ORDERING MINIMUM-PHASE SETS: NUMERICAL PROPERTIES AND SYSTEMATIC SEARCHES

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ABSTRACT

The main property of the ordered minimum-phase (OMP) sets is that they can be ordered as minimum-phase sequences. Several theoretical properties of such sets are already known. Amongst them, the set of all points having OMP property is an open non empty set. In this paper we present some numerical properties of the OMP sets, which may lead to systematic and fast exploration of permutations, in order to discover the OMP sets. Both are using lists of permutations: in the first case we have the most appropriate permutations list, in the second case we use a tabu-list of permutations.

1. INTRODUCTION

For processing usage, it is very common to describe a set of samples by mapping it as a sequence of samples [1]. However, the properties of a sequence are different from the properties of a set; many additional constraints in the resulting sequence can add into the properties of initial set [2]. On the other hand, one can use certain properties of resulting sequence to compress the information contained in a set. Thus the issue of converting a set into a sequence deserves to be investigated.

In an attempt to reduce the amount the data storage, the ordered minimum-phase (OMP) sets have been introduced [3, 4]. Their main property is that they can be ordered as minimum-phase sequences. Recall that a function is called minimum-phase function if all zeros are inside the unit open disk [5]. Based on modulus or phase of the Fourier transform, the reconstruction of a complex sequence can be possible when we know in advance that its corresponding z-transform is a minimum-phase function or maximum-phase function [6].

Several theoretical properties of such sets are already known [7]. Amongst them, we mention several:

1. The set of all points from $\mathbb{R}^{M+1}$ (or $\mathbb{C}^{M+1}$) having OMP property is an open non empty set.
2. Whenever $x(0)+x(1)+\cdots+x(M)=0$, the set $\{x(n)|n=\overline{0,M}\}$ has not OMP property.
3. Any set of real, positive and distinct numbers has OMP property.
4. Any set of three real numbers (except when their sum is zero) has OMP property.
5. Any set of complex numbers $\{x(0),x(1),x(2)\}$, which satisfies
$$\max\{|x(0)|,|x(1)|,|x(2)|\} > \min\{|x(0)|,|x(1)|,|x(2)|\} + \text{median}\{|x(0)|,|x(1)|,|x(2)|\}$$
has OMP property.
6. Any set of four real numbers $\{x(0),x(1),x(2),x(3)\}$, which differ in modulus and satisfying $x(0)x(1)x(2)x(3)>0$ and $x(0)+x(1)+x(2)+x(3)\neq 0$, has OMP property.
7. For $M=2$ (set of complex numbers) and $M=3$ (set of real numbers), we can find situations that no ordering of sets will lead to a minimum-phase sequence.

Using Schur transform (Appendix A) and performing all permutations, one can find if a set has OMP property and can also find the corresponding minimum-phase sequence. However, this may be computational expensive and fast methods would be appreciated.

The goal of this paper is to present few systematic procedures for finding the minimum-phase sequence from a given set. To this end some numerical properties of the OMP sets are discussed (Section 2), which may lead to systematic and fast exploration of permutations (Section 3), in order to discover the OMP sets. Two approaches will be presented, and both are using lists of permutations: in the first case we shall implement the most appropriate permutations list, in the second case we shall use a tabu-list of permutations.

To proceed we briefly specify the nomenclature.

Definition 1 *Let $\{x(0),x(1),\ldots,x(M)\}$ be a finite complex valued set. The set is said to have ordering minimum-phase (OMP) property if there exists a permutation*

$$\begin{pmatrix} x(0) & x(1) & \ldots & x(M) \\ y(0) & y(1) & \ldots & y(M) \end{pmatrix}$$

such that $Y(z)=y(0)+y(1)z^{-1}+\cdots+y(M)z^{-M}$ is a minimum-phase function.

The point $(x(0),x(1),\ldots,x(M))$ from $\mathbb{R}^{M+1}$ or $\mathbb{C}^{M+1}$ is said to have OMP property if the set $\{x(0),x(1),\ldots,x(M)\}$ has OMP property.

All polynomials will be considered as powers of z^{-1}.

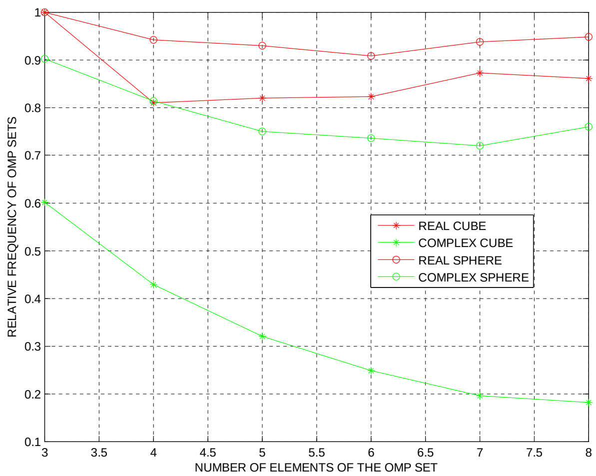

Figure 1: Relative frequency of OMP sets for real and complex sets, with uniform distribution inside a cube or inside a sphere

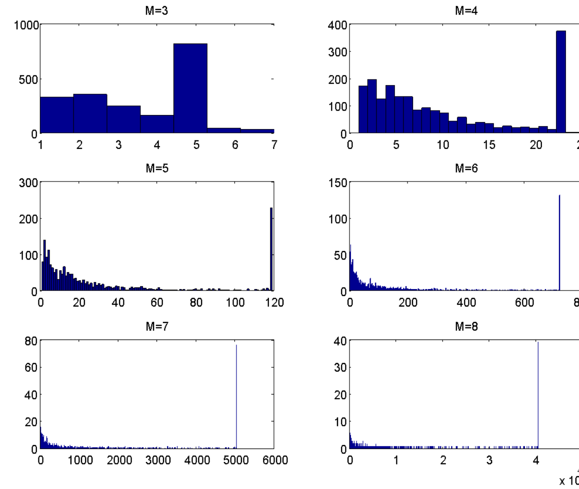

Figure 2: Histograms of minimum-phase sequences obtained from an OMP set

2. NUMERICAL PROPERTIES OF THE OMP SETS

2.1 Relative frequency of OMP sets

Our first goal was to determine the relative frequency of OMP sets. For this purpose we select the number of elements of the set from 3 to 8, and for every such selection we have generated a number of 10000 sets. Then we have verified whether any generated set has OMP property or not.

We found that OMP property is quite common, however the relative frequency depends on the way the set is generated: when data is real or complex, or if it has an uniform distribution inside a cube or inside a sphere. In the first case the real or imaginary parts are uniform distributed; in the second case the modulus and the phase are uniform distributed.

The outcomes are presented in Figure 1. Except complex cube distribution, in all shown situations the relative frequency is rather high. We also found that the relative frequency increases if the elements of the set (i.e. the coefficients of the permuted sequence) are concentrated in certain area. All these support the interest in systematic procedures for finding the minimum-phase sequence from a given set.

2.2 OMP sets and associated minimum-phase sequences

Another issue is whether an OMP set has only one or many more permutations such that associated sequence $y(n)$ is minimum-phase. We have generated 2000 minimum-phase sequences and for the corresponding OMP set, we have looked for all permutation which provided minimum-phase sequences. The histograms of these permutations are presented in Figure 2. On y axis we have the number of OMP sets having a certain number x, corresponding on x axis. We can see that the number of minimum-phase obtained from an OMP set has the properties:

1. We have sets with only one minimum-phase sequence;
2. The number of OMP sets having a certain number of minimum-phase sequences has a general tendency of decreasing;
3. The maximum number of OMP sets is obtained for a large number of minimum-phase sequences.

To conclude, we may expect that an OMP set has more than one minimum-phase sequence. However it may happen also that there will only one minimum-phase sequence for an OMP set.

2.3 Modulus distribution of OMP sets

One of the properties of minimum-phase systems which may start our discussion is the energy concentration theorem [8]. From this property, one may suppose that a condition for a certain sequence $x(n)$ to be a minimum-phase is to have its energy concentrated around origin such that:

$$|x(0)| > |x(1)| > \cdots > |x(M)| > 0. \quad (1)$$

This happens for real positive sequences, but it is not anymore valid if we skip to real sequences, with both positive and negative samples, and for complex sequences [3, 4]. However, one can assume that there is a relation between the moduli of the samples and the probability that the sequence is minimum-phase or not [9].

To this end we have generated 200000 minimum-phase sequences of length $M+1$ with $y(k) \neq 0$, $k = \overline{0,M}$. The roots of

$$Y(z) = y(0) + y(1)z^{-1} + \cdots + y(M)z^{-M}$$

have been selected with uniform random phase (between 0 and 2π) and random modulus (less than 1). Thus we obtained 200000 sets $\{x(0), x(1), \ldots, x(M)\}$ such that

$$\{x(0), x(1), \ldots, x(M)\} = \{y(0), y(1), \ldots, y(M)\}$$

and

$$|x(0)| > |x(1)| > \cdots > |x(M)| > 0.$$

For $M = 9$ the number of appearances of $x(n)$ as the k-th sample $y(k)$ in minimum-phase sequences generated is shown in Table 1. It is clear that the sample with the smallest modulus is most often as the highest order coefficient in minimum-phase function. Alternatively, the sample with the largest modulus may occupy usually the lowest order positions.

This behavior can be emphasized by mapping the Table 1 as a three-dimensional representation (Figure 3). We have

k	$x(0)$	$x(1)$	$x(2)$	$x(3)$	$x(4)$	$x(5)$	$x(6)$	$x(7)$	$x(8)$	$x(9)$
0	239882	420543	282314	55487	1736	37	1	0	0	0
1	205548	300709	328651	144509	19355	1161	67	0	0	0
2	178646	171695	237255	315489	88097	8292	502	23	1	0
3	168285	74838	85426	37095	262858	34914	2632	96	1	0
4	12616	22829	45458	69569	55079	172081	12624	484	4	0
5	62217	6814	15042	30709	56473	732752	92271	3647	75	0
6	17358	1976	4543	10605	16604	41569	861395	45166	783	1
7	1880	486	1111	2352	3586	8106	27256	935267	19861	95
8	24	100	186	312	472	1016	3088	14661	974108	6033
9	0	10	14	18	28	72	164	656	5167	993871

Table 1: *Number of appearances of $x(n)$ (moduli in decreasing order) as the k-th sample $y(k)$ in all 200000 minimum-phase sequences generated.*

also presented the corresponding images for $M = 20$ (Figure 4), and in this cases it is much clearer that the sample with the smallest modulus is most often as the highest order coefficient in minimum-phase function.

3. SYSTEMATIC PROCEDURES FOR FINDING THE MINIMUM-PHASE SEQUENCE FROM A GIVEN SET

Finding the minimum-phase sequence from a given set can be done by using Schur transform (Appendix A) and performing all permutations. With such strategy one can find if a set has OMP property and can also find the corresponding minimum-phase sequence. However, this may be computational expensive and fast methods would be appreciated.

In the following we shall present two approaches for this issue: the first uses a list of most appropriate permutations, and the second one implements a tabu-search like algorithm.

3.1 The approach based on the list of the most appropriate permutations

Based on previous experimental results (Section 2.3), we can indicate which is the most appropriate list of permutations that one can use to convert a set into a minimum-phase sequence. This approach has two parts: first we generate a list of most appropriate permutations, then we search the OMP set based on the generated list.

3.1.1 Generation of a list of most appropriate permutations

For any permutation

$$\begin{pmatrix} x(0) & x(1) & \dots & x(M) \\ y(0) & y(1) & \dots & y(M) \end{pmatrix}$$

we can estimate the probability to produce a minimum-phase sequence. This can be done by using the number of occurrences from Table 1 and assigning to each permutation a score equal with the product of the number of appearances.

As an example, consider the case when $|x(0)| > |x(1)| > \dots > |x(9)|$, and we have the following permutation

$$\begin{pmatrix} x(0) & x(1) & x(2) & x(3) & x(4) & x(5) & \dots & x(9) \\ x(1) & x(0) & x(3) & x(2) & x(4) & x(5) & \dots & x(9) \end{pmatrix}$$

which has only two transpositions. For such permutation, the score is given by (Table 1):

$$205548 \cdot 420543 \cdot 85426 \cdot 315489 \cdot 55079 \cdot$$

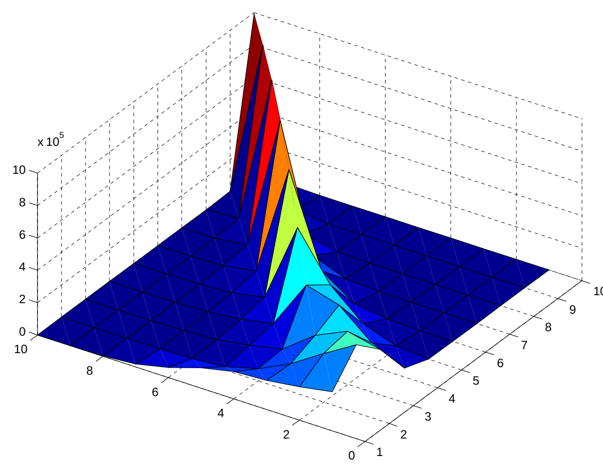

Figure 3: Number of appearances of $x(n)$ as the k-th sample $y(k)$ ($M = 9$).

$$\cdot 732752 \cdot 861395 \cdot 935267 \cdot 974108 \cdot 993871.$$

In this way we assign a score to all permutations. Having the score for all permutations and using this score in decreasing order, we can order the permutations in a list of most appropriate permutations for converting a set into a minimum-phase sequence. For instance, we found for $M = 9$ that the most appropriate 10 permutations (Table 2) are different from those provided by transpositions. Note that the natural order of permutations as given by transpositions is usually generated by software.

3.1.2 Searching by using the list of most appropriate permutations

The proposed procedure is as follows. When the set is given, we compute its sequence of moduli. Then we perform one by one the permutations given by the order from the most appropriate permutations list, until the first minimum-phase sequence is detected.

Our experiments have shown that we did not miss any OMP set by using the most appropriate permutations list procedure. However, this may happen when the procedure is not entirely respected:

1. The matrix score (Table 1) is obtained with a small number of sequences generated;

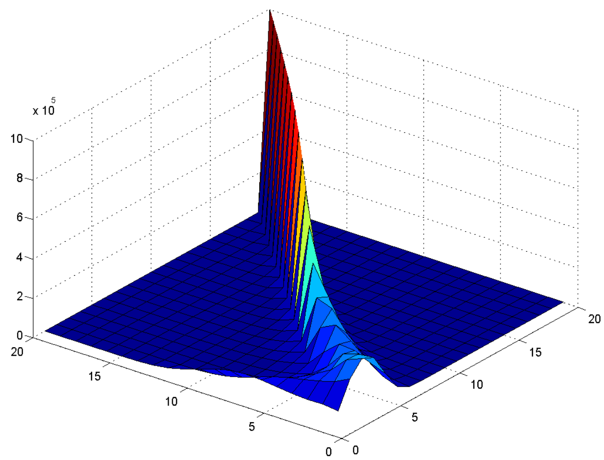

Figure 4: Number of appearances of $x(n)$ as the k-th sample $y(k)$ ($M = 19$).

2	3	1	4	5	6	7	8	9	10
2	1	3	4	5	6	7	8	9	10
2	3	4	1	5	6	7	8	9	10
1	3	2	4	5	6	7	8	9	10
3	2	1	4	5	6	7	8	9	10
1	3	2	4	5	6	7	8	9	10
3	2	4	1	5	6	7	8	9	10
3	1	2	4	5	6	7	8	9	10
2	3	4	5	1	6	7	8	9	10
2	4	3	1	5	6	7	8	9	10

Table 2: *The best first 10 permutations for $M = 9$.*

2. The search is fulfilled using a reduced list of most appropriate permutations.

For instance, we perform simulations when the best permutation list contains only the 50 first permutation (those who have together 98% of all score). In this case we get only about half of OMP correct detections.

As comparison, one can also perform permutations from the natural order permutations list, until the first minimum-phase sequence is detected. Our simulations show a significant reduction of computational effort needed by the proposed approach to detect the minimum-phase sequence. The reduction is in average about ten times for $M = 7$ and similar results have been obtained for various M ($5 \leq M \leq 12$) [9].

3.2 Tabu-Search Approach

We start by noting that the Schur transform structure may lead to a tabu-list [10]. Indeed, when a permutation does not give a minimum-phase sequence, then for a certain k, we have $\gamma_k \leq 0$. All permutations having the same first k and last k samples may be included in the tabu-list and they do not need to be considered later on as a candidate for providing a minimum-phase sequence. Thus finding a minimum-phase sequence from a given set can be implemented in a similar way as a tabu-search problem.

There is however an important difference between the tabu-search method implemented for optimization and the proposed approach to be used for finding the OMP sets. In the case of optimization there is a fixed finite length tabu-list which is updated after every iteration. Moreover, some of previous states belonging to the tabu-list must disappear when updating the tabu-list.

1	2	3	4	5	6	7	8	9	10
2	1	3	4	5	6	7	8	9	10
1	3	2	4	5	6	7	8	9	10
3	1	2	4	5	6	7	8	9	10
2	3	1	4	5	6	7	8	9	10
3	2	1	4	5	6	7	8	9	10
1	2	4	3	5	6	7	8	9	10
2	1	4	3	5	6	7	8	9	10
1	4	2	3	5	6	7	8	9	10
4	1	2	3	5	6	7	8	9	10

Table 3: *The first 10 permutations for $M = 9$ in natural order given by transpositions.*

This is not the case when tabu-search is applied for finding an OMP set; we still keep permutations on the tabu-list until we are sure that no one of possible permutations which may be derived, fits to the permutations on the list.

Alternatively, if a new candidate for the tabu-list is obtained, the length of tabu-list must be increased. It follows that when the tabu-search is used to find an OMP set, the tabu-list may be excessively large and the search for tabu permutations may be rather long. This seems a major drawback of the approach.

We have performed tests with sets having 7, 8, 9, 10 or 11 elements and in average the tabu-search approach took less time than to perform all the permutations. However, sometimes tabu-search is not the fastest method. Indeed, we have verified the source of latency, and we have discovered that the tabu-list is growing quickly.

Consequently it takes sometimes less time to compute the Schur transform for a permutation, than to compare the permutation with all the tabu permutations from the tabu-list. In addition, the tabu-list must be modified, if it is the case, and this takes time too, especially when the tabu-list is very large.

Let us consider the case of a set having 8 elements. Using all permutations we need to compute the Schur transform for 5040 times, thus the Schur transform routine is called for 5040 times. The tabu-list might be about 1000 lines for this case, and for every new permutation we have to compare this new one, partially or entirely, with all these 1000 lines.

Unfortunately we can have often sets where the γ_k coefficients are very large and consequently almost every new permutation fills a new position in the tabu-list. Thus the tabu-list must be updated almost at every iteration and in such situation the tabu-list is growing significantly.

Recent processors may compute the Schur transform even faster than searching a very long tabu-list. To reduce the period for comparison of permutations through the tabu-list, we keep a permutation in the tabu-list until we are sure that this combination cannot appear later.

In our simulations, for every M from 6 to 10, we have considered 100 sets having $M + 1$ elements. Using the tabu-search approach we have determined the OMP sets. To validate our results we have performed all permutations by transpositions, using the same testing sets. To the end, we conclude that the results for both methods were the same. Table 4 shows the average of computational time in ms, needed

M	OMP		non OMP	
	Transpositions	Tabu-Search	Transpositions	Tabu-Search
6	12.21	9.91	33.30	13.21
7	126.23	43.23	271.7	79.2
8	1019.9	265	2494.4	500.8
9	10638	2777	29854	5902
10	21056	25385	32886	27818

Table 4: The average of computational time (ms) to find if a set is an OMP set or not, using permutation by transpositions or by tabu-search method

to find if a set is an OMP set or not, using permutation by transpositions or by tabu-search method. In almost all situations, tabu-search is faster. However, the reduction is not substantial as in the case of list of appropriate permutations.

4. CONCLUSIONS

In this paper we have discussed several numerical properties of the OMP sets. We have found rather high relative frequencies of OMP sets, and an OMP set may have one or many more associated minimum-phase sequences. Moreover, we have established that the minimum-phase sequences are likely to have their moduli in decreasing order.

We have concentrated on fast systematic methods for finding an OMP set. First the approach based on the list of most appropriate permutations has been presented and this gives attractive results. It may also provide wrong decisions on OMP property, if the list is reduced or based on roughly estimations. The second method involves tabu-search approach, which does not fail, but it is not so fast. Mainly, this is a result of long time needed for comparisons on the tabu-list.

A. SCHUR TRANSFORM

Following [11], for a polynomial of degree n in z^{-1} of the form:

$$P(z) = a_0 + a_1 z^{-1} + \cdots + a_n z^{-n},$$

the reciprocal polynomial of P is defined by:

$$P^*(z) = \overline{a}_n + \overline{a}_{n-1} z^{-1} + \cdots + \overline{a}_0 z^{-n}.$$

Definition 2 *The Schur transform of the polynomial P of degree n is the polynomial TP of degree $n-1$ defined by*

$$TP(z) = \overline{a}_0 P(z) - a_n P^*(z) = \sum_{k=0}^{n-1} (\overline{a}_0 a_k - a_n \overline{a}_{n-k}) z^{-k}. \quad (2)$$

The iterated Schur transforms T^2P, T^3P, $\ldots$, T^nP are defined by:

$$T^k P = T(T^{k-1}P), \quad k = 2, 3, \ldots, n. \quad (3)$$

We set $\gamma_k = T^k P(\infty)$, for $k = 1, 2, \ldots, n$.

Theorem 1 (Schur-Cohn Algorithm) [11] *Let P be a polynomial of degree n in z^{-1}, $P \neq 0$. Then all zeros of P lie inside the open unit disk $|z| < 1$ if and only if $\gamma_k > 0$, for all $k = 1, 2, \ldots, n$.*

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