INTERPOLATION AND DECIMATION OF SPECTRALLY CORRELATED STOCHASTIC PROCESSES

Antonio Napolitano

Università di Napoli “Parthenope”, Dipartimento per le Tecnologie
Centro Direzionale di Napoli, Isola C4, I-80143, Napoli, Italy
phone: +39 081 547 6715; fax: +39 081 547 6777
email: antonio.napolitano@uniparthenope.it

ABSTRACT

The problem of interpolation and decimation of jointly spectrally correlated (SC) discrete-time stochastic processes is addressed. Jointly SC processes have Loève bifrequency cross-spectrum with spectral masses concentrated on a countable set of support curves of the bifrequency plane. Jointly almost-cyclostationary (ACS) processes are obtained as special case when the support curves are lines with unit slope. It is shown that two jointly wide-sense stationary or jointly ACS processes expanded or decimated with different rates give rise to jointly SC processes. In addition, interpolation filtering of jointly SC processes is considered and sufficient conditions for alias-free filtering are provided.

1. INTRODUCTION

In multirate digital signal processing, systems constituted by complicate interconnections of interpolators, decimators, and linear time-invariant (LTI) filters acts on input stochastic processes [2], [13]. Since interpolators and decimators are linear time-variant systems, the stationarity or nonstationarity properties of the input processes turn out to be modified at the output. For example, the expanded version of a wide-sense stationary (WSS) process is cyclostationary with period of cyclostationarity equal to the expansion factor [11].

The effects of multirate systems on second-order WSS and cyclostationary processes are analyzed in [11]. In [1], conditions that preserve at the output of an interpolation filter the wide-sense stationarity of the input sequence are derived. Moreover, conditions for the joint wide-sense stationarity of the outputs of interpolation filters are also obtained. Almost-cyclostationary (ACS) processes and higher-order statistics are considered in [5].

In [1], [5], and [11], multirate operations on a single process are addressed and no cross-statistics between a process and its expanded or decimated version are considered. Moreover, no cross-statistics are considered between processes obtained by multirate elaborations of the same process with different rates.

In the present paper, results of [1], [5], and [11], are extended to treat cross-statistics of stochastic processes elaborated with different expansion or decimation factors. Elaborations of one process with different rates or scales is encountered, for example, in tree-structured filter banks. It is shown that in several cases of interest the (joint) nonstationary behavior of such processes can be modeled by using the (jointly) spectrally correlated (SC) processes. SC processes are finite-power processes whose distinct spectral component are correlated and are characterized by Loève bifrequency spectrum with spectral masses concentrated on a countable set of support curves in the bifrequency plane [8]. The case of support lines with non necessarily unit slope is treated in [6]. ACS processes are obtained as special case of SC processes when the support curves are lines with unit slope [3]. In such a case, spectral components of the process are correlated when the frequency separation belongs to a countable set which is the set of the cycle frequencies, that is, the frequencies of the Fourier series expansion of the almost-periodically time-variant autocorrelation function. WSS processes are obtained as further specialization when the unique support line of the Loève bifrequency spectrum is the main diagonal of the principal frequency domain. In such a case, distinct spectral components are uncorrelated.

It shown that expansions with different rates transforms jointly WSS, jointly ACS, and jointly SC processes into jointly SC processes. An analogous result is obtained by decimation with different rates. The effects of interpolation filters on jointly SC processes are also considered. Sufficient conditions to assure that the Loève bifrequency cross-spectrum of the interpolated processes is a frequency-scaled alias-free version of that of the original processes are derived. The case of jointly ACS input processes is treated in detail. Furthermore, some known results for a single ACS input process are obtained as special cases.

2. SPECTRALLY CORRELATED PROCESSES

In this section, the second-order characterization of discrete-time SC processes is briefly reviewed. For the sake of generality, a joint characterization of two processes $x_1(n)$ and $x_2(n)$ in terms of cross-statistics is provided.

The discrete-time processes $x_1(n)$ and $x_2(n)$ are said to be *second-order jointly harmonizable* if their cross-correlation function can be expressed by a Fourier-Stieltjes integral

$$\begin{aligned}&\mathrm{E}\left\{x_1(n_1)\,x_2^{(*)}(n_2)\right\}\\&\quad= \int_{[-1/2,1/2]^2} e^{j2\pi[\nu_1 n_1+(-)\nu_2 n_2]}\,\mathrm{d}\gamma_{\boldsymbol{x}}(\nu_1,\nu_2)\end{aligned} \qquad (1)$$

where $\gamma_{\boldsymbol{x}}(\nu_1,\nu_2)$ is a (spectral) cross-correlation function of bounded variation [7]

$$\int_{[-1/2,1/2]^2} |\,\mathrm{d}\gamma_{\boldsymbol{x}}(\nu_1,\nu_2)| < \infty\,. \qquad (2)$$

In (1), superscript $(*)$ denotes optional complex conjugation,

This work is partially supported by the NATO Grant ICS.NUKR.CLG 983335

 ISSN 2076-1465

$_{(-)}$ is an optional minus sign which is linked to $(*)$, and subscript $\boldsymbol{x} = [x_1, x_2^{(*)}]$.

For complex-valued processes, both cross-correlation functions $\mathrm{E}\{x_1(n_1)\,x_2^*(n_2)\}$ and $\mathrm{E}\{x_1(n_1)\,x_2(n_2)\}$ must be considered for a complete second-order characterization [12]. Notation in (1) allows to treat both second-order cross-moments by considering or not the optional complex conjugation.

Let $x_1(n)$ and $x_2(n)$ be discrete-time complex-valued second-order jointly harmonizable stochastic processes. Their *Loève bifrequency cross-spectrum* is defined as [7]

$$\mathcal{S}_{\boldsymbol{x}}(\nu_1,\nu_2) \triangleq \mathrm{E}\left\{X_1(\nu_1)\,X_2^{(*)}(\nu_2)\right\} \tag{3}$$

where

$$X_i(\nu) \triangleq \sum_{n\in\mathbb{Z}} x_i(n)\,e^{-j2\pi\nu n} \tag{4}$$

is the Fourier transform of $x_i(n)$, $(i=1,2)$, and is assumed to exist (at least) in the sense of distributions [4]. It results that $\mathrm{d}\gamma_{\boldsymbol{x}}(\nu_1,\nu_2) = \mathcal{S}_{\boldsymbol{x}}(\nu_1,\nu_2)\,\mathrm{d}\nu_1\,\mathrm{d}\nu_2$.

Let us assume that $x_1(n)$ and $x_2(n)$ do not contain any additive finite-strength sinewave component. The processes are said to be *jointly spectrally correlated* if their Loève bifrequency cross-spectrum can be expressed as [8]

$$\begin{aligned} \mathcal{S}_{\boldsymbol{x}}(\nu_1,\nu_2) &= \sum_{k\in\mathbb{I}} S_{\boldsymbol{x}}^{(k)}(\nu_1)\,\widetilde{\delta}\left(\nu_2-\Psi_{\boldsymbol{x}}^{(k)}(\nu_1)\right) && \text{(5a)}\\ &= \sum_{k\in\mathbb{I}} G_{\boldsymbol{x}}^{(k)}(\nu_2)\,\widetilde{\delta}\left(\nu_1-\Phi_{\boldsymbol{x}}^{(k)}(\nu_2)\right). && \text{(5b)} \end{aligned}$$

In (5a) and (5b), $\mathbb{I}$ is a countable set and $\widetilde{\delta}(\nu) \triangleq \sum_{p\in\mathbb{Z}} \delta(\nu - p)$ is the periodic Dirac delta train with period 1. In addition, the complex valued functions $S_{\boldsymbol{x}}^{(k)}(\nu)$ and $G_{\boldsymbol{x}}^{(k)}(\nu)$, referred to as spectral cross-correlation densities, and the real-valued functions $\Psi_{\boldsymbol{x}}^{(k)}(\nu)$ and $\Phi_{\boldsymbol{x}}^{(k)}(\nu)$, referred to as spectral support functions, are periodic functions of ν with period 1. Furthermore, each $\Psi_{\boldsymbol{x}}^{(k)}(\cdot)$ is assumed to be differentiable and locally invertible in every interval of width 1, $\Phi_{\boldsymbol{x}}^{(k)}(\cdot)$ assumed differentiable is the periodic replication with period 1 of one of the local inverses and, accounting for the variable change property in the argument of the Dirac delta [14, Sec. 1.7], it results

$$\begin{aligned} S_{\boldsymbol{x}}^{(k)}(\nu_1) &= \left|\Psi_{\boldsymbol{x}}^{(k)\prime}(\nu_1)\right| G_{\boldsymbol{x}}^{(k)}\left(\Psi_{\boldsymbol{x}}^{(k)}(\nu_1)\right) && \text{(6a)}\\ G_{\boldsymbol{x}}^{(k)}(\nu_2) &= \left|\Phi_{\boldsymbol{x}}^{(k)\prime}(\nu_2)\right| S_{\boldsymbol{x}}^{(k)}\left(\Phi_{\boldsymbol{x}}^{(k)}(\nu_2)\right) && \text{(6b)} \end{aligned}$$

with superscript $'$ denoting first-order derivative.

From (5a) it follows that discrete-time jointly SC processes have Loève bifrequency cross-spectrum with spectral masses concentrated on the countable set of support curves $\nu_2 = \Psi_{\boldsymbol{x}}^{(k)}(\nu_1) \bmod 1$, $k\in\mathbb{I}$, where $\bmod 1$ is the modulo 1 operation with values in $[-1/2,1/2)$. Moreover, the spectral mass distribution is periodic with period 1 in both frequency variables ν_1 and ν_2. Without lack of generality, it can be assumed that two support curves $\nu_2 = \Psi_{\boldsymbol{x}}^{(k)}(\nu_1) \bmod 1$ and $\nu_2 = \Psi_{\boldsymbol{x}}^{(k')}(\nu_1) \bmod 1$, with $k \neq k'$, intersect at most in a finite or countable set of points (ν_1,ν_2).

Note that in the general case of nonlinear support functions $\Psi_{\boldsymbol{x}}^{(k)}(\cdot)$, for every k, two spectral cross-correlation density functions (6a) and (6b) should be considered, depending on which one of ν_1 and ν_2 is considered as independent variable in the argument of the Dirac deltas in (5a) and (5b).

SC processes are an appropriate model for signals occurring in mobile wide-band communications [8] and in the analysis of fractional Brownian motion [10].

Almost all modulated signals encountered in communications, radar, sonar, and telemetry can be modeled as almost-cyclostationary. That is, their statistical functions such as distribution functions, moments, and cumulants are almost-periodic functions of time [3]. Second-order jointly ACS signals in the wide-sense are characterized by an almost-periodic cross-correlation function. That is [3]

$$\mathrm{E}\left\{x_1(n+m)\,x_2^{(*)}(n)\right\} = \sum_{\alpha\in\mathscr{A}} R_{\boldsymbol{x}}^{\alpha}(m)\,e^{j2\pi\alpha n} \tag{7}$$

where the Fourier coefficients

$$R_{\boldsymbol{x}}^{\alpha}(m) \triangleq \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}\mathrm{E}\left\{x_1(n+m)\,x_2^{(*)}(n)\right\}e^{-j2\pi\alpha n} \tag{8}$$

are referred to as cyclic cross-correlation functions and

$$\mathscr{A} \triangleq \{\alpha\in[-1/2,1/2) : R_{\boldsymbol{x}}^{\alpha}(m) \not\equiv 0\} \tag{9}$$

is the countable set of cycle frequencies α in the principal domain $[-1/2,1/2)$. By double Fourier transforming both sides of (7), the following expression for the Loève bifrequency cross-spectrum is obtained

$$\mathcal{S}_{\boldsymbol{x}}(\nu_1,\nu_2) = \sum_{\alpha\in\mathscr{A}} S_{\boldsymbol{x}}^{\alpha}(\nu_1)\,\widetilde{\delta}(\nu_2 - {}_{(-)}(\alpha-\nu_1)) \tag{10}$$

where

$$S_{\boldsymbol{x}}^{\alpha}(\nu) = \sum_{m\in\mathbb{Z}} R_{\boldsymbol{x}}^{\alpha}(m)\,e^{-j2\pi\nu m} \tag{11}$$

are the cyclic spectra. From (10) it follows that discrete-time jointly ACS processes are obtained as special case of jointly SC processes when the spectral support curves are lines with slope ±1 in the principal frequency domain $(\nu_1,\nu_2)\in[-1/2,1/2]^2$. When $(*)$ is present, if the set $\mathscr{A}$ contains the only element $\alpha=0$, then the cross-correlation function does not depend on n and the processes x_1 and x_2 are jointly WSS. In such a case the Loève bifrequency cross-spectrum has support contained in the main diagonal of the principal frequency domain.

More generally, the processes $x_1(n)$ and $x_2^{(*)}(n)$ are said to exhibit joint almost-cyclostationarity at cycle frequency α_0 if the cross-correlation function is not necessarily an almost-periodic function of n but contains a finite-strength additive sinewave component at frequency α_0. In such a case, the cyclic cross-correlation function (8) is nonzero for $\alpha=\alpha_0$. Second-order ACS processes are those processes with almost-periodic autocorrelation function that are also SC [8], [9].

3. EXPANSION AND DECIMATION

In this section, the effects of expansion and decimation operations on jointly SC processes are analyzed. The special case of jointly ACS processes is treated in detail.

3.1 Expansion

Let $x_{Ii}(n)$ be the L_i-fold expanded version of $x_i(n)$, $(i=1,2)$. That is

$$x_{Ii}(n) \triangleq \begin{cases} x_i\left(\frac{n}{L_i}\right) & n = kL_i, k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

whose Fourier transform is

$$X_{Ii}(\nu) = X_i(\nu L_i) \,. \tag{13}$$

If $x_1(n)$ and $x_2(n)$ are jointly SC with Loève bifrequency cross-spectrum (5a), the Loève bifrequency cross-spectrum of the expanded processes $x_{I1}(n)$ and $x_{I2}(n)$ is given by

$$\begin{aligned} &\mathrm{E}\left\{X_{I1}(\nu_1)\, X_{I2}^{(*)}(\nu_2)\right\} \\ &\quad = \mathrm{E}\left\{X_1(\nu_1 L_1)\, X_2^{(*)}(\nu_2 L_2)\right\} \\ &\quad = \sum_{k\in\mathbb{I}} S_{\boldsymbol{x}}^{(k)}(\nu_1 L_1)\, \widetilde{\delta}\left(\nu_2 L_2 - \Psi_{\boldsymbol{x}}^{(k)}(\nu_1 L_1)\right) \\ &\quad = \sum_{k\in\mathbb{I}} S_{\boldsymbol{x}}^{(k)}(\nu_1 L_1)\, \frac{1}{L_2} \sum_{h\in\mathbb{Z}} \delta\left(\nu_2 - \frac{h}{L_2} - \frac{1}{L_2}\Psi_{\boldsymbol{x}}^{(k)}(\nu_1 L_1)\right) \end{aligned} \tag{14}$$

where, in the third equality, the scaling property of the Dirac delta is used [14, Sec. 1.7]. From (14) it follows that the expanded processes are jointly SC. In addition, their Loève bifrequency cross-spectrum is periodic in ν_1 with period $1/L_1$ and in ν_2 with period $1/L_2$.

In the special case of $x_1(n)$ and $x_2(n)$ jointly ACS, accounting for (10), the Loève bifrequency cross-spectrum (14) of the expanded processes $x_{I1}(n)$ and $x_{I2}(n)$ specializes into

$$\begin{aligned} &\mathrm{E}\left\{X_{I1}(\nu_1)\, X_{I2}^{(*)}(\nu_2)\right\} \\ &\quad = \sum_{\alpha\in\mathscr{A}} S_{\boldsymbol{x}}^{\alpha}(\nu_1 L_1)\, \widetilde{\delta}(\nu_2 L_2 - {\scriptstyle(-)}(\alpha - \nu_1 L_1)) \\ &\quad = \sum_{\alpha\in\mathscr{A}} S_{\boldsymbol{x}}^{\alpha}(\nu_1 L_1)\, \frac{1}{L_2} \sum_{h\in\mathbb{Z}} \delta\left(\nu_2 - \frac{h}{L_2} - {\scriptstyle(-)}\left(\frac{\alpha}{L_2} - \frac{L_1}{L_2}\nu_1\right)\right). \end{aligned} \tag{15}$$

Thus, $\mathrm{E}\{X_{I1}(\nu_1)\, X_{I2}^{(*)}(\nu_2)\}$ is periodic in ν_1 with period $1/L_1$ and in ν_2 with period $1/L_2$. In addition, from (15) it follows that the Loève bifrequency cross-spectrum of the expanded processes $x_{I1}(n)$ and $x_{I2}(n)$ in the principal domain $(\nu_1,\nu_2) \in [-1/2,1/2)^2$ has support curves (${\scriptstyle(-)}$ present)

$$\nu_2 = (L_1/L_2)\nu_1 - \alpha/L_2 \qquad \alpha \in \mathscr{A} \tag{16}$$

that is, lines with slope L_1/L_2. Consequently, $x_{I1}(n)$ and $x_{I2}(n)$ are jointly SC. In particular, considering a single process $x_1(n) \equiv x_2(n) \equiv x(n)$, by taking $L_1 = 1$, that is $x_{I1}(n) \equiv x(n)$, it follows that the ACS process $x(n)$ and its expanded version $x_{I2}(n)$ are jointly SC with support lines with slope $1/L_2$. In the special case of $x(n)$ WSS, $x(n)$ and $x_{I2}(n)$ are jointly SC with support constituted by a unique line. These results constitute a strong motivation to treat the problem of multirate processing within the framework of the SC processes. In fact, even in the case of WSS input process, the joint characterization of input and output processes involves jointly SC processes.

By double inverse Fourier transforming both sides of (15), it can be shown that when $x_1(n)$ and $x_2(n)$ are jointly ACS, for $L_1 \neq L_2$ the cross-correlation function of $x_{I1}(n)$ and $x_{I2}(n)$ does not contain any finite-strength additive sinewave component. That is, $x_{I1}(n)$ and $x_{I2}^{(*)}(n)$ do not exhibit joint almost-cyclostationarity at any cycle frequency. In particular, their time-averaged cross-correlation is identically zero. Moreover, $x_{I1}(n)$ and $x_{I2}(n)$ are jointly ACS if and only if $L_1 = L_2 = L$. In such a case, the cyclic cross-spectra of $x_{I1}(n)$ and $x_{I2}(n)$ are given by

$$S^{\alpha}_{x_{I1}x_{I2}^{(*)}}(\nu) = \frac{1}{L} S_{\boldsymbol{x}}^{\alpha L}(\nu L) \qquad \alpha L \in \mathscr{A} \,. \tag{17}$$

In the special case of $x_1 \equiv x_2$, (17) reduces to [5, eq. (34)] specialized to second order.

3.2 Decimation

Let $x_{Di}(n)$ be the decimated version of $x_i(n)$, with decimation factor M_i $(i=1,2)$. That is,

$$x_{Di}(n) = x_i(nM_i) \,. \tag{18}$$

Its Fourier transform is

$$X_{Di}(\nu) = \frac{1}{M_i} \sum_{p=0}^{M_i-1} X_i\left(\frac{\nu - p}{M_i}\right). \tag{19}$$

Therefore, if $x_1(n)$ and $x_2(n)$ are jointly SC with Loève bifrequency cross-spectrum (5a), by using (19), the Loève bifrequency cross-spectrum of $x_{D1}(n)$ and $x_{D2}(n)$ can be expressed as

$$\begin{aligned} &\mathrm{E}\left\{X_{D1}(\nu_1)\, X_{D2}^{(*)}(\nu_2)\right\} \\ &\quad = \frac{1}{M_1 M_2} \sum_{p_1=0}^{M_1-1} \sum_{p_2=0}^{M_2-1} \sum_{k\in\mathbb{I}} S_{\boldsymbol{x}}^{(k)}\left(\frac{\nu_1 - p_1}{M_1}\right) \\ &\qquad \widetilde{\delta}\left(\frac{\nu_2 - p_2}{M_2} - \Psi_{\boldsymbol{x}}^{(k)}\left(\frac{\nu_1 - p_1}{M_1}\right)\right) \\ &\quad = \frac{1}{M_1} \sum_{p_1=0}^{M_1-1} \sum_{k\in\mathbb{I}} S_{\boldsymbol{x}}^{(k)}\left(\frac{\nu_1 - p_1}{M_1}\right) \\ &\qquad \sum_{h\in\mathbb{Z}} \delta\left(\nu_2 - h - M_2 \Psi_{\boldsymbol{x}}^{(k)}\left(\frac{\nu_1 - p_1}{M_1}\right)\right) \end{aligned} \tag{20}$$

where, in the second equality, the scaling property of the Dirac delta [14, Sec. 1.7] is used. From (20) it follows that the decimated versions of jointly SC processes are jointly SC. In addition, their Loève bifrequency cross-spectrum is periodic with period 1 in both variables ν_1 and ν_2.

In the special case where $x_1(n)$ and $x_2(n)$ are jointly ACS, then

$$\begin{aligned} &\mathrm{E}\left\{X_{D1}(\nu_1)\, X_{D2}^{(*)}(\nu_2)\right\} \\ &\quad = \frac{1}{M_1} \sum_{p_1=0}^{M_1-1} \sum_{\alpha\in\mathscr{A}} S_{\boldsymbol{x}}^{\alpha}\left(\frac{\nu_1 - p_1}{M_1}\right) \\ &\qquad \sum_{h\in\mathbb{Z}} \delta\left(\nu_2 - h - {\scriptstyle(-)}\left(\alpha M_2 - (\nu_1 - p_1)\frac{M_2}{M_1}\right)\right). \end{aligned} \tag{21}$$

The delta train is periodic in both ν_1 and ν_2 with period 1. In addition, from (21) it follows that the Loève bifrequency cross-spectrum of the decimated processes $x_{D1}(n)$ and $x_{D2}(n)$ in the principal domain $(\nu_1,\nu_2) \in [-1/2,1/2)^2$ has support curves ($(-)$ present)

$$\begin{aligned} &\nu_2 = (M_2/M_1)\nu_1 - \alpha M_2 - (M_2/M_1)p_1 \\ &\alpha \in \mathscr{A},\ p_1 \in \{0,1,\ldots,M_1-1\} \end{aligned} \tag{22}$$

that is, lines with slope M_2/M_1. Consequently, the decimated processes $x_{D1}(n)$ and $x_{D2}(n)$ are jointly SC. In particular, considering a single process $x_1(n) \equiv x_2(n) = x(n)$, by taking $M_1 = 1$, that is $x_{D1}(n) \equiv x(n)$, it follows that the ACS process $x(n)$ and its decimated version $x_{D2}(n)$ are jointly SC with support lines with slope M_2. In the special case of $x(n)$ WSS, the support line is unique.

From (21) it follows that $x_{D1}(n)$ and $x_{D2}(n)$ are jointly ACS if and only if $M_1 = M_2 = M$. In such a case, their cyclic cross-spectrum can be expressed as

$$S^{\alpha}_{x_{D1}x^{(*)}_{D2}}(\nu) = \frac{1}{M}\sum_{p=0}^{M-1}\sum_{q=0}^{M-1} S_{\boldsymbol{x}}^{(\alpha-q)/M}\left(\frac{\nu-p}{M}\right). \tag{23}$$

In the special case of $x_1 \equiv x_2$, (23) reduces to [5, eq. (24)] specialized to second order.

4. INTERPOLATION FILTERS

In this section, the effects of interpolation filters on jointly SC processes are analyzed and sufficient conditions are established to assure that the Loève bifrequency cross-spectrum of the interpolated processes $y_1(n)$ and $y_2(n)$ is a frequency-scaled image-free version of that of $x_1(n)$ and $x_2(n)$.

Fig. 1 presents two interpolation filters. Each interpolation filter $(i = 1,2)$ is constituted by a L_i-fold interpolator followed by a LTI filter with bandwidth W_i whose purpose is to obtain image-free interpolation [13]. An additional LTI filter with bandwidth B_i precedes the L_i-fold interpolator to strictly bandlimit the input process $x_{\mathrm{in},i}(n)$ in order to avoid overlapping among images in the Loève bifrequency cross-spectrum.

If $x_{\mathrm{in},1}(n)$ and $x_{\mathrm{in},2}(n)$ are jointly SC processes, then $x_1(n)$ and $x_2(n)$ are jointly SC processes strictly bandlimited with bandwidths B_1 and B_2, respectively. Using (14), the Loève bifrequency cross-spectrum of $y_1(n)$ and $y_2(n)$ is given by

$$\begin{aligned} &\mathrm{E}\left\{Y_1(\nu_1)\,Y_2^{(*)}(\nu_2)\right\} \\ &\quad= H_{W_1}(\nu_1)\,H_{W_2}(\nu_2)\,\mathrm{E}\left\{X_{I1}(\nu_1)\,X_{I2}^{(*)}(\nu_2)\right\} \\ &\quad= H_{W_1}(\nu_1)\,H_{W_2}(\nu_2)\sum_{k\in\mathbb{I}} S_{\boldsymbol{x}}^{(k)}(\nu_1 L_1) \\ &\qquad\quad \widetilde{\delta}\left(\nu_2 L_2 - \Psi_{\boldsymbol{x}}^{(k)}(\nu_1 L_1)\right) \end{aligned} \tag{24}$$

where $H_{W_i}(\nu)$ is the Fourier transform of the ideal low-pass filter $h_{W_i}(n)$.

Accounting for the band-limitedness of $x_1(n)$ and $x_2(n)$, the support of each term in the sum over $k \in \mathbb{I}$ in (24) is such that

$$\begin{aligned} &\mathrm{supp}\left\{S_{\boldsymbol{x}}^{(k)}(\nu_1 L_1)\,\widetilde{\delta}\left(\nu_2 L_2 - \Psi_{\boldsymbol{x}}^{(k)}(\nu_1 L_1)\right)\right\} \\ &\quad\subseteq \Big\{(\nu_1,\nu_2) \in \mathbb{R}\times\mathbb{R} : |(\nu_1 L_1) \bmod 1| \leqslant B_1, \\ &\qquad\quad \left|\Psi_{\boldsymbol{x}}^{(k)}(\nu_1 L_1)\right| \leqslant B_2,\ \nu_2 L_2 = \Psi_{\boldsymbol{x}}^{(k)}(\nu_1 L_1) \bmod 1\Big\} \\ &\quad= \bigcup_{\ell_1\in\mathbb{Z}}\bigcup_{\ell_2\in\mathbb{Z}} \Gamma_{\ell_1\ell_2}^{(k)} \end{aligned} \tag{25}$$

where

$$\begin{aligned} \Gamma_{\ell_1\ell_2}^{(k)} \triangleq\ &\Big\{(\nu_1,\nu_2) \in \mathbb{R}\times\mathbb{R} : |\nu_1 - \ell_1/L_1| \leqslant B_1/L_1, \\ &|\nu_2 - \ell_2/L_2| \leqslant B_2/L_2,\ \nu_2 - \ell_2/L_2 = \frac{1}{L_2}\Psi_{\boldsymbol{x}}^{(k)}(\nu_1 L_1)\Big\} \end{aligned} \tag{26}$$

In addition, we also have

$$\mathrm{supp}\left\{H_{W_1}(\nu_1)\,H_{W_2}(\nu_2)\right\} = \bigcup_{m_1\in\mathbb{Z}}\bigcup_{m_2\in\mathbb{Z}} \Delta_{m_1m_2} \tag{27}$$

where

$$\begin{aligned} \Delta_{m_1m_2} \triangleq\ &\Big\{(\nu_1,\nu_2) \in \mathbb{R}\times\mathbb{R} : |\nu_1 - m_1| \leqslant W_1, \\ &|\nu_2 - m_2| \leqslant W_2,\Big\}. \end{aligned} \tag{28}$$

Since $B_i \leqslant 1/2$, $(i = 1,2)$, images do not overlap. Furthermore, since $W_i \leqslant 1/2$, $(i = 1,2)$, then the filter supports do not overlap. If $B_i/L_i \leqslant W_i$, $(i = 1,2)$, then the low-pass filters $H_{W_1}(\nu_1)$ and $H_{W_2}(\nu_2)$ capture the main images, i.e., those centered in $(\nu_1,\nu_2) = (m_1,m_2) \in \mathbb{Z}^2$. That is, $\forall k \in \mathbb{I}$ it results

$$\Gamma_{m_1L_1,\,m_2L_2}^{(k)} \subseteq \Delta_{m_1m_2}\,. \tag{29}$$

Moreover, if $W_i \leqslant 1/(2L_i)$, $(i = 1,2)$, then the low-pass filters $H_{W_1}(\nu_1)$ and $H_{W_2}(\nu_2)$ do not capture images different from the main ones. That is, $\forall k \in \mathbb{I}$ we have

$$\Gamma_{\ell_1\ell_2}^{(k)} \cap \Delta_{m_1m_2} = \emptyset \ \text{ for }\ (\ell_1,\ell_2) \neq (m_1L_1, m_2L_2)\,. \tag{30}$$

The sufficient conditions such that (29)–(30) hold can be summarized into

$$\frac{B_i}{L_i} \leqslant W_i \leqslant \frac{1}{2L_i} \quad i = 1,2 \tag{31}$$

which is therefore a sufficient condition to assure that the Loève bifrequency cross-spectrum $\mathrm{E}\{Y_1(\nu_1)\,Y_2^{(*)}(\nu_2)\}$ is a frequency-scaled image-free version of $\mathrm{E}\{X_1(\nu_1)\,X_2^{(*)}(\nu_2)\}$.

Condition (31) is independent of the shape of the support curves $\nu_2 = \Psi_{\boldsymbol{x}}^{(k)}(\nu_1)$. Moreover, in the special case of a single ACS process and $L_1 = L_2 = L$, is less restrictive than [5, eq. (56)] specialized to second-order. In fact, [5, eq. (56)] assures the lack of images in the densities of spectral correlation (the cyclic spectra) for $\nu_1 \in [-1/(2L), 1/(2L)]$ and cycle frequencies $\alpha \in [-1/(2L), 1/(2L)]$.

It can be shown that

$$B_1 \leqslant 1/2 \quad \text{and} \quad 1 \geqslant B_2 + \max\Big\{B_2, \sup_k \left|\Psi_{\boldsymbol{x}}^{(k)}(\pm 1/2)\right|\Big\} \tag{32}$$

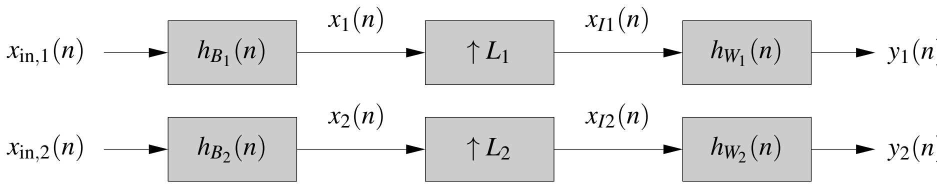

Figure 1: Interpolation Filters.

is a sufficient condition such that the density of Loève bifrequency cross-spectrum (24) along every support curve is a frequency-scaled image-free version of the corresponding density of the Loève bifrequency cross-spectrum of x_1 and x_2 for every $\nu_1 \in [-1/(2L_1), 1/(2L_1)]$. In (32), "+" or "-" sign should be taken if $\Psi_{\boldsymbol{x}}^{(k)}(\cdot)$ is increasing or decreasing, respectively.

5. CONCLUSION

The joint statistical characterization of processes elaborated with different scales or rates is provided within the framework of the (jointly) SC processes. By using the Loève bifrequency cross-spectrum as a tool, it is shown that jointly WSS, ACS, and SC processes expanded or decimated by different factors give rise to jointly SC processes. For interpolation filters, sufficient conditions on the expansion factors and the LTI system bandwidths are derived such that the Loève bifrequency cross-spectrum of the interpolated processes is a frequency-scaled alias-free version of that of the original processes. The derived results are useful in the joint statistical characterization of processes in systems with different rates as those occurring in tree-structured filter banks.

REFERENCES

[1] S. Akkarakaran and P.P. Vaidyanathan, "Bifrequency and bispectrum maps: A new look at multirate systems with stochastic inputs", *IEEE Transactions on Signal Processing*, vol. 48, n. 3, pp. 723-736, March 2000.

[2] R.E. Crochiere and L.R. Rabiner, "Interpolation and decimation of digital signals–A tutorial review," *Proceedings of the IEEE*, vol. 69, pp. 300-331, March 1981.

[3] W.A. Gardner, A. Napolitano, and L. Paura, "Cyclostationarity: Half a century of research," *Signal Processing*, vol. 86, n. 4, pp. 639-697, April 2006.

[4] I.M. Gel'fand and N.Ya. Vilenkin, *Generalized Functions, Vol. 4, Applications of Harmonic Analysis*, Academic Press, New York, 1964.

[5] L. Izzo and A. Napolitano, "Multirate processing of time-series exhibiting higher-order cyclostationarity", *IEEE Transactions on Signal Processing*, vol. 46, pp. 429-439, February 1998.

[6] K.-S. Lii and M. Rosenblatt, "Spectral analysis for harmonizable processes," *The Annals of Statistics*, vol. 30, N. 1, pp. 258-297, 2002.

[7] M. Loève, *Probability Theory*. Van Nostrand, Princeton, NJ, 1963.

[8] A. Napolitano, "Uncertainty in measurements on spectrally correlated stochastic processes", *IEEE Transactions on Information Theory*, vol. 49, pp. 2172-2191, September 2003.

[9] A. Napolitano, "Discrete-time estimation of second-order statistics of generalized almost-cyclostationary processes," *IEEE Transactions on Signal Processing*, vol. 57, n. 5, pp. 1670-1688, May 2009.

[10] T.A. Øigård, L.L. Scharf, and A. Hanssen, "Spectral correlations of fractional Brownian motion," *Physical Review E*, E74 031114, 2006.

[11] V.P. Sathe and P.P. Vaidyanathan, "Effects of multirate systems on the statistical properties of random signals," *IEEE Trans. Signal Processing,* vol. 41, pp. 131-146, January 1993.

[12] P.J. Schreier and L.L. Scharf, "Second-order analysis of improper complex random vectors and processes," *IEEE Transactions on Signal Processing*, vol. 51, n. 3, pp. 714-725, March 2003.

[13] P.P. Vaidyanathan, "Multirate digital filters, filter banks, polyphase networks, and applications: A tutorial review," *Proceedings of the IEEE*, vol. 78, pp. 56-93, January 1990.

[14] A.H. Zemanian, *Distribution theory and transform analysis*. New York: Dover, 1987.