MULTIPLIERLESS IMPLEMENTATION OF GENERALIZED COMB FILTERS (GCF) BASED ON CHEBYSHEV POLYNOMIALS

Alfonso Fernandez-Vazquez[1] *and Gordana Jovanovic Dolecek*[2]

[1]School of Computer Engineering (ESCOM)
National Polytechnic Institute (IPN)
Mexico City, 07738, Mexico

[2]Department of Electronics
National Institute of Astrophysics, Optics and Electronics (INAOE)
Puebla, 72240, Mexico

emails: {afernan,gordana}@ieee.org

ABSTRACT

This paper addresses the multiplier free implementation of the Generalized Comb Filters (GCF) based on Chebyshev polynomials. In particular, we focus on GCF rational transfer function implementation for any decimation factor. Our approach ensures the perfect pole-zero cancelation for any GCF filter order. The closed form equation for the implementation complexity in terms of the numbers of two-input adders and shifts is also provided.

1. INTRODUCTION

The cascaded-integrator-comb (CIC) filter, proposed by Hogenauer [1], is the simplest multiplierless decimation filter, which is usually used at the first decimation stage. The filter must have a low passband droop and a high attenuation within the folding bands (bands around the comb zeros). Unfortunately, CIC filter has a high passband droop and a low attenuation in the folding bands. There have been proposed different methods for compensating the passband droop as well as for improving the stopband characteristic, for example [2-6].

Recently, a generalization of the CIC filter (GCF) is proposed by Laddomada [6] to improve the attenuation as well as to span the folding bands. Consequently GCFs filters have a high quantization noise rejection within the folding bands.

The rational transfer function of the GCF filter is expressed as [6]

$$H_{\mathrm{GCF}_N}(z) = A_N \left(\frac{1-z^{-D}}{1-z^{-1}}\right)^p \prod_{n=1}^{m} \frac{1-2\cos(D\alpha_n)z^{-D}+z^{-2D}}{1-2\cos(\alpha_n)z^{-1}+z^{-2}}, \quad (1)$$

where A_N and D are the normalization constant and the decimation factor, respectively. The rotation parameters α_n, $n = 1,\ldots,m$, are chosen such that the minimum attenuation within folding bands is maximized [6]. A useful value for α_n is $q_n\pi/\nu D$, where ν is a positive integer factor and q_n is a real value in the range $[-1,1]$, [6]. Consequently, $|\alpha_n| < 1$.

The order of the CGF is given by $N = 2m + p$, with

$$p = \begin{cases} 1, & N \text{ odd;} \\ 0, & N \text{ even.} \end{cases} \quad (2)$$

Unlike the CIC filter, the complexity of the CGF decimation filter is high because of the presence of multipliers in both nominator and denominator of (1). Yet another problem is the pole-zero cancelation of (1) when the coefficients have a finite precision.

In order to overcome this problem, in [7] the authors proposed an efficient multiplierless architecture for the GCF filters based on trigonometric identities which leads to the perfect pole-zero cancelation of the multiplierless transfer function (1).

In this paper we consider a general approach to solve the same problem, i.e., the multiplierless rational GCF transfer function implementation using the Chebychev polynomials. The motivation to use Chebyshev polynomials is twofold: First we can easily generalize the procedure for any given GCF order, and second, we can obtain the closed form equation for the required number of adders and shifts.

The rest of the paper is organized as follows. Next Section gives a brief overview of Chebyshev polynomials. The proposed approach is introduced in Section 3. In Section 4, we present the implementation complexity in terms of number of two-input adders and number of shifts. Discussion and results are presented in last section.

2. OVERVIEW OF CHEBYSHEV POLYNOMIALS

This section defines Chebychev polynomials and introduces an important property, which is used in Section 3 to implement generalized comb filter in a multiplierless form.

Formally, Chebyshev polynomial of order D, $C_D(x)$, can be defined by the following recursive equation:

$$C_D(x) = 2xC_{D-1}(x) - C_{D-2}(x), \quad (3)$$

where $C_1(x) = x$ and $C_0(x) = 0$. Table 1 presents Chebyshev polynomials for $D = 0,\ldots,10$. Observe that x is a common factor of $C_D(x)$ when D is odd.

D	$C_D(x)$
0	1
1	x
2	$2x^2-1$
3	$x(4x^2-3)$
4	$8x^4-8x^2+1$
5	$x(16x^4-20x^2+5)$
6	$32x^6-48x^4+18x^2-1$
7	$x(64x^6-112x^4+56x^2-7)$
8	$128x^8-256x^6+160x^4-32x^2+1$
9	$x(256x^8-576x^6+432x^4-120x^2+9)$
10	$512x^{10}-1280x^8+1120x^6-400x^4+50x^2-1$

Table 1: Examples of Chebyshev polynomials.

An alternative form to define Chebyshev polynomials is [8]

$$C_D(x) = \begin{cases} \cos(D\cos^{-1}(x)), & |x| \leq 1; \\ \cosh(D\cosh^{-1}(x)), & |x| > 1. \end{cases} \tag{4}$$

We now consider how to efficiently compute $C_D(x)$. Our goal is to express the higher order Chebyshev polynomials in terms of second order polynomials. To this end, consider the following relation [8]:

$$C_{n+m}(x) = 2C_n(x)C_m(x) - C_{n-m}(x), \qquad n \geq m, \tag{5}$$

Using $m = n$ in (5), we obtain

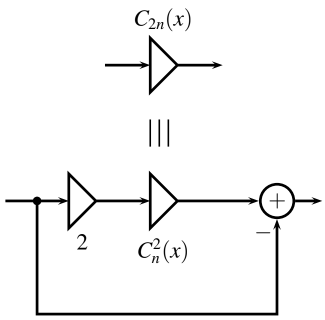
$$C_{2n}(x) = 2C_n^2(x) - 1. \tag{6}$$

In a similar way, replacing m with $n-1$ in (5), we have

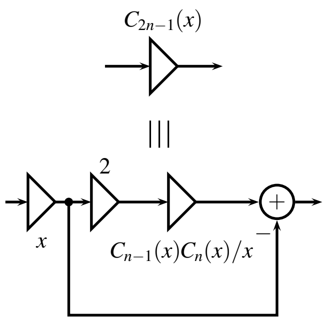
$$C_{2n-1}(x) = x\left(2\frac{C_n(x)C_{n-1}(x)}{x} - 1\right). \tag{7}$$

Note that the product $C_n(x)C_{n-1}(x)$ in (7) is divided by x. If n is even, then x is a common factor of $C_{n-1}(x)$. Similarly, if n is odd, $C_n(x)$ will have a common factor x. Therefore in both cases the product $C_n(x)C_{n-1}(x)$ will cancel x in the denominator.

Figure 1 illustrates the implementation of $C_D(x)$ based on equations (6) and (7).



(a) D even ($D = 2n$).



(b) D odd ($D = 2n-1$).

Figure 1: Efficient implementation of $C_D(x)$.

We can express in a simpler way any Chebyshev polynomial, applying recursively (6) and (7), as shown in the following example.

Considering $D = 7$, we have $D = 2n - 1 = 7$ resulting in $n = 4$. From (7), $C_7(x)$ is expressed as

$$C_7(x) = x\Big(2C_4(x)C_3(x)/x - 1\Big). \tag{8}$$

Using (6), we have

$$C_4(x) = 2C_2^2(x) - 1. \tag{9}$$

Replacing (9) into (8), we obtain

$$C_7(x) = x\Big(2\big(2C_2^2(x) - 1\big)C_3(x)/x - 1\Big). \tag{10}$$

Similarly, using (7), we have

$$C_3(x) = x\Big(2C_2(x) - 1\Big). \tag{11}$$

Substituting (11) into (10), we express (8) in terms of $C_2(x)$

$$C_7(x) = x\Big(2\left(2C_2^2(x) - 1\right)\left(2C_2(x) - 1\right) - 1\Big). \tag{12}$$

Finally, the desired result is obtained by replacing $C_2(x)$ from Table 1 into (12)

$$C_7(x) = x\Big(2\Big(2\left(2x^2 - 1\right)^2 - 1\Big)\left(2\left(2x^2 - 1\right) - 1\right) - 1\Big). \tag{13}$$

Figure 2 illustrates the implementation of $C_7(x)$ based on (13). We have indicated the polynomials $C_2(x)$, $C_3(x)/x$, and $C_4(x)$ with dashed lines. The benefit of this presentation is that all coefficients are equal to two.

3. GCF AND CHEBYSHEV POLYNOMIALS

This section relates the GCF decimation filter and the Chebyshev polynomials.

Denote the cosine term in the denominator of (1) as

$$x_n = \cos(\alpha_n). \tag{14}$$

Using (4), the cosine term (14) can be rewritten as

$$\cos(D\alpha_n) = \cos(D\cos^{-1}(x_n)). \tag{15}$$

Note that (15) can be expressed using the Chebyshev polynomial of order D, i.e., $C_D(x_n)$.

Using (14) and (15), we rewrite (1) as

$$\begin{aligned} H_{\mathrm{GCF}_N}(z) =& A_N\left(\frac{1-z^{-D}}{1-z^{-1}}\right)^p \\ & \prod_{n=1}^{m}\frac{1-2\cos(D\cos^{-1}x_n)z^{-D}+z^{-2D}}{1-2x_nz^{-1}+z^{-2}} \\ =& A_N\left(\frac{1-z^{-D}}{1-z^{-1}}\right)^p\prod_{n=1}^{m}\frac{1-2C_D(x_n)z^{-D}+z^{-2D}}{1-2x_nz^{-1}+z^{-2}}. \end{aligned} \tag{16}$$

The obtained result (16) shows that the problem to implement the coefficients in (1) is reduced to the problem of the computation of the Chebyshev polynomials of order D.

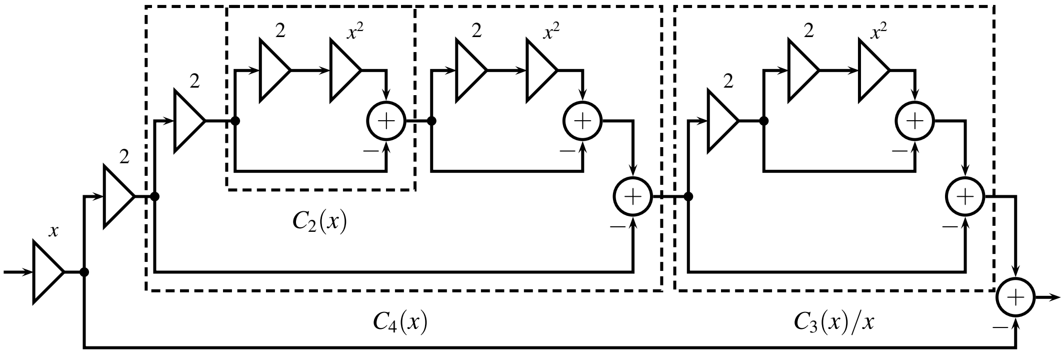


Figure 2: Implementation of $C_7(x)$ based on (13).

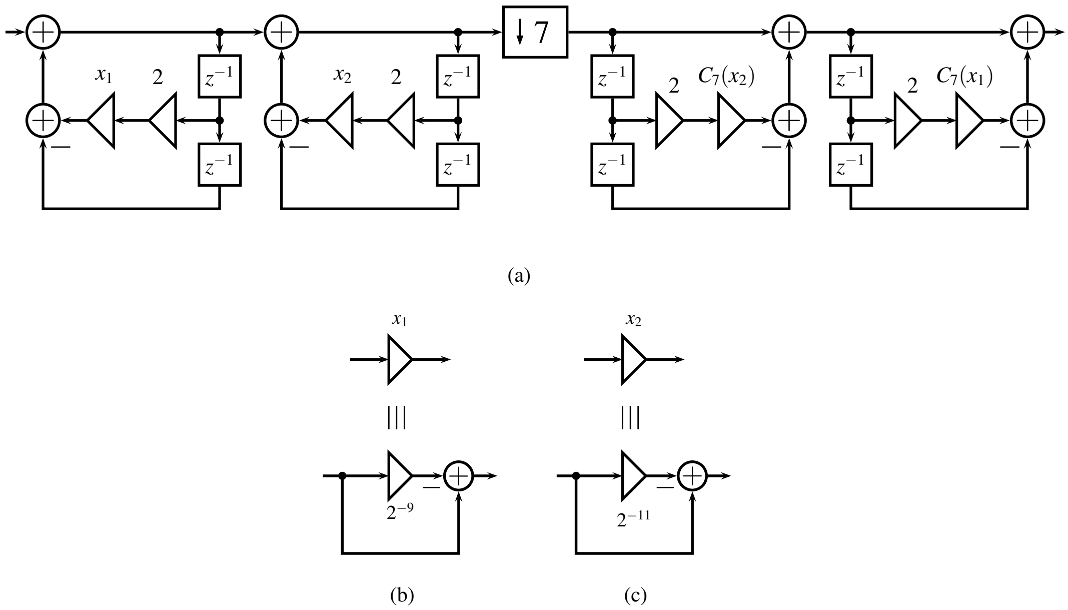


Figure 3: Decimation filter structure in Example 1.

Now, we turn our attention to the multiplierless implementation of the value x_n. For $D \geq 2$ and $\nu \geq 2$, we approximate $\cos(\alpha_n)$ using two first terms of Taylor polynomial, that is,

$$x_n = \cos(\alpha_n) \approx 1 - \alpha_n^2/2. \tag{17}$$

Rounding the constant $\alpha_n^2/2$ as a power of two, i.e., 2^{k_n}, $n = 1, \ldots, m$, equation (17) becomes

$$x_n = 1 - 2^{k_n}. \tag{18}$$

From (17) and (18), the value of k_n can be estimated as

$$k_n = \lfloor \log_2(1 - \cos\alpha_n) \rfloor, \tag{19}$$

where $\lfloor \cdot \rfloor$ stands for the floor function.

Next example illustrates the application of the Chebyshev polynomial in GCF.

Example 1. We design a generalized comb filter using the following specifications: The decimation factor D is equal to seven, the order of the GCF is four and the integer ν is four. Therefore, using the optimized values of q_1 and q_2 given in [6], we have $q_1 = 0.35$ and $q_2 = 0.88$.

The rational transfer function of GCF is

$$H_{\mathrm{GCF}_4}(z) = A_4 \frac{1 - 2C_7(x_1)z^{-7} + z^{-14}}{1 - 2x_1 z^{-1} + z^{-2}} \frac{1 - 2C_7(x_2)z^{-7} + z^{-14}}{1 - 2x_2 z^{-1} + z^{-2}}, \tag{20}$$

Using (19), we have $k_1 = -9$ and $k_2 = -11$. Consequently, $x_1 = 1 - 2^{-9}$ and $x_2 = 1 - 2^{-11}$.

Substituting the resulting values of x_1 and x_2 into (13) gives $C_7(x_1) = 0.905783498857570$ and $C_7(x_2) = 0.976167542024052$.

The implementation of the resulting decimation filter is shown in Fig. 3(a). The coefficients $C_7(x_1)$ and $C_7(x_2)$ are implemented as shown in Fig. 2. Additionally, the implementations of x_1 and x_2 are shown in Figs. 3(b) and 3(c), respectively.

Figure 4 shows the pole/zero pattern of the proposed GCF. Note the perfect pole-zero cancelation around the frequency $\omega = 0$. The magnitude responses of the GCF$_4$ and proposed quantized GCF$_4$ are shown in Fig. 5. Also note that the proposed filter has a higher attenuations in the folding bands than the GCF. However the GCF folding bands are wider.

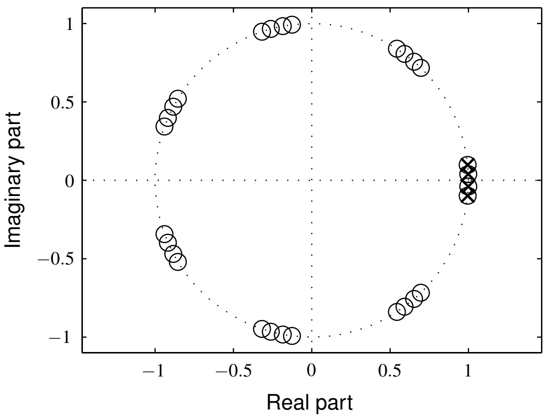


Figure 4: Pole/zero pattern of the GCF in Example 1.

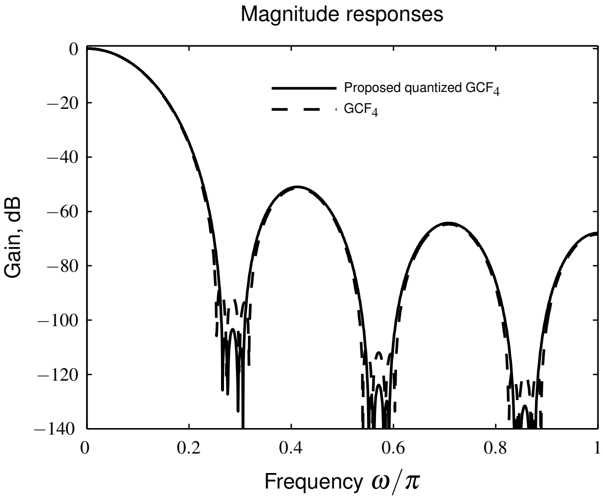


Figure 5: Magnitude responses of the GCF and quantized GCF in Example 1.

4. IMPLEMENTATION COMPLEXITY

In this section we turn our attention to the complexity of the proposed GCF implementation.

We first consider the complexity of x_n. From (18), we observe that x_n requires

- one two-input adder and
- one shift.

Next we consider the complexity of second order Chebyshev polynomial,

$$C_2(x_n) = 2x_n^2 - 1$$
$$= 2\left(1 - 2^{k_n}\right)^2 - 1. \quad (21)$$

It requires

- three two-input adders and
- three shifts.

Similarly, the Chebyshev polynomial

$$C_3(x_n) = x_n\left(2\left(2x_n^2 - 1\right) - 1\right)$$
$$= \left(1 - 2^{k_n}\right)\left(2\left(2\left(1 - 2^{k_n}\right)^2 - 1\right) - 1\right) \quad (22)$$

needs

- five two-input adders and
- five shifts.

Generally the D-order Chebysev polynomial needs

- $2D - 1$ two-input adders and
- $2D - 1$ shifts.

Finally, using the previous result for the computation of the complexity of D-order Chebyshev polynomial and using (16), we find the following expression of the complexity of the of generalized comb filter GCF of order N in terms of two-input adders and shifts. The implementation needs N_A two-input adders and N_S shifts, where

$$N_A = 2N + \left\lfloor \frac{N}{2} \right\rfloor (2D - 1), \quad (23)$$

$$N_S = 2\left\lfloor \frac{N}{2} \right\rfloor + \left\lfloor \frac{N}{2} \right\rfloor (2D - 1). \quad (24)$$

Observe that, for a given order N, the values of N_A and N_S are linear functions of D as illustrated in Figs. 6 and 7.

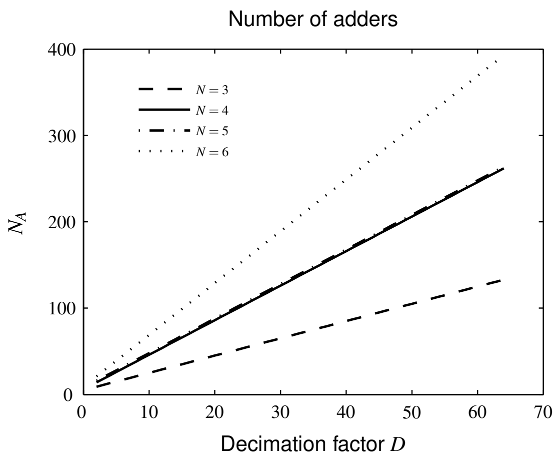


Figure 6: Number of adders.

Using (23) and (24), we find that the GCF decimation structure in Example 1 requires 34 two-input adders and 34 shifts.

5. DISCUSSION OF RESULTS

The proposed method presents a general approach to the problem of multiplierless GCF implementation solved in [7]. The pole-zero cancelation is achieved in both methods.

The advantage of our approach, based on Chebyshev polynomials, is the closed form equations for the implementation of $\cos(D\alpha_n)$ for any degree D. Additionally the equations to compute the implementation complexity for any D are provided.

The method [7] based on trigonometric identities gives the closed form equation for $\cos(D\alpha)$, $D = 2,3,4,8$, which

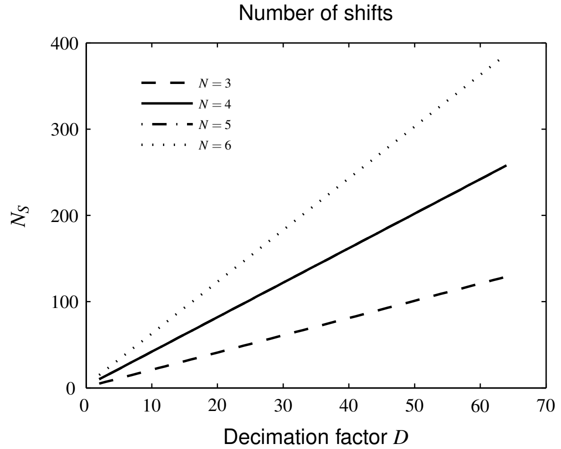


Figure 7: Number of shifts.

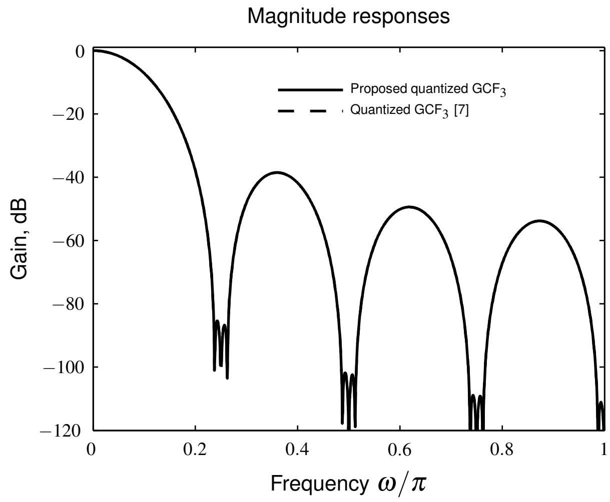


Figure 8: Magnitude responses of the designed filter in Example 2 and the proposed in [7].

is the same as Chebyshev polynomials in Table 1 when $x = \cos\alpha$. However the proposed approach uses the efficient implementation of Chebyshev polynomials based on (5) and (6). As a consequence the proposed approach results in less complexity as shown in the following example.

Example 2. Now, we design the GCF filter using the specifications given in [7], i.e., $N = 3$, $D = 8$, $\nu = 8$, $q_1 = 0.79$.

The resulting transfer function is

$$H_{\mathrm{GCF}_3}(z) = A_3 \frac{1 - z^{-8}}{1 - z^{-1}} \frac{1 - 2C_8(x_1)z^{-8} + z^{-16}}{1 - 2x_1 z^{-1} + z^{-2}}, \qquad (25)$$

where $x_1 = 1 - 2^{-11}$.

Applying recursively (6) to $C_8(x_1)$, we have

$$C_8(x_1) = 2\left(2(2x_1 - 1)^2 - 1\right)^2 - 1. \qquad (26)$$

According to (23) and (24), the implementation of the resulting structure involves 21 adders and 17 shifts. However, method [7] requires 32 adders and 24 shifts. Therefore the proposed approach saves 11 adders and 7 shifts.

Figure 8 shows the magnitude responses of the quantized GCF_3 using Chebyshev polynomials and the method [7]. Note that the magnitude responses are equal.

6. CONCLUSIONS

This paper presents a general approach to the multiplierless implementation of recursive GCF filters, based on Chebyshev polynomials. The perfect pole-zero cancelation is guaranteed since the coefficients of the numerator and denominator are related with a corresponding Chebyshev polynomial of order D. The advantage of the proposed approach is the provided closed form equations for the computation of the adders and shifts in terms of the order of the GCF_N and the decimation factor D. Another advantage is the closed form equation for the implementation of the multiplier $\cos(D\alpha_n)$.

Acknowledgments

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