ANALYSIS OF NORMALIZED CORRELATION ALGORITHM FOR ADAPTIVE FILTERS IN IMPULSIVE NOISE ENVIRONMENTS

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ABSTRACT

This paper proposes normalized correlation algorithm (NCA) for complex-domain adaptive filters with Gaussian inputs. Stochastic models are presented for two types of impulse noise intruding adaptive filters: one in observation noise and another at filter input. Performance analysis of the NCA is developed to derive difference equations for calculating transient and steady-state convergence behavior. Through experiment with simulations and theoretical calculations of filter convergence for some examples, we demonstrate high robustness of the NCA in impulsive noise environments. Good agreement between simulated and theoretical convergence proves the validity of the analysis.

1. INTRODUCTION

Without adaptive filtering technology, virtually no recent advanced services could have been offered by many latest information and communication systems in which adaptive filters play a crucial role in economically realizing essential functions and required performance. In fact, they are used in broadband internet access systems, digital mobile systems and digital TV broadcasting systems, to name a few.

Historically, the first adaptation algorithm practically applied to adaptive filters was the LMS algorithm which has born many "children" such as the NLMS algorithm, the sign algorithm (SA), the sign-sign algorithm (SSA), etc. Among these algorithms, the LMS algorithm is most intensively studied in the literature, e.g. [1], [2]. Many implementers are attracted to adopt the LMS algorithm in their systems, because its performance and detailed design practices have already been well studied and established.

Although the LMS algorithm exhibits fastest convergence speed for a required level of steady-state error, it is known to be vulnerable in nature to disturbances, e.g. impulse noise that intrudes adaptive filtering systems [3], [4]. Two types of impulse noise are identified: one is present in observation noise and another at filter input.

In adaptive filters defined in the real-number domain, the SA [5], the signed regressor LMS algorithm (SRA) [6] and others [7], [8] are proven effective in combating such impulse noise that seriously degrades the filter performance. Among them, the SSA is known to be highly robust against *both* types of impulse noise stated above [9]. However, the SSA has a drawback of considerably slow convergence compared with the LMS algorithm. In the complex-number domain, the author studied least mean modulus (LMM) algorithm [10] and correlation phase algorithm (CΦA) [11] which are counterparts of the SA and SSA, respectively.

In this paper, we propose a "normalized" type adaptation algorithm named *normalized correlation algorithm* (NCA) defined in the complex-number domain that is expected to be highly robust against both the impulsive observation noise and impulse noise at filter input.

The remaining part of the paper is organized as follows. In Section 2, the proposed NCA is formulated. Section 3 presents stochastic models for two types of impulse noise that intrude adaptive filters, and provides simulated performance for the NCA in the presence of impulse noise. In Section 4, we develop transient and steady-state analysis of the NCA with Gaussian inputs, deriving difference equations for tap weight misalignment for a small number as well as for a large number of tap weights. Section 5 provides results of experiment with simulations and theoretical calculations of filter convergence in terms of excess mean square error that demonstrate the robustness of the NCA in impulsive noise environments and prove the validity of the analysis. Finally, Section 6 concludes the paper.

2. NORMALIZED CORRELATION ALGORITHM

In this section, we formulate the proposed NCA.

Let our complex-domain adaptive filter be of an FIR-type for use in identification of unknown stationary systems. First we define correlation between the conjugated error signal $e^*(n)$ and the filter reference input at the kth tap $a(n-k)$ as

$$z_k(n) = e^*(n)\, a(n-k) \quad k = 0, 1, \cdots, N-1. \qquad (1)$$

Then, we form a correlation vector

$$\mathbf{z}(n) = [z_0(n) \cdots z_k(n) \cdots z_{N-1}(n)]^T = e^*(n)\,\mathbf{a}(n) \qquad (2)$$

with $\mathbf{a}(n) = [a(n) \cdots a(n-k) \cdots a(n-N+1)]^T$ being the filter reference input vector. Using (2), let the update equation of tap weight vector $\mathbf{c}(n)$ be given by

$$\mathbf{c}(n+1) = \mathbf{c}(n) + \alpha_c\, \mathbf{z}(n) / D[\mathbf{z}(n)], \qquad (3)$$

where α_c is the step size and the denominator on the right-hand side is given by

$$D[\mathbf{z}(n)] = \sum_{l=0}^{N-1} |\, z_l(n)\,| \qquad (4)$$

which is defined as a sum of moduli of the correlation (1) over the filter taps. Note that $D[\mathbf{z}(n)]$ can alternatively be denoted by $|\,\mathbf{z}(n)\,|$ or $\|\,\mathbf{z}(n)\,\|_1$ ("1-norm"). Since in (3) the correlation is divided by the normalizing factor (4), we name this adaptation algorithm *normalized correlation algorithm* (NCA).

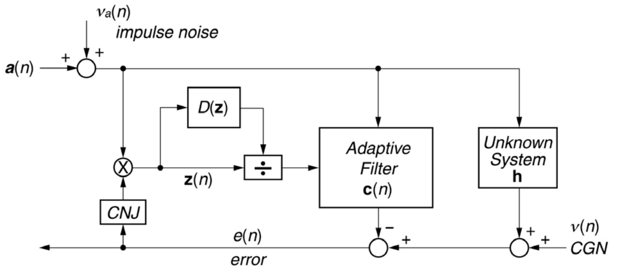

Fig.1 Schematic diagram for the NCA.

In the above formulae, n is the discrete time instant, N is the number of tap weights and $a(n)$ is the filter reference input of a complex-valued Gaussian process, colored in general. The error signal is given by $e(n) = \epsilon(n) + v(n)$, $\epsilon(n) = \boldsymbol{\theta}^H(n)\,\mathbf{a}(n)$ is the excess error, $\boldsymbol{\theta}(n) = \mathbf{h} - \mathbf{c}(n)$ is the tap weight misalignment vector, $\mathbf{h}$ is the response vector of the unknown stationary system and $v(n)$ is the additive observation (measurement) noise. Fig. 1 is a schematic diagram of an adaptive filter using the NCA.

3. IMPULSE NOISE MODELS AND NCA

In this section, we present impulse noise models that are stochastic in nature. We identify two types of impulse noise that intrude adaptive filtering systems: one in observation noise and another at filter input. Simulated convergence for the NCA in the presence of impulse noise is shown.

3.1 Impulsive Observation Noise

The impulse noise found in the additive observation noise is often modelled as *contaminated Gaussian noise* (CGN) [12] that is mathematically a combination of two independent Gaussian noise sources, i.e.,

$v^{(0)}(n)$: Gaussian noise source #0 with variance $\sigma^2 v^{(0)}$ and probability of occurrence $pv^{(0)}$,

$v^{(1)}(n)$: Gaussian noise source #1 with variance $\sigma^2 v^{(1)}$ and probability of occurrence $pv^{(1)}$.

Note that $pv^{(0)} + pv^{(1)} = 1$ holds. Usually, $\sigma^2 v^{(1)} >> \sigma^2 v^{(0)}$ and $pv^{(1)} < pv^{(0)}$. For "pure" Gaussian noise, $pv^{(1)} = 0$ and $\sigma^2 v = \sigma^2 v^{(0)}$.

3.2 Impulse Noise at Filter Input

The "noisy" filter input $b(n)$ with impulse noise added is given by

$$b(n) = a(n) + \tau(n)\, va(n) \text{ [11]},$$

where $\tau(n)$ is an independent Bernoulli random variable that governs the occurrence of the impulse noise. $\tau(n)$ takes 1 with probability of occurrence pva and 0 with $1 - pva$. The impulse noise $va(n)$ itself is assumed to be a White & Gaussian process independent of $a(n)$, and its variance is $\sigma^2 va = E[|va(n)|^2]/2$.

3.3 NCA in the Presence of Impulse Noise – Simulation Results –

Based upon the models above, we run simulations of filter convergence for *Example #1* below in the absence as well as in the presence of *either* type of impulse noise. Simulation result is an ensemble average of square excess error $<|\epsilon(n)|^2>/2$ over 1000 independent runs of filter convergence.

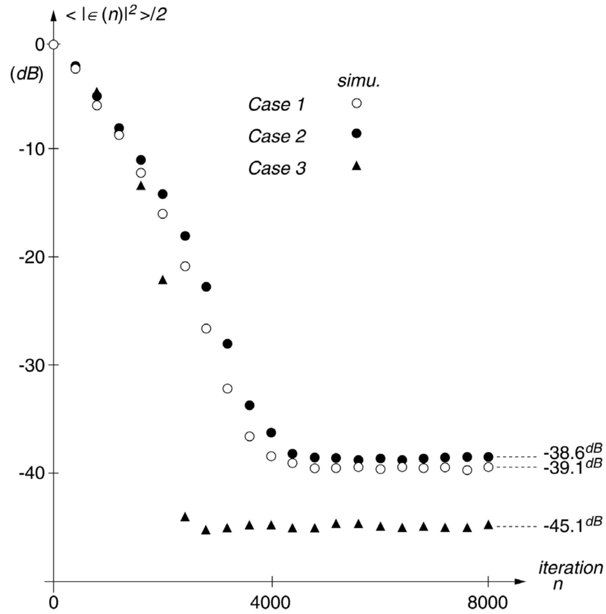

Fig. 2 Adaptive filter convergence – NCA.
(*Example #1*, $N = 4$, Cases 1, 2 & 3)

Example #1 $N = 4$

filter reference input: AR1 Gaussian process with variance $\sigma^2_a = 1$ (0 dB) and regression coefficient $\eta = 0.5$

step size: $\alpha_c = 2^{-9}$

Case 1: "pure" Gaussian noise $\sigma^2_v = 0.01$
no impulse noise at filter input

Case 2: CGN $\sigma^2_v{}^{(0)} = 0.01$; $p_v^{(0)} = 0.9$
$\sigma^2_v{}^{(1)} = 10$; $p_v^{(1)} = 0.1$
no impulse noise at filter input

Case 3: "pure" Gaussian noise $\sigma^2_v = 0.01$
impulse noise at filter input
$\sigma^2 va = 1000$; $pva = 0.1$

Fig. 2 shows simulated filter convergence for *Example #1*. In Case 1 (no impulse noise) the steady-state ensemble average $<|\epsilon(\infty)|^2>/2$ is –39.1 dB, whereas in Case 2 (CGN) this value is –38.6 dB, the increase in the steady-state square excess error being only half a dB. In Case 3 (impulse noise at filter input), $<|\epsilon(\infty)|^2>/2$ = –45.1 dB which is even smaller than that for Case 1 by 6 dB. These simulation results demonstrate the high robustness of the NCA against either type of impulse noise. Filter convergence in the presence of *both* types of impulse noise will be presented in Section 5.

4. PERFORMANCE ANALYSIS

In this section, transient and steady-state performance of filter convergence for the NCA is analyzed. For ease of analysis, we assume *absence* of impulse noise. Due to space limitations, detailed derivation process cannot be fully described, but only main results are summarized. However, the validity of the analysis in this section is verified through experiment in Section 5

4.1 Assumptions

For the analysis to be developed in this section and for the experiment in the next section, we make the following assumptions.

A1: The filter reference input $a(n) = a_R(n) + ja_I(n)$ is a zero mean complex-valued stationary Gaussian process, colored in general, with variance $\sigma^2 a = E[a^2_R(n)] = E[a^2_I(n)]$. $\mathbf{a}_R(n)$ and $\mathbf{a}_I(n)$ are mutually independent vectors with zero mean, covariance matrix $\mathbf{Ra} = E[\mathbf{a}_R(n)\mathbf{a}^T_R(n)] = E[\mathbf{a}_I(n)\mathbf{a}^T_I(n)]$, and $E[a_R(n-k)a_I(n-\kappa)] = 0$ for any k and κ. Note that $E[\mathbf{a}(n)\mathbf{a}^H(n)] = 2\,\mathbf{Ra}$ and $E[|\,a(n)\,|^2] = 2\,\sigma^2 a$.

A2: The additive observation noise (CGN in general) $v(n) = v_R(n) + jv_I(n)$ is stationary and independent of the filter input with variance $\sigma^2 v = E[v^2_R(n)] = E[v^2_I(n)]$.

A3: The impulse noise at filter input is an independent White & Gaussian process with variance $\sigma^2 va$ and probability of occurrence pva.

A4: The filter reference input $\mathbf{a}(n)$ and the tap weights $\mathbf{c}(n)$ are mutually independent. (*Independence Assumption*)

A5: The error $e(n)$ and the filter input $\mathbf{a}(n)$ are jointly Gaussian distributed [5], [6].

Assumption **A4** is frequently used in many papers for ease of analysis.

4.2 Difference Equations for Tap Weight Misalignment

From (3) we derive the following update equation for the tap weight misalignment vector $\theta(n)$.

$$\theta(n+1) = \theta(n) - \alpha_c\, e^*(n) / |\,e(n)\,| \times \mathbf{a}(n) / D[\mathbf{a}(n)], \quad (5)$$

where $D[\mathbf{a}(n)] = \sum_{l=0}^{N-1} |\,a(n-l)\,|$. Then, from (5), under the assumptions above we derive a set of difference equations for the mean vector $\mathbf{m}(n) = E[\theta(n)]$ and the second-order moment matrix $\mathbf{K}(n) = E[\theta(n)\theta^H(n)]$:

$$\mathbf{m}(n+1) = \mathbf{m}(n) - \alpha_c\, \mathbf{p}(n) \quad (6)$$

$$\mathbf{K}(n+1) = \mathbf{K}(n) - \alpha_c\,[\mathbf{V}(n) + \mathbf{V}^H(n)] + \alpha^2_c\, \mathbf{Ta}, \quad (7)$$

where $\mathbf{p}(n) = E\{e^*(n)/|\,e(n)\,| \times \mathbf{a}(n)/D[\mathbf{a}(n)]\}$, $\mathbf{V}(n) = E\{e^*(n)/|\,e(n)\,| \times \mathbf{a}(n)/D[\mathbf{a}(n)] \times \theta^H(n)\}$ and $\mathbf{Ta} = E[\mathbf{a}(n)\mathbf{a}^H(n)/D[\mathbf{a}(n)]^2]$.

As a measure of how accurately the adaptive filter identifies the unknown system, we often use *excess mean square error* (EMSE) as defined by

$$\varepsilon(n) = E[|\,\epsilon(n)\,|^2]/2 = \mathrm{tr}[\mathbf{Ra}\,\mathbf{K}(n)]$$

with tr(·) being trace of a matrix.

Now, let us calculate the following conditional expectation.

$$E\{e^*(n)/|\,e(n)\,| \times \mathbf{a}(n) / D[\mathbf{a}(n)] \,\big|\, \theta(n)\}$$
$$= (2/\pi)^{1/2} \int_0^{\infty} E[e^*(n)/|\,e(n)\,| \times \mathbf{a}(n) \exp\{-(u^2/2) D[\mathbf{a}(n)]^2\}]du.$$

For further calculation, we approximate $D[\mathbf{a}(n)]^2$ by $\|\,\mathbf{a}(n)\,\|^2 = \mathbf{a}^H(n)\mathbf{a}(n)$ as follows.

$$D[\mathbf{a}(n)]^2 \cong \beta^2\, \mathbf{a}^H(n)\mathbf{a}(n), \quad (8)$$

where detailed calculation of a coefficient β^2 is given in **APPENDIX**. For example, when $N = 4$ and the filter input is an AR1 Gaussian process with regression coefficient $\eta = 0.5$, we calculate $\beta^2 \cong 3.444912$. Using (8), we rewrite

$$E\{e^*(n)/|\,e(n)\,| \times \mathbf{a}(n) / D[\mathbf{a}(n)] \,\big|\, \theta(n)\}$$
$$\cong (2/\pi)^{1/2} \int_0^{\infty} E\{e^*(n)/|\,e(n)\,| \times \mathbf{a}(n) \exp[-(u^2/2)\beta^2 \mathbf{a}^H(n)\mathbf{a}(n)]\}du.$$

Recognizing that $\mathbf{a}(n)$ is a complex-valued Gaussian random vector and applying the method in [13] under Assumption **A5**, we finally calculate

$$E\{e^*(n)/|\,e(n)\,| \times \mathbf{a}(n) / D[\mathbf{a}(n)] \,\big|\, \theta(n)\}$$
$$\cong (2/\pi)^{1/2} \int_0^{\infty} (\pi/2)^{1/2}\, \sigma^{-1}_D(u, n)\, |\,\mathbf{A}(u)\,|^{-1} \mathbf{D}(u)\, du\, \theta(n),$$

where $\sigma^2_D(u, n) = \mathrm{tr}[\mathbf{D}(u)\mathbf{K}(n)] + \sigma^2 v$, $\mathbf{D}(u) = \mathbf{A}^{-1}(u)\mathbf{Ra}$, $\mathbf{A}(u) = \mathbf{I} + u^2\beta^2\mathbf{Ra}$ and $|\,\mathbf{A}(u)\,|$ is the determinant of the matrix $\mathbf{A}(u)$.

Then, we find in (6) and (7)

$$\mathbf{p}(n) = \mathbf{W}(n)\, \mathbf{m}(n),$$
$$\mathbf{V}(n) = \mathbf{W}(n)\, \mathbf{K}(n)$$

with

$$\mathbf{W}(n) \cong \int_0^{\infty} \sigma^{-1}_D(u, n)\, |\,\mathbf{A}(u)\,|^{-1} \mathbf{D}(u)\, du. \quad (9)$$

Next, in (7) we calculate in a similar manner

$$\mathbf{Ta} \cong 2 \int_0^{\infty} u\, |\,\mathbf{A}(u)\,|^{-1} \mathbf{D}(u)\, du. \quad (10)$$

Using (9) and (10) in (6) and (7), we can iteratively calculate filter convergence in terms of EMSE.

4.3 Analysis for a Large Number of Tap Weights

Since $\mathbf{W}(n)$ in (9) does depend on the value of $\mathbf{K}(n)$, we need to perform numerical integration with respect to u at each n. This requires a large amount of computation during adaptation calculation, particularly for a large N. If N is very large, say N>20, we can approximately calculate $\mathbf{W}(n)$ as follows.

For N>>1, expectation for $e(n)$ and $\mathbf{a}(n)$ can be calculated separately. Thus,

$$E\{e^*(n)/|\,e(n)\,| \times \mathbf{a}(n) / D[\mathbf{a}(n)] \,\big|\, \theta(n)\}$$
$$\approx E[e^*(n)/|\,e(n)\,| \times \mathbf{a}(n) \,\big|\, \theta(n)] \,/\, E\{D[\mathbf{a}(n)]\}$$
$$\approx (\pi/2)^{1/2}\, \sigma^{-1}_e(n)\, \mathbf{Ra}\, \theta(n) \,/\, E[\textstyle\sum_{l=0}^{N-1} |\,a(n-l)\,|]$$
$$\approx (\pi/2)^{1/2}\, \sigma^{-1}_e(n) \,/\, [(\pi/2)^{1/2}\sigma_a N] \times \mathbf{Ra}\, \theta(n)$$

from which

$$\mathbf{W}(n) \approx \mathbf{Ra} \,/\, [\sigma_e(n)\, \sigma_a N] \quad (11)$$

results. Here, $\sigma^2_e(n) = \varepsilon(n) + \sigma^2 v$ is the error variance.

4.4 Steady-State Solution

We assume that the filter converges as $n \to \infty$. Then, eq. (7) yields a matrix equation

$$\mathbf{W}(\infty)\mathbf{K}(\infty) + \mathbf{K}(\infty)\mathbf{W}(\infty) = \alpha_c\, \mathbf{Ta},$$

whence we solve, noting that $\mathbf{K}(\infty)$, $\mathbf{W}(\infty)$ and $\mathbf{Ta}$ are symmetric matrices,

$$\mathbf{K}(\infty) = (\alpha_c/2)\, \mathbf{W}^{-1}(\infty)\mathbf{Ta}.$$

Since $\mathbf{W}(\infty)$ depends on the value of $\mathbf{K}(\infty)$, we can solve $\mathbf{K}(\infty)$ only iteratively with an appropriate initial guess. With $\mathbf{K}(\infty)$ solved, we easily obtain the steady-state EMSE $\varepsilon(\infty)$. If we use (11), we find

$$\varepsilon(\infty) \approx (\alpha_c/2)\, \sigma_a N\, \mathrm{tr}(\mathbf{Ta})\, \sigma_e(\infty).$$

Since this is a quadratic equation, we can easily solve for $\varepsilon(\infty)$ analytically and explicitly.

5. EXPERIMENT

In this section, experiment is carried out with simulations and theoretical calculations of adaptive filter convergence for the NCA. The effectiveness of the proposed algorithm as well as the validity of the analysis in Section 4 are demonstrated.

Results of experiment are presented where we compare simulated and theoretically calculated filter convergence in the absence of impulse noise. The theoretical convergence is calculated in terms of EMSE using the difference equations developed in the previous section. We also present simulation results in the presence of both types of impulse noise.

Three examples are carefully prepared as given below, where *Example #2* is basically the same as *Example #1* in Subsection 3.3

Example #2 $N = 4$

filter reference input: AR1 Gaussian process with variance $\sigma^2_a = 1$ (0 dB) and regression coefficient $\eta = 0.5$

step size: $\alpha_c = 2^{-9}$

Case 1: "pure" Gaussian noise $\sigma^2_v = 0.01$
no impulse noise at filter input

Case 4: CGN $\sigma^{2\,(0)}_v = 0.01$; $p_v^{(0)} = 0.9$
$\sigma^{2\,(1)}_v = 10$; $p_v^{(1)} = 0.1$
impulse noise at filter input
$\sigma^2 va = 1000$; $pva = 0.1$

Example #3 $N = 2$

filter reference input: AR1 Gaussian process with variance $\sigma^2_a = 1$ (0 dB) and regression coefficient $\eta = 0$

step size: $\alpha_c = 2^{-8}$

Case 1: "pure" Gaussian noise $\sigma^2_v = 0.001$
no impulse noise at filter input

Case 4: CGN $\sigma^{2\,(0)}_v = 0.001$; $p_v^{(0)} = 0.9$
$\sigma^{2\,(1)}_v = 0.1$; $p_v^{(1)} = 0.1$
impulse noise at filter input
$\sigma^2 va = 100$; $pva = 0.1$

Example #4 $N = 32$

filter reference input: AR1 Gaussian process with variance $\sigma^2_a = 1$ (0 dB) and regression coefficient $\eta = 0.9$

step size: $\alpha_c = 2^{-6}$

Case 1: "pure" Gaussian noise $\sigma^2_v = 1$
no impulse noise at filter input

Case 4: CGN $\sigma^{2\,(0)}_v = 1$; $p_v^{(0)} = 0.95$
$\sigma^{2\,(1)}_v = 100$; $p_v^{(1)} = 0.05$
impulse noise at filter input
$\sigma^2 va = 100$; $pva = 0.05$

Results of experiment for *Examples #2*, *#3* and *#4* are shown in Figs. 3, 4 and 5, respectively. For *Example #4* (large N), $\mathbf{W}(n)$ in (11) is used. In the figures we observe good agreement between simulated and theoretically calculated convergence (Case 1) that proves the validity of the analysis in Section 4. In Case 4, simulated steady-state squared excess error $<|\epsilon(\infty)|^2>/2$ is again smaller than that in Case 1, demonstrating the robustness of the NCA against *both* types of impulse noise.

6. CONCLUSION

In this paper, we have derived normalized correlation algorithm (NCA) to be applied to complex-domain adaptive filters with Gaussian inputs for robust filtering in severe impulsive noise environments. Stochastic models for two types of impulsive noise have been presented: one for impulsive observation noise and another for impulse noise at filter input.

We have developed rigorous analysis of the NCA for a small number of tap weights N as well as approximate analysis for a large N for calculating transient and steady-state convergence behavior in terms of EMSE.

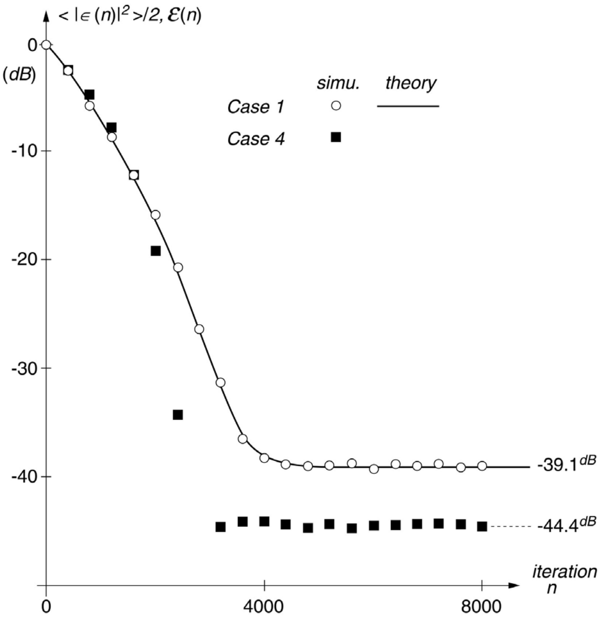

Fig. 3 Adaptive filter convergence – NCA. (*Example #2*, $N = 4$, Cases 1 & 4)

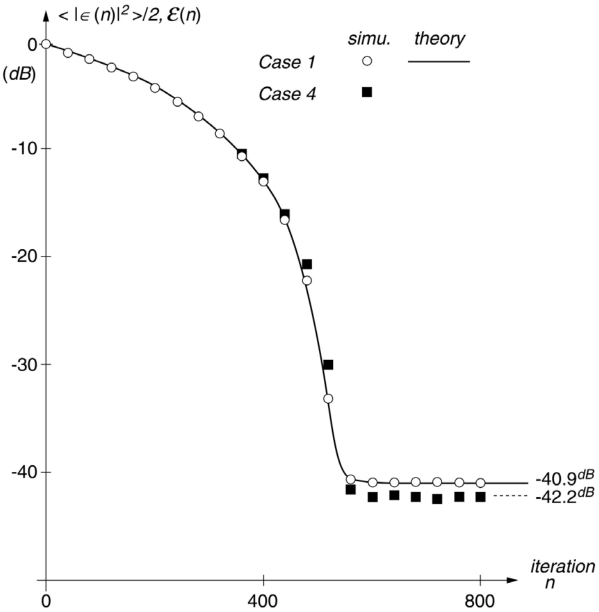

Fig. 4 Adaptive filter convergence – NCA. (*Example #3*, $N = 2$, Cases 1 & 4)

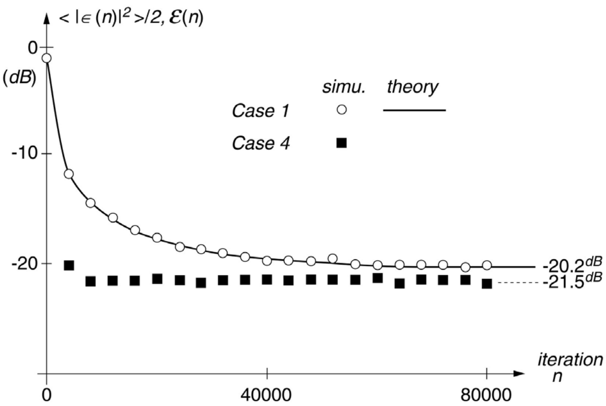

Fig. 5 Adaptive filter convergence – NCA. (*Example #4*, $N = 32$, Cases 1 & 4)

Through experiment with simulations, we have demonstrated that the NCA is highly robust in the presence of both types of impulse noise.

In the absence of impulse noise, good agreement between simulations and theoretical calculations has proven the validity of the analysis. Theoretical analysis of the NCA in the presence of both types of impulse noise is for further study. Also left as a future work is improvement in the filter convergence speed for the NCA, while preserving its robustness in impulsive noise environments.

APPENDIX

Calculation of β^2 for Approximation $D[\mathbf{a}(n)]^2 \cong \beta^2\, \mathbf{a}^H(n)\mathbf{a}(n)$

The simplest way to find β^2 is to take expectation of both sides of the above equation, i.e.,

$$E\{[\sum_{l=0}^{N-1} |\, a(n-l)\,|]^2\} \cong \beta^2 E[\sum_{l=0}^{N-1} |\, a(n-l)\,|^2]. \quad (12)$$

First, the left-hand side of (12) can be written as

$$E[\sum_{l=0}^{N-1}\sum_{\lambda=0}^{N-1} |\, a(n-l)\,|\,|\, a(n-\lambda)\,|]$$
$$= E[\sum_{l=0}^{N-1} |\, a(n-l)\,|^2] + E[\sum_{l=0}^{N-1}\sum_{\lambda\neq l} |\, a(n-l)\,|\,|\, a(n-\lambda)\,|]$$
$$= \sum_{l=0}^{N-1} E[|\, a(n-l)\,|^2] + 2\sum_{l=0}^{N-2}\sum_{\lambda>l} E[|\, a(n-l)\, a(n-\lambda)\,|].$$

Since $\sum_{l=0}^{N-1} E[|\, a(n-l)\,|^2] = 2\sigma_a^2 N$, we find

$$\beta^2 \cong 1 + N^{-1}\sum_{l=0}^{N-2}\sum_{\lambda>l} E[|\, a(n-l)\, a(n-\lambda)\,|] / \sigma_a^2.$$

Now, $|\, a(n-l)\, a(n-\lambda)\,|$ is a modulus of product of two correlated complex-valued Gaussian random variables. The author derived a probability density function of this modulus, naming its distribution *Gaussian Product Modulus Distribution* [14]. Referring to [14, 3.1(b)], we calculate

$$E[|\, a(n-l)\, a(n-\lambda)\,|] / \sigma_a^2 = L(\sin^2\alpha_{l\lambda}),$$

where $\sin\alpha_{l\lambda} = R_{a l\lambda}/\sigma_a^2$ and for $0 \leq m < 1$ we define a function

$$L(m) = (1-m)^2 \int_0^{\pi/2} (1+2m\sin^2\varphi)(1-m\sin^2\varphi)^{-5/2}\, d\varphi$$

which belongs to a family of *Elliptic Integrals*.

For details of calculation, let us define $\xi = \xi_R + j\xi_I = a(n-l)/\sigma_a$, $\eta = \eta_R + j\eta_I = a(n-\lambda)/\sigma_a$ and $E(\xi_R\,\eta_R) = E(\xi_I\,\eta_I) = \sin\alpha$. Referring to [14], we find

$$E(|\xi\,\eta|) = 8\cos^4\alpha\, \pi^{-1}\int_0^{\pi} d\varphi \int_0^{\infty} du\, u^2/(u^2+1-2u\sin\alpha\cos\varphi)^3.$$

Here, we integrate

$$\int_0^{\infty} du\, u^2/(u^2+1-2u\sin\alpha\cos\varphi)^3 = (1/8)(1-\sin^2\alpha\cos^2\varphi)^{-3/2}$$
$$\times[\pi/2 + \arcsin(\sin\alpha\cos\varphi) + \sin\alpha\cos\varphi(1-\sin^2\alpha\cos^2\varphi)^{1/2}]$$
$$+(1/4)\sin\alpha\cos\varphi$$
$$+(1/8)\sin^2\alpha\cos^2\varphi(1-\sin^2\alpha\cos^2\varphi)^{-5/2}$$
$$\times[3\pi/2 + 3\arcsin(\sin\alpha\cos\varphi) + 3\sin\alpha\cos\varphi(1-\sin^2\alpha\cos^2\varphi)^{1/2}$$
$$+2\sin\alpha\cos\varphi(1-\sin^2\alpha\cos^2\varphi)^{3/2}].$$

Recognizing that for a function $f(\cos\varphi)$

$$\int_0^{\pi} f(\cos\varphi)\, d\varphi = \int_0^{\pi/2} [f(\cos\varphi) + f(-\cos\varphi)]\, d\varphi$$

holds, we calculate

$$E(|\xi\,\eta|) = \cos^4\alpha \int_0^{\pi/2}$$
$$[(1-\sin^2\alpha\cos^2\varphi)^{-3/2} + 3\sin^2\alpha\cos^2\varphi(1-\sin^2\alpha\cos^2\varphi)^{-5/2}]\, d\varphi$$
$$= L(\sin^2\alpha).$$

Then, we finally obtain

$$\beta^2 \cong 1 + N^{-1}\sum_{l=0}^{N-2}\sum_{\lambda=l+1}^{N-1} L(\sin^2\alpha_{l\lambda}).$$

Note that $L(0) = \pi/2$ and $L(m) \to 2$ as $m \to 1$.

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