# A NEW ERROR BOUND FOR MIMO DISCRETE-TIME STATE-SPACE TRANSVERSAL ESTIMATORS

*Yuriy S. Shmaliy, Oscar Ibarra-Manzano*

Electronics Department, Guanajuato University
Ctra. Salamanca-Valle, 3.5+1.8km, Palo-Blanco, 36855, Salamanca, Mexico
phone: + 52 (464) 647-01-95, email: shmaliy@ugto.mx,
web: www.ingenierias.ugto.mx

## ABSTRACT

We address an exact noise power gain (NPG) matrix for the $p$-shift linear time-variant transversal finite impulse response (FIR) estimator intended for filtering ($p = 0$), $p$-step prediction ($p > 0$), and $p$-lag smoothing ($p < 0$) of discrete-time $K$-state space system models with $M$ states measured. We also propose a new error bound (EB) formed in the three-sigma sense with the NPG and measurement noise variance. A fast iterative algorithm for NPG and EB of the $p$-shift unbiased FIR estimator is provided along. It is demonstrated that the unbiased FIR and Kalman estimates well range within a gap between EB and −EB.

## 1. INTRODUCTION

A distinct advantage of the transversal finite impulse response (FIR) estimators against the recursive infinite impulse response (IIR) ones is the imbedded bounded input/bounded output (BIBO) stability. This fact was mentioned by Johnson in [1], while extending the optimal Wiener filter theory to finite discrete time $n$, and supported with an asymptotic formula for the output noise power (variance) $\sigma^2_{\text{out}}$. Soon after, in [2], Blum generalized Johnson's result and showed that there exists an exact ratio of $\sigma^2_{\text{out}}$ to the input noise power $\sigma^2_{\text{in}}$. This ratio is now known as the noise power gain (NPG).

During decades, the NPG metric has been invoked to many investigations as a convenient measure of noise reduction and denoising. After Blum [2], Trench proved in [3] that NPG for white Gaussian noise is the sum of the square coefficients of the finite impulse response (FIR) filter gain $h(n)$ of length $N$,

$$NPG = \frac{\sigma^2_{\text{out}}}{\sigma^2_{\text{in}}} = \sum_{n=0}^{N-1} h^2(n)\,.$$

From [2], we also learn that polynomial signals with white Gaussian noise make NPG equal to the FIR filter gain at zero. These and other useful properties of NPG associated with polynomial signals were recently outlined in [4–6]. We meet NPG as a characteristic of FIR filters, predictors, smoothers, and differentiators in [4–13] and in many other papers. In a somewhat sophisticated way the concept of NPG was used in [14] while analyzing an unbiased impulse response estimator and in [15] to characterize errors in adaptive filters. Some authors determine NPG via the noise transfer function [16], following Trench [3] and the development made by Kuo [17]. It can also be noticed that NPG suitable for unbiased filtering is a special case of the noise figure [18] commonly used in wireless communications.

The aforementioned results relate to single input/single output (SISO) estimators. One of us discussed in [19] the NPG matrix for single input/multiple output (SIMO) ones. Still no NPG form was addressed for multiple input/multiple output (MIMO) estimators associated with $M$ states measured in the $K$-state model. The estimate error bound (EB) was not proposed for the MIMO estimators as well.

Below, we develop NPG for the $p$-shift time-variant MIMO estimator intended for filtering ($p = 0$), $p$-step prediction ($p > 0$), and $p$-lag smoothing ($p < 0$) of linear discrete time-varying state-space models. Based upon the NPG form proposed, we specify the EB in the three-sigma sense via the measurement noise variance and investigate it numerically. We show that the unbiased FIR and standard Kalman estimates range well within a gap between EB and −EB.

## 2. SIGNAL MODEL

Consider a time-varying model, measured in the presence of additive noise and represented with the state and observation equations, respectively,

$$\mathbf{x}_n = \mathbf{A}_n\mathbf{x}_{n-1} + \mathbf{B}_n\mathbf{w}_n\,, \tag{1}$$

$$\mathbf{y}_n = \mathbf{C}_n\mathbf{x}_n + \mathbf{D}_n\mathbf{v}_n\,, \tag{2}$$

where $\mathbf{x}_n \in \Re^K$ and $\mathbf{y}_n \in \Re^M$ are the state and observation vectors, respectively. Noise $\mathbf{w}_n \in \Re^P$ is zero mean, $E\{\mathbf{w}_n\} = \mathbf{0}$, with any distribution and known covariance function. Noise $\mathbf{v}_n \in \Re^M$ is also zero mean, $E\{\mathbf{v}_n\} = \mathbf{0}$, but represented in white Gaussian approximation with known variances, $\sigma_1^2, \ldots, \sigma_M^2$. Vectors $\mathbf{w}_n$ and $\mathbf{v}_n$ are supposed to be mutually independent and uncorrelated, $E\{\mathbf{w}_i\mathbf{v}_j^T\} = \mathbf{0}$, for all $i$ and $j$. Here, $\mathbf{A}_n \in \Re^{K\times K}$, $\mathbf{C}_n \in \Re^{M\times K}$, $\mathbf{B}_n \in \Re^{K\times P}$, and $\mathbf{D}_n \in \Re^{M\times M}$.

On a finite interval of $N$ points, from $m = n-N+1$ to $n$, the $p$-shift estimate of $\mathbf{x}_n$ can be found via the convolution [21] with the $K\times MN$ FIR gain $\mathbf{H} \triangleq \mathbf{H}(n,N,p)$ [19,20]. For such an estimator, the estimate noise $\mathbf{e}_n$ (output) caused by $\mathbf{v}_n$ (input) can be determined at $n+p$ as

$$\mathbf{e}_{n+p} = \mathbf{H}\mathbf{V}_{n,m}\,, \tag{3}$$

where $\mathbf{H}$ must be substituted with $\mathbf{H}\mathbf{D}_{n,m}$ if $\mathbf{D}_n$ is not identity, $\mathbf{D}_{n,m} = \text{diag}\Big(\underbrace{\mathbf{D}_n\,\mathbf{D}_{n-1}\,\ldots\,\mathbf{D}_m}_{N}\Big)$, and the $MN\times 1$ observation noise vector is

$$\mathbf{V}_{n,m} = \left[\mathbf{v}_n^T\,\mathbf{v}_{n-1}^T\,\ldots\,\mathbf{v}_m^T\right]^T. \tag{4}$$

The estimate noise $K\times K$ covariance matrix $\mathbf{J} \triangleq \mathbf{J}(n,N,p)$ can thus be written as

$$\mathbf{J} = E\{\mathbf{e}_{n+p}\mathbf{e}_{n+p}^T\} \quad (5a)$$
$$= \mathbf{H}E\{\mathbf{V}_{n,m}\mathbf{V}_{n,m}^T\}\mathbf{H}^T , \quad (5b)$$

where $E\{x\}$ means an average of $x$.

The problem now formulates as follows. We would like to investigate (5b) and derive an exact NPG matrix in order to evaluate the main and cross denoising effects in the estimator channels. We also wish to find a computationally efficient form for NPG and consider a typical example.

### 2.1 Noise Power Gain in State Space

In order to specify NPG in state space, (5b) can be transformed for the white Gaussian components in $\mathbf{V}_{n,m}$ to

$$\mathbf{J} = \mathbf{H}\Phi\mathbf{H}^T , \quad (6)$$

where

$$\Phi = E\{\mathbf{V}_{n,m}\mathbf{V}_{n,m}^T\} \quad (7a)$$
$$= \text{diag}\Big(\underbrace{\sigma_1^2 \dots \sigma_M^2\ \sigma_1^2 \dots \sigma_M^2 \dots \sigma_1^2 \dots \sigma_M^2}_{MN}\Big) . \quad (7b)$$

By (7b), the covariance matrix (6) acquires an equivalent form of

$$\mathbf{J} = \sigma_1^2\mathbf{K}_1 + \sigma_2^2\mathbf{K}_2 + \cdots + \sigma_M^2\mathbf{K}_M , \quad (8)$$

in which $\mathbf{K}_k \triangleq \mathbf{K}_k(n,N,p)$, $k \in [1,M]$, is the generic $K \times K$ NPG matrix associated with the $k$th measured state,

$$\mathbf{K}_k = \mathbf{H}_k\mathbf{H}_k^T , \quad (9)$$

where the generic $K \times N$ gain $\mathbf{H}_k \triangleq \mathbf{H}_k(n,N,p)$ is composed by $K$th columns of $\mathbf{H}$ starting with the $k$th one. The generic NPG matrix is hence formed as $\mathbf{K}_k = \frac{1}{\sigma_k^2}\mathbf{J}_k$, where $\mathbf{J}_k$ is associated with the $k$th constituent in the series (8).

Most generally, $\mathbf{H}_k$ can be written as

$$\mathbf{H}_k = \left[\ \mathbf{h}_{k1}^T \quad \dots \quad \mathbf{h}_{kk}^T \quad \dots \quad \mathbf{h}_{kK}^T\ \right]^T , \quad (10)$$

where $\mathbf{h}_{kv} \triangleq \mathbf{h}_{kv}(n,N,p)$, $v \in [1,K]$, is

$$\mathbf{h}_{kv} = \left[\ h_{kv0} \quad h_{kv1} \quad \dots \quad h_{kv(N-1)}\ \right] \quad (11)$$

and the $p$-shift generic component $h_{kvi} \triangleq h_{kvi}(n,N,p)$, $i \in [0,N-1]$, represents an estimator channel gaining the $k$th input to the $v$th output at $n$. The $K \times K$ NPG matrix (9) can hence be represented in the form of

$$\mathbf{K}_k = \begin{bmatrix} K_{k(11)} & \dots & K_{k(1k)} & \dots & K_{k(1K)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ K_{k(k1)} & \dots & K_{k(kk)} & \dots & K_{k(kK)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ K_{k(K1)} & \dots & K_{k(Kk)} & \dots & K_{k(KK)} \end{bmatrix} . \quad (12)$$

The main component $K_{k(kk)} = \mathbf{h}_{kk}\mathbf{h}_{kk}^T$ occupies a central place in (12), representing the $k$-to-$k$ channel. Other important ones, $K_{k(vv)} = \mathbf{h}_{kv}\mathbf{h}_{kv}^T$, $v \neq k$, placed on the main diagonal characterize the $k$-to-$v$ channels. The remaining ones $K_{k(vg)} = \mathbf{h}_{kv}\mathbf{h}_{kg}^T$, $g \neq k \neq v$, $g \in [1,K]$, play rather an auxiliary role. They represent interactions in the estimator channels and complete the noise reduction picture.

## 3. ERROR BOUND

Provided $\mathbf{K}_k$, the estimate error bound (EB) can be specialized in the three-sigma sense as follows:

$$EB_{k(vg)}(n,N,p) = 3\sigma_k K_{k(vg)}^{1/2}(n,N,p) , \quad (13)$$

where $\sigma_k$ is the measurement noise standard deviation, $K_{k(vg)}$ is a component in (12), and an index $k(vg)$ means that EB is specified for the interacting $v$th and $g$th estimator channels via measurement of the $k$th state.

By the components of (12) placed on the main diagonal, the relevant value $EB_{k(vv)}(n,N,p)$ characterizes denoising in the $k-$to$-v$ channel.

In what follows, we specify (12) and (13) for the time-variant unbiased FIR estimator and compare its estimates with the Kalman ones, considering a simple example.

### 3.1 EB for the Unbiased FIR Estimator

Observing (8), one infers that $\mathbf{J}$ becomes zero valued if all of the generic gains (9) acquire zeroth components. Because noise prevents such an ideal situation, estimators are commonly optimized in different sense such as the minimum MSE, minimum variance, or minimum bias. Below, we derive NPG for the unbiased FIR estimator with $\mathbf{D}_n$ identity.

It follows from [19, Eq. (33)] that the unbiased gain for time-invariant models is the product of a power of $\mathbf{A}$ and an auxiliary matrix composed with $\mathbf{A}$ and $\mathbf{C}$. If to substitute the former with the multiplication of the time-variant matrices $\mathbf{A}_n$ and $\mathbf{C}$ with $\mathbf{C}_n$ then the gain becomes suitable for time-varying models; that is,

$$\bar{\mathbf{H}} = \prod_{i=0}^{n+p-m-1} \mathbf{A}_{n+p-i}(\mathbf{C}_{n,m}^T\mathbf{C}_{n,m})^{-1}\mathbf{C}_{n,m}^T , \quad (14)$$

where the $MN \times K$ matrix $\mathbf{C}_{n,m}$ is

$$\mathbf{C}_{n,m} = \begin{bmatrix} \mathbf{C}_n \prod_{i=0}^{n-m-1} \mathbf{A}_{n-i} \\ \mathbf{C}_{n-1} \prod_{i=1}^{n-m-1} \mathbf{A}_{n-i} \\ \vdots \\ \mathbf{C}_{m+1}\mathbf{A}_{m+1} \\ \mathbf{C}_m \end{bmatrix} . \quad (15)$$

Saving the $k$th row in each $\mathbf{C}_i$, $i \in [m,n]$, as $\tilde{\mathbf{C}}_i$ allows us to specify the generic gain via (14) as

$$\bar{\mathbf{H}}_k = \prod_{i=0}^{n+p-m-1} \mathbf{A}_{n+p-i}(\tilde{\mathbf{C}}_{n,m}^T\tilde{\mathbf{C}}_{n,m})^{-1}\tilde{\mathbf{C}}_{n,m}^T , \quad (16)$$

in which $\tilde{\mathbf{C}}_{n,m}$ is $\mathbf{C}_{n,m}$ thinned by $\tilde{\mathbf{C}}_i$. Then substituting (16) to (9) gives us the generic NPG matrix

$$\bar{\mathbf{K}}_k = \prod_{i=0}^{n+p-m-1} \mathbf{A}_{n+p-i}(\tilde{\mathbf{C}}_{n,m}^T \tilde{\mathbf{C}}_{n,m})^{-1} \left( \prod_{i=0}^{n+p-m-1} \mathbf{A}_{n+p-i} \right)^T . \quad (17)$$

Further representing (17) as (12) determines the required components. Because NPG must be provided in line with the estimate, a computational problem may arise in the batch form (17) when $N$ is large. The following theorem states a fast iterative algorithm for (17), which proof is postponed to Appendix A,

**Theorem 1** *Given the generic NPG matrix (17), then its fast computation can be provided iteratively by*

$$\bar{\mathbf{K}}_k(l,N,p) = \mathbf{A}_{l+p}[\mathbf{\Theta}_l + \bar{\mathbf{K}}_k^{-1}(l-1,N,p)]^{-1}\mathbf{A}_{l+p}^T, \quad (18)$$

*where $l$ ranges from $K$ to $n$, an initial gain value $\bar{\mathbf{K}}_k(K-1,N,p)$ is computed using (17), $\mathbf{\Theta}_l$ is given by (A.10), and the true gain corresponds to $l = n$.*

Provided fast computation of NPG, by (18) (theorem 1), the error bounds for main and interacting channels can easily be computed employing (13).

## 4. EXAMPLE

To illustrate efficiency of the error bound (13) proposed, we consider below a simple case of (1) and (2) with measurement of the first state saving only the first term in (8). The model is specialized with $\mathbf{B}_n = \mathbf{I}$, $\mathbf{C}_n = [1\ 0]$, $\mathbf{D}_n$ identity, and

$$\mathbf{A}_n = \begin{bmatrix} 1 & (1+d_n)\tau \\ 0 & 1 \end{bmatrix}, \quad (19)$$

where we let $d_n = 5$ for $160 \leqslant n \leqslant 200$ and $d_n = 0$ otherwise. The variances of the independent and uncorrelated zero mean white noise sequences in the first state $x_{1n}$ and second state $x_{2n}$ were allowed to be $\sigma_{x1}^2 = 10^{-4}$ and $\sigma_{x2}^2 = 4 \times 10^{-6}/\mathrm{s}^2$, respectively. Measurement of the first state was organized with noise having the variance $\sigma_1^2 = 0.225$.

The process was simulated at 400 points. The time-invariant and time-varying unbiased FIR and Kalman filters were applied in order to estimate the first state. In the time-varying case, we used (19) and the time-invariant one was organized by letting $d_n = 0$ in (19) over all measurement. Following [22], optimum averaging intervals were found to be $N_{\mathrm{opt}} = 29$ and $N_{\mathrm{opt}} = 14$ beyond and within the variation region, respectively.

Figure 1 illustrates typical errors in the time-invariant (Fig. 1a) and time-varying (Fig. 1b) filtering. Here, EB (dashed) was calculated by (13) with $k = 1$ and $v = 1$. As can be seen, mismatch between the estimator and model in the region of variations causes errors in the time-invariant estimates to exceed the bounds (Fig. 1a). In the time-varying case (Fig. 1b), the estimate well range within an gap between EB and $-$EB. One can notice that there is no substantial difference between the errors produced by the Kalman and unbiased FIR estimates (Fig. 1), since the latter is near optimal. This observation suggests that EB can be applied for Kalman filtering as well.

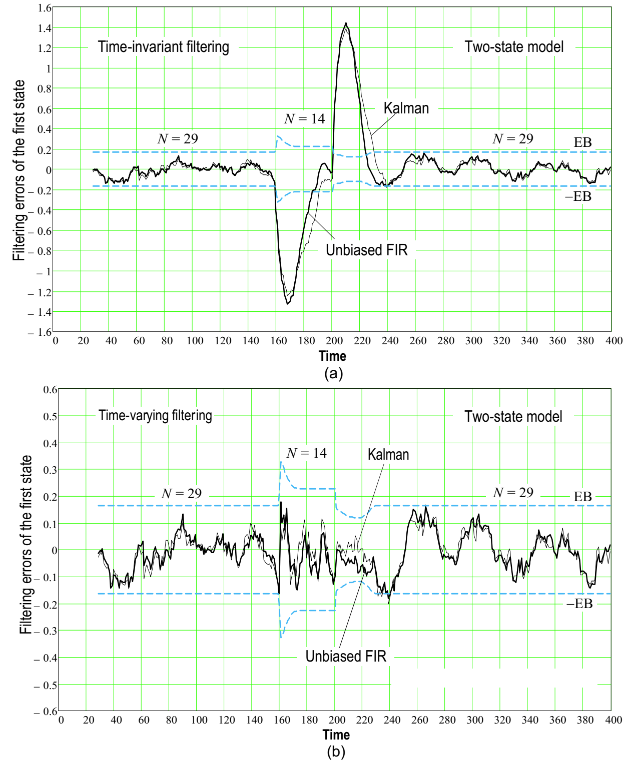


Figure 1: Typical estimate errors of the first state produced by the two-state unbiased FIR and Kalman filters: (a) time-invariant filtering and (b) time-varying filtering. EB and $-$EB are dashed.

## 5. CONCLUSION

In this paper, we specified the NPG matrix for a $p$-shift discrete time-variant state-space MIMO FIR estimator. A computationally efficient iterative algorithm for the generic NPG was provided along. Employing the concept of NPG, we also specialized the error bound EB (13) in the three-sigma sense.

The $K_{k(kk)}$ component is principle in the generic NPG matrix (12). It characterizes denoising in the estimator $k$$-$to$-$$k$ channel. Other critical ones $K_{k(vv)}$ placed on the main diagonal of (12) characterize the $k$-to-$v$, $v \neq k$, channel. The remaining components $K_{k(vg)}$, $k \neq v \neq g$, represent interactions in the estimator channels and can be invoked to complete the noise reduction picture, although it has rather a theoretical meaning. A numerical example given for $k = v = g = 1$ confirms that the estimate errors are efficiently bounded with EB (13).

Overall, one may deduce that EB formed via NPG and the measurement noise variance can serve as an efficient measure of errors in optimal and suboptimal transversal estimators. They may also be used to bound the recursive Kalman estimates. Therefore, deeper studies of EB should certainly be a special topic for further investigation.

## A. ITERATIVE COMPUTATION OF THE GENERIC NPG MATRIX

Consider (17), substitute $n$ with an iterative variable $l$ and rewrite $\bar{\mathbf{K}}_k(n,N,p)$ (17) as

$$\bar{\mathbf{K}}_k(l,N,p) = \prod_{i=0}^{l+p-m-1} \mathbf{A}_{l+p-i}(\tilde{\mathbf{C}}_{l,m}^T \tilde{\mathbf{C}}_{l,m})^{-1} \times \left( \prod_{i=0}^{l+p-m-1} \mathbf{A}_{l+p-i} \right)^T , \quad \text{(A.1)}$$

where $l$ ranges from $K$ to $n$, because the inverse in (A.1) does not exist with $l < K$. Assign

$$\mathbf{P}_l^{-1} = \tilde{\mathbf{C}}_{l,m}^T \tilde{\mathbf{C}}_{l,m} , \quad \text{(A.2)}$$

employ the matrix inversion lemma [23],

$$(\mathbf{B}+\mathbf{D})^{-1} = \mathbf{B}^{-1} - \mathbf{B}^{-1}(\mathbf{I}+\mathbf{D}\mathbf{B}^{-1})^{-1}\mathbf{D}\mathbf{B}^{-1} , \quad \text{(A.3)}$$

and represent $\mathbf{P}_l$ via (15) as follows:

$$\mathbf{P}_l = \mathbf{P}_{l-1} - \mathbf{P}_{l-1} \left[ \mathbf{I} + \left( \prod_{i=0}^{l-m-1} \mathbf{A}_{l-i} \right)^T \tilde{\mathbf{C}}_l^T \tilde{\mathbf{C}}_l \times \prod_{i=0}^{l-m-1} \mathbf{A}_{l-i} \mathbf{P}_{l-1} \right]^{-1} \left( \prod_{i=0}^{l-m-1} \mathbf{A}_{l-i} \right)^T \times \tilde{\mathbf{C}}_l^T \tilde{\mathbf{C}}_l \prod_{i=0}^{l-m-1} \mathbf{A}_{l-i} \mathbf{P}_{l-1} . \quad \text{(A.4)}$$

Introduce

$$\mathbf{F}_l = \prod_{i=0}^{l-m-1} \mathbf{A}_{l-i} \mathbf{P}_l \left( \prod_{i=0}^{l-m-1} \mathbf{A}_{l-i} \right)^T ,$$

$$\mathbf{F}_{l-1} = \prod_{i=1}^{l-m-1} \mathbf{A}_{l-i} \mathbf{P}_{l-1} \left( \prod_{i=1}^{l-m-1} \mathbf{A}_{l-i} \right)^T ,$$

apply to (A.4), assign $\mathbf{\Xi}_l = \mathbf{A}_l^T \tilde{\mathbf{C}}_l^T \tilde{\mathbf{C}}_l \mathbf{A}_l$, follow [19], and arrive at the recursive form of

$$\mathbf{F}_l = \mathbf{A}_l \mathbf{F}_{l-1} \mathbf{A}_l^T - \mathbf{A}_l \mathbf{F}_{l-1} (\mathbf{I} + \mathbf{\Xi}_l \mathbf{F}_{l-1})^{-1} \times \mathbf{\Xi}_l \mathbf{F}_{l-1} \mathbf{A}_l^T . \quad \text{(A.5)}$$

Next, refer to (A.2) and (A.5) and rewrite (A.1) as

$$\bar{\mathbf{K}}_k(l,N,p) = \mathbf{\Gamma}_l \mathbf{A}_l \mathbf{F}_{l-1} \mathbf{A}_l^T \mathbf{\Gamma}_l^T - \mathbf{\Gamma}_l \mathbf{A}_l \mathbf{F}_{l-1} \times (\mathbf{I} + \mathbf{\Xi}_l \mathbf{F}_{l-1})^{-1} \mathbf{\Xi}_l \mathbf{F}_{l-1} \mathbf{A}_l^T \mathbf{\Gamma}_l^T , \quad \text{(A.6)}$$

where $\mathbf{\Gamma}_l \triangleq \mathbf{\Gamma}_l(p)$ depends on $p$ as

$$\mathbf{\Gamma}_l = \begin{cases} \left( \prod_{i=0}^{|p|-1} \mathbf{A}_{l-i} \right)^{-1} , & p < 0 \quad \text{(smoothing)} \\ \mathbf{I} , & p = 0 \quad \text{(filtering)} \\ \prod_{i=0}^{p-1} \mathbf{A}_{l+p-i} , & p > 0 \quad \text{(prediction)} \end{cases} . \quad \text{(A.7)}$$

The gain at $l-1$ can now be written as

$$\bar{\mathbf{K}}_k(l-1,N,p) = \mathbf{A}_{l+p}^{-1} \mathbf{\Gamma}_l \mathbf{A}_l \mathbf{F}_{l-1} \mathbf{A}_l^T \mathbf{\Gamma}_l^T \mathbf{A}_{l+p}^{-T} \quad \text{(A.8)}$$

that allows us to transform (A.6) to

$$\bar{\mathbf{K}}_k(l,N,p) = \mathbf{A}_{l+p} \bar{\mathbf{K}}_k(l-1,N,p) \mathbf{A}_{l+p}^T - \mathbf{A}_{l+p} \bar{\mathbf{K}}_k(l-1,N,p) \times [\mathbf{I} + \mathbf{\Theta}_l \bar{\mathbf{K}}_k(l-1,N,p)]^{-1} \mathbf{\Theta}_l \bar{\mathbf{K}}_k(l-1,N,p) \mathbf{A}_{l+p}^T , \quad \text{(A.9)}$$

where $\mathbf{\Theta}_l$ is given by

$$\mathbf{\Theta}_l = \mathbf{A}_{l+p}^T \mathbf{\Gamma}_l^{-T} \tilde{\mathbf{C}}_l^T \tilde{\mathbf{C}}_l \mathbf{\Gamma}_l^{-1} \mathbf{A}_{l+p} . \quad \text{(A.10)}$$

Note that, for time-invariant models, (A.9) transforms to [19, Eq. (58)]. By (A.3), the form (A.9) readily converts to (18).

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