum of the interpolated processes \( y_1(n) \) and \( y_2(n) \) is a frequency-scaled image-free version of that of \( x_1(n) \) and \( x_2(n) \).

Fig. 2 presents two interpolation filters. Each interpolation filter \((i = 1, 2)\) is constituted by a \( L_i \)-fold interpolator followed by a LTI filter with bandwidth \( W_i \) whose purpose is to obtain image-free interpolation [13]. An additional LTI filter with bandwidth \( B_i \) precedes the \( L_i \)-fold interpolator to strictly bandlimit the input process \( x_{in_1}(n) \) in order to avoid overlapping among images in the Loe\`e birefringence cross-spectrum.

If \( x_{in_1}(n) \) and \( x_{in_2}(n) \) are jointly SC processes, then \( x_1(n) \) and \( x_2(n) \) are jointly SC processes strictly bandlimited with bandwidths \( B_1 \) and \( B_2 \), respectively. Using (14), the Loe\`e birefringence cross-spectrum of \( y_1(n) \) and \( y_2(n) \) is given by

\[
E\left\{ Y_1(v_1) Y_2^{(s)}(v_2) \right\} = H_{W_1}(v_1) H_{W_2}(v_2) E\left\{ X_1(n) X_2^{(s)}(v) \right\} = H_{W_1}(v_1) H_{W_2}(v_2) \sum_{k \in \mathbb{Z}} S^{(k)}(v_1 L_1) \tilde{\Psi}^{(k)}(v_2 L_2 - \Psi^{(k)}(v_1 L_1))
\]

(24)

where \( H_{W_i}(v) \) is the Fourier transform of the ideal low-pass filter \( h_{W_i}(n) \).

Accounting for the band-limitedness of \( x_1(n) \) and \( x_2(n) \), the support of each term in the sum over \( k \in \mathbb{Z} \) in (24) is such that

\[
\text{supp}\left\{ S^{(k)}(v_1 L_1) \tilde{\Psi}^{(k)}(v_2 L_2 - \Psi^{(k)}(v_1 L_1)) \right\} \subseteq \left\{ (v_1, v_2) \in \mathbb{R} \times \mathbb{R} : |(v_1 L_1) \mod 1 | \leq B_1, \right\}
\]

\[
\left| \Psi^{(k)}(v_1 L_1) \right| \leq B_2, v_2 L_2 = \frac{1}{L_2} \Psi^{(k)}(v_1 L_1) \mod 1 \right\}
\]

\[
= \bigcup_{\ell_1, \ell_2} \Gamma^{(k)}_{\ell_1, \ell_2}
\]

(25)

where

\[
\Gamma^{(k)}_{\ell_1, \ell_2} = \left\{ (v_1, v_2) \in \mathbb{R} \times \mathbb{R} : |v_1 - \ell_1 / L_1| \leq B_1/L_1, \right\}
\]

\[
|v_2 - \ell_2 / L_2| \leq B_2/L_2, v_2 - \ell_2 / L_2 = \frac{1}{L_2} \Psi^{(k)}(v_1 L_1) \}
\]

(26)

In addition, we also have

\[
\text{supp}\left\{ H_{W_1}(v_1) H_{W_2}(v_2) \right\} = \bigcup_{m_1 \in \mathbb{Z}} \bigcup_{m_2 \in \mathbb{Z}} \Delta_{m_1 m_2}
\]

(27)

where

\[
\Delta_{m_1 m_2} = \left\{ (v_1, v_2) \in \mathbb{R} \times \mathbb{R} : |v_1 - m_1| \leq W_1, \right\}
\]

\[
|v_2 - m_2| \leq W_2 \}
\]

(28)

Since \( B_i < 1/2, (i = 1, 2) \), images do not overlap. Furthermore, since \( W_i < 1/2, (i = 1, 2) \), then the filter supports do not overlap. If \( B_i/L_i < W_i, (i = 1, 2) \), then the low-pass filters \( H_{W_1}(v_1) \) and \( H_{W_2}(v_2) \) capture the main images, i.e., those centered in \( (v_1, v_2) \equiv (m_1, m_2) \in \mathbb{Z}^2 \). That is, \( \forall k \in \mathbb{Z} \) it results

\[
\Gamma^{(k)}_{m_1, m_2} \subseteq \Delta_{m_1 m_2}
\]

(29)

Moreover, if \( W_i < 1/(2L_i), (i = 1, 2) \), then the low-pass filters \( H_{W_1}(v_1) \) and \( H_{W_2}(v_2) \) do not capture images different from the main ones. That is, \( \forall k \in \mathbb{Z} \) we have

\[
\Gamma^{(k)}_{\ell_1, \ell_2} \cap \Delta_{m_1 m_2} = \emptyset \text{ for } (\ell_1, \ell_2) \neq (m_1 L_1, m_2 L_2).
\]

(30)

The sufficient conditions such that (29)–(30) hold can be summarized into

\[
\frac{B_i}{L_i} \leq W_i < \frac{1}{2L_i} \quad i = 1, 2
\]

(31)

which is therefore a sufficient condition to assure that the Loe\`e birefringence cross-spectrum \( E\{Y_1(v_1) Y_2^{(s)}(v_2)\} \) is a frequency-scaled image-free version of \( E\{X_1(v_1) X_2^{(s)}(v_2)\} \).

Condition (31) is independent of the shape of the support curves \( v_2 = \Psi^{(k)}(v_1) \). Moreover, in the special case of a single ACS process and \( L_1 = L_2 = L \), is less restrictive than [5, eq. (56)] specialized to second-order. In fact, [5, eq. (56)] assures the lack of images in the densities of spectral correlation (the cyclic spectra) for \( v_1 \in [-1/(2L), 1/(2L)] \) and cycle frequencies \( \alpha \in [-1/(2L), 1/(2L)] \).

It can be shown that

\[
B_1 \leq 1/2 \quad \text{and } 1 \geq B_2 + \max_k \left\{ B_2, \sup \left| \Psi^{(k)}(\pm 1/2) \right| \right\}
\]

(32)
is a sufficient condition such that the density of Loève bifrequency cross-spectrum (24) along every support curve is a frequency-scaled image-free version of the corresponding density of the Loève bifrequency cross-spectrum of $x_1$ and $x_2$ for every $\nu_1 \in [-1/(2L_1), 1/(2L_1)]$. In (32), “+” or “−” sign should be taken if $\Psi^{(k)}(\cdot)$ is increasing or decreasing, respectively.

5. CONCLUSION
The joint statistical characterization of processes elaborated with different scales or rates is provided within the framework of the (jointly) SC processes. By using the Loève bifrequency cross-spectrum as a tool, it is shown that jointly WSS, ACS, and SC processes expanded or decimated by different factors give rise to jointly SC processes. For interpolation filters, sufficient conditions on the expansion factors and the LTI system bandwidths are derived such that the Loève bifrequency cross-spectrum of the interpolated processes is a frequency-scaled alias-free version of that of the original processes. The derived results are useful in the joint statistical characterization of processes in systems with different rates as those occurring in tree-structured filter banks.

REFERENCES