HIGHER ORDER DIRECTION FINDING FOR ARBITRARY NONCIRCULAR SOURCES : THE NC-2Q-MUSIC ALGORITHM

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ABSTRACT

These two last decades, Higher Order (HO) High Resolution (HR) Direction Finding (DF) algorithms, such as $2q$-MUSIC ($q \geq 1$), exploiting the information contained in the HO circular cumulants of the data, have been developed for non Gaussian sources to overcome some limitations of second order (SO) methods. However, for $2q$-th order noncircular sources such as $M-PSK$ sources with $M \leq 2q$, strong gains in performance may be obtained by taking into account the information contained in both the $2q$-th order circular and noncircular cumulants of the data, giving rise to noncircular $2q$-th order DF algorithms. As Noncircular HO DF methods are very scarce, the purpose of this paper is to introduce and to analyze some performance of the NonCircular $2q$-MUSIC (NC-$2q$-MUSIC) method ($q \geq 1$), for mixtures of arbitrary sources ($2q$-th order circular or not), typical of operational contexts.

1. INTRODUCTION

Angle Of Arrival (AOA) estimation using an array of sensors finds applications in many fields, such as radar, sonar and wireless communications in particular. For more than two decades, HO high resolution DF methods, exploiting the information contained in the HO circular cumulants of the data, have been developed for non Gaussian sources [12] to overcome some limitations of SO methods such as MUSIC [13]. Among these methods, the $2q$-MUSIC method ($q \geq 1$) [5], which exploits the information contained in the circular $2q$-th order cumulants of the data [2], [11] is probably the most popular. This method may process a number of sources greater than the number of sensors and has a resolution and a robustness to modelling errors which increases with q [5]. However, for $2q$-th order noncircular sources such as M-PSK sources with $M \leq 2q$, omnipresent in radio communications contexts, the information contained in the $2q$-th order circular cumulants of the data is not exhaustive and some information is also contained in the $2q$-th order noncircular cumulants of the data. In such conditions, strong gains in performance may be obtained by taking into account the information contained in both the $2q$-th order circular and noncircular cumulants of the data, giving rise to noncircular $2q$-th order DF methods. Several noncircular DF methods have been developed this last decade at the SO ($q = 1$), mainly for rectilinear sources [3], [8], [9] except [1] which also considers the case of nonrectilinear sources. However, for $q > 1$, only one noncircular DF method, presented in [10], seems to be available but under the restrictive condition of rectilinear sources. The purpose of this paper is to overcome this limitation by introducing and by analyzing some performance of the noncircular $2q-$MUSIC (NC-$2q$-MUSIC) method, for mixtures of arbitrary sources ($2q$-th order circular or not), typical of operational contexts. The development of the NC-$2q$-MUSIC method for arbitrary noncircular sources is based on a new tool introduced in this paper and corresponding to the decomposition of a source in a finite or infinite number of rectilinear sources. The problem is formulated in section 2. Higher order statistics of the data jointly with the new tool are introduced in section 3. The NC$-2q-$MUSIC algorithm for arbitrary sources is presented in section 4. Computer simulations showing off the interest of the proposed method are presented in section 5. Section 6 concludes the paper.

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2. MODEL AND PROBLEM FORMULATION

Let us assume that a noisy mixture of P narrow-band sources of AOAs $\mathbf{\Theta}_i$, $1 \leq i \leq P$, is received by an array of N sensors. The associated observation vector, $\mathbf{x}(t)$, whose components $x_n(t)$, $1 \leq n \leq N$, are the complex envelopes of the signals at the output of the sensors, is then given by

$$\mathbf{x}(t) = \sum_{i=1}^{P} \mathbf{a}(\mathbf{\Theta}_i) \; m_i(t) + \mathbf{n}(t), \tag{1}$$

where $\mathbf{n}(t)$ is an additive noise vector which is supposed to be spatially white, circular and Gaussian, and $m_i(t)$ is the complex envelope of the ith source and $\mathbf{a}(\mathbf{\Theta})$ is the array response in direction $\mathbf{\Theta}$ (or steering vector). In the case of plane wave and homogeneous array, this steering vector is

$$\mathbf{a}(\mathbf{\Theta}) = \begin{bmatrix} a_1(\mathbf{\Theta}) \\ \vdots \\ a_N(\mathbf{\Theta}) \end{bmatrix} \quad \text{with} \quad a_n(\mathbf{\Theta}) = \exp\left(j \frac{2\pi\left((\mathbf{p}_n)^T \mathbf{k}(\mathbf{\Theta})\right)}{\lambda} \right) \tag{2}$$

where $\mathbf{p}_n$ and $\mathbf{k}(\mathbf{\Theta})$ are the location vector of the nth antenna and the wave vector respectively such that

$$\mathbf{k}(\mathbf{\Theta}) = \begin{bmatrix} \cos(\theta)\cos(\Delta) \\ \sin(\theta)\cos(\Delta) \\ \sin(\Delta) \end{bmatrix}$$

where θ and Δ are the azimuth and elevation angles respectively. To simplify the following analysis, the signals $m_i(t)$ are assumed to be statistically independent, but no particular assumption is made on their circular or noncircular properties contrary to [10]. The purpose of this paper is to develop an extension of the $2q$-MUSIC ($q \geq 1$) algorithm which takes benefit of the potential SO or HO noncircularity of the sources but which can also be adaptable to any circular sources. To this aim we analyze in section 3 the algebraic structure of the HO statistics of the extended data $\tilde{\mathbf{x}}(t) = [\ \mathbf{x}(t)^T \quad \mathbf{x}(t)^H\ ]^T$, where $(.^H)$ and $(.^T)$ denote the transpose conjugate and transpose symbols respectively and we derive from this analysis the NC-$2q$-MUSIC algorithm in section 4. Impact of modelling errors on the NC-$2q$-MUSIC algorithm will be presented elsewhere.

3. FULL $2Q$-TH ORDER STATISTICS OF THE DATA

We present in this section the full $2q$-th order statistics of the data and we analyze some properties of the latter, required to develop the NC-$2q$-MUSIC algorithm in section 4.

3.1 Full 2q-th order statistics of $\mathbf{x}(t)$

The full $2q$-th order statistics of $\mathbf{x}(t)$ correspond to the set of both the circular and the noncircular $2q$-th order cumulants of $\mathbf{x}(t)$, i.e. to the circular $2q$-th order cumulants of the extended vector $\tilde{\mathbf{x}}(t)$. According to (1) and the fact that $m_i(t) = (\mathbf{h}_{IQ})^H \mathbf{m}_i(t)$ where $(\mathbf{h}_{IQ})^H = [\ 1 \quad j\ ]$ and $\mathbf{m}_i(t)^T = [\ \text{real}(m_i(t)) \quad \text{imag}(m_i(t))\ ]$, the signal $\tilde{\mathbf{x}}(t)$ can be written as

$$\tilde{\mathbf{x}}(t) = \sum_{i=1}^{P} \mathbf{A}(\mathbf{\Theta}_i)\ \mathbf{m}_i(t) + \tilde{\mathbf{n}}(t) \quad (3)$$

where

$$\mathbf{A}(\mathbf{\Theta}_i) = \begin{bmatrix} \mathbf{a}(\mathbf{\Theta}_i)\,\mathbf{h}_{IQ}^H \\ \mathbf{a}(\mathbf{\Theta}_i)^*\ \mathbf{h}_{IQ}^T \end{bmatrix} = \begin{bmatrix} \mathbf{a}(\mathbf{\Theta}_i) & j\mathbf{a}(\mathbf{\Theta}_i) \\ \mathbf{a}(\mathbf{\Theta}_i)^* & -j\mathbf{a}(\mathbf{\Theta}_i)^* \end{bmatrix} \quad (4)$$

* means conjugate and $\tilde{\mathbf{n}}(t) = [\ \mathbf{n}(t)^T \quad \mathbf{n}(t)^H\ ]^T$. Note that matrix $\mathbf{A}(\mathbf{\Theta}_i)$ of (4) has the following properties

1. $\mathbf{A}(\mathbf{\Theta}_i)^H \mathbf{A}(\mathbf{\Theta}_i) = \left(\mathbf{a}(\mathbf{\Theta}_i)^H \mathbf{a}(\mathbf{\Theta}_i)\right) \times \mathbf{I}_2$ (5)
2. $\mathbf{A}(\mathbf{\Theta}_i)^* = \mathbf{T}\mathbf{A}(\mathbf{\Theta}_i)$ with $\mathbf{T} = \begin{bmatrix} 0 & \mathbf{I}_N \\ \mathbf{I}_N & 0 \end{bmatrix}$ (6)

Thus, the columns of $\mathbf{A}(\mathbf{\Theta}_i)$ are orthogonal and the matrices $\mathbf{A}(\mathbf{\Theta}_i)^*$ and $\mathbf{A}(\mathbf{\Theta}_i)$ are equal through a permutation matrix. The circular $2q$-th order statistics of $\tilde{\mathbf{x}}(t)$ correspond to the elements $< cum\left(\tilde{x}_{i_1}(t), ..., \tilde{x}_{i_q}(t), \tilde{x}_{i_{q+1}}(t)^*, ..., \tilde{x}_{i_{2q}}(t)^*\right) >$ for $(1 \le i_j \le 2N)$, where $\tilde{x}_j(t)$ is the j−th component of $\tilde{\mathbf{x}}(t)$, the symbol $<.>$ corresponds to the time-averaged operation and $cum(.)$ the cumulant. In a similar way as for the $2q$−th-order circular cumulants of the vector $\mathbf{x}(t)$, the latter entries can be arranged in the $\mathbf{C}_{2q,\tilde{\mathbf{x}}}$ matrix in different ways, indexed by the same integer l such that $(1 \le l \le q)$ [5]. However, we easily deduce from (6) that all the arrangements in the $\mathbf{C}_{2q,\tilde{\mathbf{x}}}$ matrix are equivalent and we choose for the following the natural arrangement defined by $\mathbf{C}_{2q,\tilde{\mathbf{x}}}(I,J) =< cum\left(\tilde{x}_{i_1}(t), ..., \tilde{x}_{i_q}(t), \tilde{x}_{i_{q+1}}(t)^*, ..., \tilde{x}_{i_{2q}}(t)^*\right) >$ with $I = L_0$ and $J = L_q$ where

$$L_l = \sum_{k=1}^{q} N^{q-k}\left(i_{l+k} - 1\right) + 1 \quad (7)$$

The NC-$2q$-MUSIC method presented in section 4 has to exploit the information contained in $\mathbf{C}_{2q,\tilde{\mathbf{x}}}$. Under the previous assumptions, we deduce from (3) that in the presence of statistically independent signals $m_i(t)$, the matrix $\mathbf{C}_{2q,\tilde{\mathbf{x}}}$ can be written as

$$\mathbf{C}_{2q,\tilde{\mathbf{x}}} = \sum_{i=1}^{P} \mathbf{A}(\mathbf{\Theta}_i)^{\otimes q}\,\mathbf{C}_{2q,\mathbf{m}_i}\left(\mathbf{A}(\mathbf{\Theta}_i)^{\otimes q}\right)^H + \mathbf{C}_{2q,\tilde{\mathbf{n}}} \quad (8)$$

where $\mathbf{C}_{2q,\tilde{\mathbf{n}}} = \sigma^2\mathbf{I}_2\delta(q-1)$, $\delta(.)$ is the Kronecker symbol, $\otimes$ the Kronecker product, σ^2 the power of the noise per sensor, $\mathbf{C}_{2q,\mathbf{m}_i}$ is the $2q$-th order statistical matrix of $\mathbf{m}_i(t)$ and $\mathbf{A}^{\otimes q} = \mathbf{A} \otimes \cdots \otimes \mathbf{A}$ with $q-1$ kronecker products. To get more insight into the signal and noise subspaces of $\mathbf{C}_{2q,\tilde{\mathbf{x}}}$, it is necessary to analyze the algebraic structure, and in particular the rank, of both the $2^q \times 2^q$ matrix $\mathbf{C}_{2q,\mathbf{m}_i}$ and the $(2N)^q \times 2^q$ matrix $\mathbf{A}(\mathbf{\Theta}_i)$.

3.2 Algebraic structure of $\mathbf{C}_{2q,\mathbf{m}}$

We analyze in this section the algebraic properties of the time-averaged $2q$-th order statistical matrix $\mathbf{C}_{2q,\mathbf{m}}$ of $\mathbf{m}(t)^T = [\ \text{real}(m(t)) \quad \text{imag}(m(t))\ ] = [\ u_1(t) \quad u_2(t)\ ]$ associated with the complex envelope $m(t) = u_1(t) + ju_2(t)$, where $u_1(t)$ and $u_2(t)$ are real-valued quantities. Let us recall that the elements of $\mathbf{C}_{2q,\mathbf{m}}$ are the quantities $< cum\left(u_{i_1}(t), ..., u_{i_q}(t), u_{i_{q+1}}(t), ..., u_{i_{2q}}(t)\right) >$ for $(1 \le i_j \le 2)$.These elements are arranged in the $\mathbf{C}_{2q,\mathbf{m}}$ matrix in a natural way described in the previous section for $N = 2$. The main purpose of this section is to evaluate the potential rank of $\mathbf{C}_{2q,\mathbf{m}}$ for a given $m(t)$. To this aim, two methods are considered. The first one consists to evaluate the number of different rows of $\mathbf{C}_{2q,\mathbf{m}}$whereas the second one deduces the rank from available results about the generic rank of q−th order real tensors.

3.2.1 ***Method 1 :*** *Evaluation of the number of different rows of* $\mathbf{C}_{2q,\mathbf{m}}$

The row $I = L_0$ of $\mathbf{C}_{2q,\mathbf{m}}$ is associated with the q−uplet $(i_1, \ldots, i_q)$ through (7) with $N = 2$. As the $2q$-th order cumulants are invariant by any permutation of the indices, we deduce that when two q−uplets $(i_1, \ldots, i_q)$ and $(j_1, \ldots, j_q)$ are invariant by permutation, the associated rows are identical. Such two q−uplets will be qualified as redundant q−uplets. Then the maximal number of different rows of $\mathbf{C}_{2q,\mathbf{m}}$ corresponds to the number of nonredundant q−uplets of indices $(i_1, \ldots, i_q)$. As the cumulants and the multiplication operation have the same permutation invariance property, the maximal number of nonredundant rows of $\mathbf{C}_{2q,\mathbf{m}}$ also corresponds to the maximal number of nonredundant products $u_{i_1}(t) \times ... \times u_{i_q}(t)$, i.e. the maximal number of nonredundant components of $\mathbf{m}(t)^{\otimes q}$. It is then straightforward to verify that, for arbitrary nonzero signals $u_1(t)$ and $u_2(t)$, the nonredundant elements of $\mathbf{m}(t)^{\otimes q}$ corresponds to the $(q+1)$ elements $u_1(t)^{q-I+1}u_2(t)^{I-1}$ for $1 \le I \le q+1$. A similar reasoning may be done for the columns of $\mathbf{C}_{2q,\mathbf{m}}$. We deduce from this analysis that the maximal value of the rank of $\mathbf{C}_{2q,\mathbf{m}}$ is $(q+1)$ and this rank is reached in particular for circular source. At the opposite for noncircular sources, the rank of $\mathbf{C}_{2q,\mathbf{m}}$ may decrease below $(q+1)$. It is in particular the case for a rectilinear source for which $u_1(t)$ and $u_2(t)$ are proportional ($u_2(t) = \alpha\ u_1(t)$, where α is a real scalar). In this case, the $(q+1)$ elements $u_1(t)^{q-I+1}u_2(t)^{I-1}$ for $1 \le I \le q+1$ correspond to the $(q+1)$ elements $\alpha^{I-1}u_1(t)^q$, which are all proportional to $u_1(t)^q$. Then the $(q+1)$ associated rows of $\mathbf{C}_{2q,\mathbf{m}}$ become also proportional and the rank of $\mathbf{C}_{2q,\mathbf{m}}$ is only 1. As a summary we obtain

$$1 \le rank\left(\mathbf{C}_{2q,\mathbf{m}}\right) \le q+1 \quad (9)$$

Note that the relation between $\mathbf{m}(t)^{\otimes q}$ and the associated non-redundant vector $\bar{\mathbf{m}}(t)$ is

$$\mathbf{m}(t)^{\otimes q} = \mathbf{\Gamma}^q_{2^q,q+1}\,\bar{\mathbf{m}}(t)^q \quad (10)$$

$$\mathbf{\Gamma}_{2^q,q+1} = \left(\mathbf{I}_2 \otimes \mathbf{\Gamma}_{2^{q-1},q}\right)\begin{bmatrix} \mathbf{I}_q & \mathbf{0}_{q,1} \\ \mathbf{0}_{q,1} & \mathbf{I}_q \end{bmatrix} \quad (11)$$

where $\bar{\mathbf{m}}(t)^q(I) = u_1(t)^{q-I+1}u_2(t)^{I-1}$ and $\mathbf{\Gamma}_{2^q,q+1}$ is a $(2^q) \times (q+1)$ full rank matrix composed by 0 and 1 where

$\mathbf{\Gamma}_{1,1} = \mathbf{I}_2$.The relation between $\mathbf{C}_{2q,\mathbf{m}}$ and the matrix $\mathbf{C}_{2q,\tilde{\mathbf{m}}_{IQ}}$ composed by the non redundant elements of $\mathbf{C}_{2q,\mathbf{m}}$ is then

$$\mathbf{C}_{2q,\mathbf{m}} = \mathbf{\Gamma}_{2^q,q+1}\mathbf{C}_{2q,\tilde{\mathbf{m}}_{IQ}} \left(\mathbf{\Gamma}_{2^q,q+1}\right)^T \tag{12}$$

3.2.2 *Method 2 : Exploitation of results about generic rank of $2q$-th order real tensors*

Expression (9) can be obtained from available results about the generic rank of $q-$th order real tensors. Indeed, according to [6], the maximal number of components of a $q-$th order real tensor with indices $(i_1, ..., i_q)$ such that $i_1 \le ... \le i_q$ and $1 \le i_k \le N$ is

$$D(q, N) = C^q_{N+q-1} = \frac{(N+q-1)!}{q!\,(N-1)!}. \tag{13}$$

This is equivalent to identify the number of $q-sets$ $(i_1, ..., i_q)$ such that $(i_1 \le ... \le i_q)$ and $(1 \le i_k \le N)$. Applying this result for $N = 2$, we obtain the maximal number of components of $\mathbf{C}_{2q,\mathbf{m}}$, given by $D(q, 2) = q + 1$.

3.3 Concept of $2q$-th order k–IQ-rank source

The results of section 3.2 show that depending on the $2q$th-order correlation between the real and imaginary parts of a given complex envelope $m(t) = u_1(t) + ju_2(t)$, the latter generates a rank k which may be comprised between 1 and $(q+1)$ in the $2q-$th order covariance matrix $\mathbf{C}_{2q,\mathbf{m}}$ of $\mathbf{m}(t) = [\ u_1(t) \quad u_2(t)\ ]^T$. As the components of $\mathbf{m}(t)$ are the I and Q components of $m(t)$, we qualify such a source as a $2q-$**th order** $k - IQ-$**rank source**.

A particular family of $2q$-th order $k - IQ-$rank source corresponds to the family of $k-$rectilinear sources with $(1 \le k \le q+1)$, whose complex envelope $m(t)$ corresponds to a mixture of k statistically independent rectilinear sources i $(1 \le i \le k)$. Indeed, for such a source, assuming that the complex envelope of the ith rectilinear source is proportional to the real signal $v_i(t)$, the complex envelope $m(t)$ can be written as

$$m(t) \quad = \sum_{i=1}^{k} v_i(t) \exp(j\Phi_i) = \mathbf{h}(\mathbf{\Phi})^H \mathbf{v}(t) \tag{14}$$

where the signals $v_i(t)$ are real and statistically independent, Φ_i is the phase of the ith source and where $\mathbf{h}(\mathbf{\Phi})^H = [\ \exp(j\Phi_1) \ \cdots \ \exp(j\Phi_k)\ ]$ and $\mathbf{v}(t)^T = [\ v_1(t) \ \cdots \ v_k(t)\ ]$.

Note that the complex envelope $m(t)$ of an arbitrary source can be written as (14) with $k = 2$ where $v_1(t) = u_1(t)$, $v_2(t) = u_2(t)$, $\Phi_1 = 0$, $\Phi_2 = \pi/2$ and where $u_1(t)$ and $u_2(t)$ are not necessarily statistically independent. In this context

$$\begin{aligned}&\mathbf{m}(t) = \mathbf{v}(t)\\ &(\mathbf{h}_{IQ})^H = \mathbf{h}\left(\left[\ 0 \quad \pi/2\ \right]^T\right)^H\end{aligned} \tag{15}$$

A particular example of *k-rectilinear source* is the Phase Shift Keying (PSK) modulated source with $2k$ states, called $2k-$PSK source. A $k-rectilinear\ source$ such that $k \ge q+1$ is a *$2q$-th order $(q+1) - IQ-rank$ source.* A $2q-$th order circular source [2], [11] $m(t)$, for which $< cum[m(t)^{\varepsilon_1}, \cdots, m(t)^{\varepsilon_l}, m(t)^{\varepsilon_{l+1}}, \cdots, m(t)^{\varepsilon_{2q}}] >$ is zero for $l \ne q$, where $\varepsilon_i = 1$ for $(1 \le i \le l)$ and $\varepsilon_i = -1$ for $(l+1 \le i \le 2q)$ with the convention $m^1 = m$ and $m^{-1} = m^*$, is a *$2q$-th order $(q+1) - IQ-rank$ source.* As a consequence of this result, a *$2q$-th order $k-IQ-rank$ source* with $k < q+1$ is necessarily a $2q-$th order noncircular source.

For example a BPSK source or a ASK source, which are rectilinear sources, are $2q$-th order $1 - IQ-$rank source and $2q-$th order noncircular sources whatever the value q. A QPSK source, which is a $2-$rectilinear source (or a bi-rectilinear source), is a 2^{nd} order $2 - IQ-$rank and circular source and a $2q-$th order $2 - IQ-$rank and noncircular source for $q > 1$. A source $m(t)$ for which $u_1(t) = \text{real}(m(t))$ and $u_2(t) = imag(m(t))$ are statistically independent is a $2-$rectilinear source, as shown by (15), and is then necessarily $2q-$th order noncircular for $q > 1$ and is also 2^{nd} order noncircular if $E[u_1(t)^2] \ne E[u_2(t)^2]$.

3.4 Alternative expression of $\mathbf{C}_{2q,\tilde{\mathbf{x}}}$

In order to estimate the angle $\mathbf{\Theta}_i$ of the source with a MUSIC approach from $\mathbf{C}_{2q,\tilde{\mathbf{x}}}$, the purpose of this section is to analyze the signal subspace structure of the matrix $\mathbf{C}_{2q,\tilde{\mathbf{x}}}$. This analysis is difficult for arbitrary sources and will be presented elsewhere. In this paper, we limit the analysis to mixtures of either k_i-**rectilinear** sources with $k_i \le q+1$ or general $2q-$**th order** $(q+1) - IQ-$**rank source**.

3.4.1 *Case of P k_i-rectilinear sources with $k_i \le q+1$*

In the presence of $k_i-rectilinear$ statistically independent sources such that $k_i \le q+1$ for $(1 \le i \le P)$ the signal $m_i(t)$ is given by (14) where $m(t)$, k, $\mathbf{v}(t)$ are replaced by $m_i(t)$, k_i, $\mathbf{v}_i(t)$ respectively. Under this assumption, it is straighforward to verify that $\mathbf{C}_{2q,\tilde{\mathbf{x}}}$ can be written as

$$\begin{aligned}\mathbf{C}_{2q,\tilde{\mathbf{x}}} \quad &= \sum_{i=1}^{P}\sum_{l=1}^{k_i} c_{2q,v_{il}} \mathbf{b}(\mathbf{\Theta}_i, \Phi_{il})^{\otimes q} \left(\mathbf{b}(\mathbf{\Theta}_i, \Phi_{il})^{\otimes q}\right)^H\\ &+\mathbf{C}_{2q,\tilde{\mathbf{n}}}\end{aligned} \tag{16}$$

where $c_{2q,v_{il}}$ is the time average of the $2q-$th order cumulant of the lth component of $\mathbf{v}_i(t)$ and

$$\mathbf{b}(\mathbf{\Theta}, \Phi) = \begin{bmatrix} \mathbf{a}(\mathbf{\Theta}) & \exp(j\Phi) \\ \mathbf{a}(\mathbf{\Theta})^* & \exp(-j\Phi) \end{bmatrix}. \tag{17}$$

Expression (16) shows that vectors $\mathbf{b}(\mathbf{\Theta}_i, \Phi_{il})^{\otimes q}$ for $(1 \le l \le k_i)(1 \le i \le P)$ span the signal subspace of $\mathbf{C}_{2q,\tilde{\mathbf{x}}}$, whose rank r is such that

$$P \le r = \sum_{i=1}^{P} k_i \le P(q+1)$$

provided that vectors $\mathbf{b}(\mathbf{\Theta}_i, \Phi_{il})^{\otimes q}$ are linearly independent.

3.4.2 *Case of P $2q-$**th order**-$(q+1) - IQ-$**rank** source*

In the presence of P statistically independent sources, we deduce from (3) that $\mathbf{C}_{2q,\tilde{\mathbf{x}}}$ is given by (8) where $\mathbf{A}(\mathbf{\Theta}_i) = [\ \mathbf{b}(\mathbf{\Theta}_i, 0) \quad \mathbf{b}(\mathbf{\Theta}_i, \pi/2)\ ]$. Using (9) (10) and (12), we deduce that the signal subspace of $\mathbf{C}_{2q,\tilde{\mathbf{x}}}$ is spanned by the columns of matrices $(\mathbf{A}(\mathbf{\Theta}_i)^{\otimes q}) \times \mathbf{\Gamma}^q_{2^q,q+1}$ for $1 \le i \le P$. In the presence of P sources which are $2q-$**th order**-$(q+1) - IQ-$**rank**, it is possible to show that the signal subspace of $\mathbf{C}_{2q,\tilde{\mathbf{x}}}$ is spanned by particular linear combinations of the columns of $\mathbf{A}(\mathbf{\Theta}_i)^{\otimes q}$ which contain $[\mathbf{b}(\mathbf{\Theta}_i, 0)^{\otimes q}$ and $\mathbf{b}(\mathbf{\Theta}_i, \pi/2)^{\otimes q}]$ for $(1 \le i \le P)$ because the rank of $\mathbf{\Gamma}^q_{2^q,q+1}$ is $q+1$.

A consequence of the results of this section 3.4 is that for each source i of an arbitrary mixture of statistically independent $2q-$**th order**-$(q+1) - IQ-$**rank** source and $k-rectilinear$ sources such that $k_i \le q+1$, vector of the kind $\mathbf{b}(\mathbf{\Theta}_i, \Phi_{il})^{\otimes q}$ belong to the signal subspace for each source i.

4. THE NC-2Q-MUSIC ALGORITHMS

4.1 The NC1-2q-MUSIC algorithm

As vectors of the kind $\mathbf{b}(\mathbf{\Theta}_i, \Phi_{ij})^{\otimes q}$ for $(1 \leq i \leq P)$ belong to the signal subspace of $\mathbf{C}_{2q,\tilde{\mathbf{x}}}$ for arbitrary mixtures of $2q-$**th order-**$(q+1) - IQ-$**rank** and k_i-*rectilinear* sources such that $k_i \leq q+1$, we conjecture about the fact that this result remains true whatever the kind of sources in the mixture (this conjecture seems to be verified by simulations). In this context, the directions $\mathbf{\Theta}_i$ and the phases, Φ_{ij}, can be estimated with a MUSIC like algorithm [5], giving rise to the $NC1-2q-$MUSIC algorithm defined by

$$\left(\hat{\mathbf{\Theta}}_i, \hat{\Phi}_{ij}\right) = \min_{(\mathbf{\Theta},\Phi)} \left(J\left(\mathbf{\Theta}, \Phi\right)\right) \tag{18}$$

$$J\left(\mathbf{\Theta}, \Phi\right) = \frac{\left(\mathbf{b}\left(\mathbf{\Theta}, \mathbf{\Phi}\right)^{\otimes q}\right)^H \mathbf{\Pi}_{2q}\ \mathbf{b}\left(\mathbf{\Theta}, \mathbf{\Phi}\right)^{\otimes q}}{\left(\mathbf{b}\left(\mathbf{\Theta}, \mathbf{\Phi}\right)^{\otimes q}\right)^H \mathbf{b}\left(\mathbf{\Theta}, \mathbf{\Phi}\right)^{\otimes q}} \tag{19}$$

where $\mathbf{\Theta}$ is the vector of AOA of a source (azimuth and site), $\mathbf{\Phi}$ is a scalar phase term and $\mathbf{\Pi}_{2q}$ is the orthogonal projector on the noise subspace of $\mathbf{C}_{2q,\tilde{\mathbf{x}}}$ defined by

$$\mathbf{\Pi}_{2q} = \sum_{k=K+1}^{(2N)^q} \mathbf{e}_i \left(\mathbf{e}_i\right)^H$$

where $\mathbf{e}_i$ are the eigenvectors of $\mathbf{C}_{2q,\tilde{\mathbf{x}}}$ associated to the smallest eigen values and K is the rank of $\mathbf{C}_{2q,\tilde{\mathbf{x}}}$. The implementation of this algorithm requires a search on the phases and AOA parameters, which may be associated with a high complexity.

4.2 The NC2-2q-MUSIC algorithm

In order to reduce the computation cost of NC1-2q-MUSIC algorithm, we may prefer to estimate the phases directly from de AOA of the sources and to use an algorithm, called NC$2-2q-$MUSIC algorithm, which only implements a search procedure in AOAs. It can be shown that the $2q^{th}$ order steering vector $\mathbf{b}\left(\mathbf{\Theta}, \mathbf{\Phi}\right)^{\otimes q}$ can be rewrite as

$$\mathbf{b}\left(\mathbf{\Theta}, \Phi\right)^{\otimes q} = \quad \mathbf{U}^q\left(\mathbf{\Theta}\right)\ \mathbf{e}^q\left(\Phi\right) \tag{20}$$

$$\begin{cases} \mathbf{U}^q\left(\mathbf{\Theta}\right) = \left[\begin{array}{ccc} \mathbf{b}\left(\mathbf{\Theta}\right)^{q,0} & \cdots & \mathbf{b}\left(\mathbf{\Theta}\right)^{q,q} \end{array}\right] \\ \mathbf{e}^q\left(\Phi\right) = \left[\begin{array}{cccc} z^q & z^{q-2} & \cdots & z^{-q} \end{array}\right]^T \end{cases}$$

where $z = \exp\left(j\Phi\right)$ and

$$\mathbf{b}\left(\mathbf{\Theta}\right)^{q,k} = \quad \mathbf{U}_{q,k}\ \left(\mathbf{a}\left(\mathbf{\Theta}\right)^{\otimes q-k} \otimes \left(\mathbf{a}\left(\mathbf{\Theta}\right)^*\right)^{\otimes k}\right) \tag{21}$$

where $\mathbf{U}_{q,k}$ are $(2N)^q \times (q+1)$ permutation matrices. According to [7], the criterion of (18) can be reduced to

$$\hat{\mathbf{\Theta}}_i = \min_{(\mathbf{\Theta},\Phi)} \left(J_{opt}\left(\mathbf{\Theta}\right)\right) \tag{22}$$

$$J_{opt}\left(\mathbf{\Theta}\right) = \frac{\det\left(\mathbf{Q}_{q,1}\left(\mathbf{\Theta}\right)\right)}{\det\left(\mathbf{Q}_{q,2}\left(\mathbf{\Theta}\right)\right)} \tag{23}$$

where

$$\begin{aligned} \mathbf{Q}_{q,1}\left(\mathbf{\Theta}\right) &= \quad \left(\mathbf{U}^q\left(\mathbf{\Theta}\right)\right)^H \mathbf{\Pi}_{2q}\ \mathbf{U}^q\left(\mathbf{\Theta}\right) \\ \mathbf{Q}_{q,2}\left(\mathbf{\Theta}\right) &= \quad \left(\mathbf{U}^q\left(\mathbf{\Theta}\right)\right)^H\ \mathbf{U}^q\left(\mathbf{\Theta}\right) \end{aligned}$$

The criterion is then reduced to an AOA optimization.

4.3 Identifiability

We evaluate in this section the maximal number of sources that can be processed by the NC$1-2q-$MUSIC algorithm. This maximal number is obtained when all the sources are rectilinear. In this case, the HO Virtual Array (VA) theory presented in [4] for the $2q-$th order circular statistics of $\mathbf{x}(t)$ can be easily extended to the $2q-$th order circular statistics of $\tilde{\mathbf{x}}(t)$ and will be presented elsewhere. Using this extension and noting N_{2q}^{NC} the number of different virtual sensors (VSs) of the VA for the NC $2q$th-order array processing problem, it can be shown that

$$N_{2q}^{NC} \quad = \sum_{l=0}^{q} N_{2q}^{l}$$

where N_{2q}^{l} is the number of different VSs of the VA associated with the circular $2q$th-order array processing problem for the lth arrangement [4]. In practical applications, this number depends on the structure of the array manifold. In particular, when the array manifold has no particular structure, which is the case when there is a strong coupling between the array and the metallic support on which the array is installed (plane. . .) it is possible to show that

$$N_{2q}^{NC} \leq N_{2q,\max}^{NC}\left(US\right) \quad = \sum_{l=0}^{q} D(q-l, N) \times D(l, N)$$

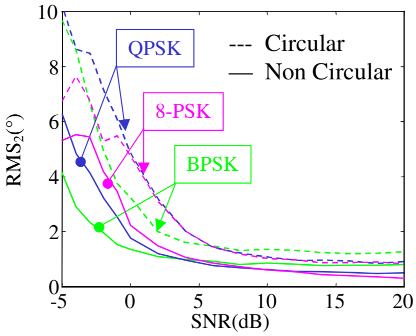
where US means UnStructured and $D(q, N)$ is defined by (13). When the array manifold has a particular structure, without mutual coupling and coupling between the array and the metallic support, the upper bound $N_{2q,\max}^{NC}\left(S\right)$, where S means structured, becomes lower than $N_{2q,\max}^{NC}\left(US\right)$.

In all cases, the maximal number of sources which may be processed by the NC$1-2q-$MUSIC algorithm is $P_{max} = N_{2q}^{NC} - 1$ rectilinear sources. For $k-rectilinear$ sources such that $k \leq q+1$, we obtain $P_{max} = (N_{2q}^{NC} - 1)/k$.

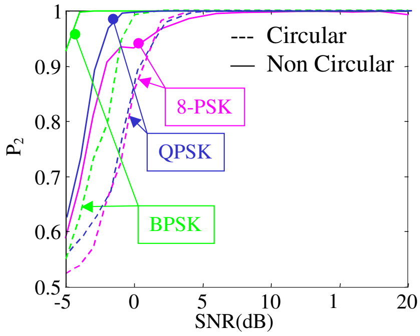
5. SIMULATIONS

We illustrate in this section the performance of the NC$2-2q-$MUSIC algorithm for $q = 2$. To this aim we consider a mixture of $P = 2$ statistically independent sources, having the same power and impinging on a uniform circular array of $N = 3$ sensors with a radius $R = 0.2\lambda$, where λ is the wavelenght. The angles of arrival of the two sources are $\mathbf{\Theta}_1 = (\theta_1 = 60°, \Delta_1 = 0°)$ and $\mathbf{\Theta}_2 = (\theta_2 = 90°, \Delta_2 = 0°)$ whereas their phases are such that $\Phi_1 = 0°$ and $\Phi_2 = 45°$ respectively. The signals are $n - PSK$ signals with the same square pulse shaped filter and with a symbol duration $T_s = 2T_e$ where T_e is the sample period. The first source is a BPSK($n = 2$) source, whereas the second source is either a BPSK, a QPSK($n = 4$) or a 8-PSK($n = 8$) source. Under these assumptions Figures 5 and 2 show respectively the variations of the probability of acceptable estimate and the Root Mean Square error (RMSE) of the AOA estimate of the source 2 and 1 respectively as a function of the SNR of the sources when either the classical $4 - MUSIC$ or the $NC2-4-MUSIC$ algorithm is used. Let us recall that the probability of acceptable estimate for the source p is defined by $P_p = \Pr(J_{opt}(\hat{\theta}_p) < \eta)$ where $\eta = 0.1$ is a threshold that remove the outliers, whereas the RMSE RMS_p for the source p is defined by $(RMS_p)^2 = E[(\hat{\theta}_p - \theta_p)^2 | J_{opt}\left(\hat{\theta}_p\right) < \eta]$. For the simulations, the 4^{th} order statistics are estimated from $K = 1000$ snapshots $\mathbf{x}(t_k)$ for $(1 \leq k \leq K)$ and the number of realizations is $L = 200$. Note the capability of the $NC2-4-MUSIC$ algorithm to estimate the AOA of all the sources in the mixture, even nonrectilinear sources, and the better performances of this algorithm with respect to

4 – *MUSIC* for both 4^{th}-order noncircular sources (BPSK, QPSK) and circular (8-PSK) sources.

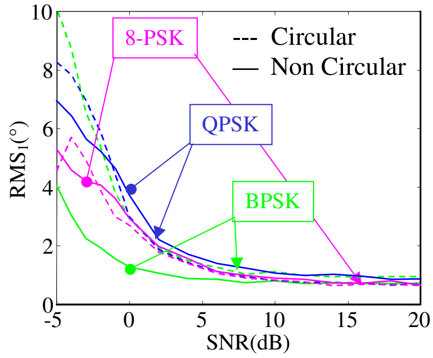


(a) RMS error

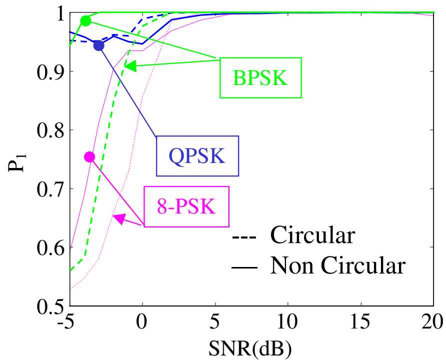


(b) Probability of detection

Figure 1: Performances of a source (BPSK, QPSK or 8-PSK) of direction $\theta_2 = 90°$ and phase $\Phi_2 = 45°$ in presence of a BPSK of direction $\theta_1 = 60°$ and phase $\Phi_1 = 0°$. Comparison between the classical and non-circular MUSIC-4 algorithms.



(a) RMS error



(b) Probability of detection

Figure 2: Performances of a source BPSK of direction $\theta_1 = 60°$ and phase $\Phi_1 = 0°$ in presence of a (BPSK, QPSK or 8-PSK) of direction $\theta_2 = 90°$ and phase $\Phi_2 = 45°$. Comparison between the classical and non-circular MUSIC-4 algorithms.

6. CONCLUSIONS

In this paper, two noncircular $2q^{th}$ order extensions of the $2q-$MUSIC algorithm have been presented for $q \geq 1$ without any assumption about the noncircularity properties of the sources not limited to PSK sources. For a given array of sensors and a given value of q, these NC extensions allow to process much more sources than the $2q-$MUSIC algorithm and improve the performance of AOA estimation for $2q^{th}$-order NC sources, not necessarily rectilinear. To our knowledge, the proposed algorithms are the first algorithm which can both take benefit of the potential noncircularity of the sources and accommodate with circular sources and this is true for arbitrary value of q.

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