NONLINEAR ADAPTIVE FILTERING TECHNIQUES WITH MULTIPLE KERNELS

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ABSTRACT

In this paper, we propose a novel approach using multiple kernels to nonlinear adaptive filtering problems. We present two types of multi-kernel adaptive filtering algorithms, both of which are based on the kernel normalized least mean square (KNLMS) algorithm (Richard et al., 2009). One is a simple generalization of KNLMS, adopting the coherence criterion for dictionary selection. The other is derived by applying the adaptive proximal forward-backward splitting method to a certain squared distance function penalized by a weighted block $\ell_1$ norm. The latter algorithm operates the weighted block soft-thresholding which encourages the sparsity of dictionary at the block level. Numerical examples demonstrate the efficacy of the proposed approach.

1. INTRODUCTION

Kernel has been proven an attractive tool in adaptive filtering and online learning when the desired response is a nonlinear function of input data [1–7]; see [8] for a comprehensive introduction to kernel adaptive filtering. The major advantages of the kernel-based approach include that it involves (i) no local minima unlike the neural network approach, and (ii) a marginal number of parameters unlike the Volterra series based approach. One of its central issues is that the size of dictionary grows linearly with the number of input data observed. This obviously conflicts with the limitations of memory and computational resource/time. Several sparsification techniques have been proposed and investigated for updating the dictionary in such a way that only dominant ones remain among basis vectors.

Kernel design is another important issue, as witnessed by the successive studies on kernel selection; see [9–11] among others. Those previous studies have been done exclusively for batch processing, and the techniques developed are not suitable for online processing. In the literature of kernel adaptive filtering or online learning, a reasonable kernel has been assumed available prior to adaptation, which is however not always possible.

This paper proposes efficient nonlinear adaptive filtering techniques using a set of kernels, typically Gaussian with different kernel parameters. Although they are other possible approaches to exploiting multiple kernels, we restrict ourselves to extending the kernel normalized least mean square (KNLMS) algorithm [6] in the current study. First, we derive a multi-kernel NLMS algorithm adopting a coherence-based criterion for the dictionary construction. Second, we present another multi-kernel NLM algorithm based on iterative use of metric projection and weighted block soft-thresholding. The latter algorithm is derived by applying the adaptive proximal forward-backward splitting method [12] to the following cost function: a certain squared distance (smooth) plus a weighted block $\ell_1$ norm (nonsmooth). The weighted block soft-thresholding promotes sparsity of the dictionary at the block level.

Let $u_0 \in \mathcal{U}$ be an input vector at time instant $n \in \mathbb{N}$, and $d_n \in \mathbb{R}$ the desired response depending nonlinearly on $u_n$. Here $\mathcal{U}$ is a compact subset of the $L$ dimensional Euclidean space $\mathbb{R}^L$. The task is to find the nonlinear dependency in an online fashion with the measurements $(u_n, d_n)$ arriving sequentially.

In the kernel adaptive filtering approach, a filter (or an estimator) is modeled as an element of the reproducing kernel Hilbert space (RKHS) associated with a kernel $k: \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}$ [8, 13–15]. One of the celebrated examples is the Gaussian kernel defined as follows: $k(x, y) = \exp(-\alpha ||x - y||^2)$. For the sake of simple notation, we denote by $|| \cdot ||$ the Euclidean norm in any dimension space. It is known that the Gaussian kernel has the universal approximation property [8]. Data sparsification is necessary to limit the computational costs to a manageable amount. Let $f_{n-1} := \{a_1, a_2, \ldots, a_{n-1}\} \subset \{0, 1, \ldots, n-1\}$ indicate the dictionary $(k(\cdot, a_j))_{j \in f_{n-1}}$ that forms a nonlinear estimator at time $n$, where $r_{n-1}$ denotes the size of dictionary. The estimator producing an estimate of $d_n$ is given as follows:

$$v_n(u) := \sum_{j \in f_{n-1}} h_{j,n} k(u, a_j), \quad u \in \mathcal{U},$$

where $h_{j,n} \in \mathbb{R}$ for $j \in f_{n-1}$, are the parameters determined by an adaptive algorithm with the set of input-output data $(u_j, d_j)_{j \in f_{n-1}}$. Define $h_{\theta} := [h_{a_0,n}, h_{a_1,n}, \ldots, h_{a_{n-1},n}]^T \in \mathbb{R}^{n-1}$ and $k_\theta := [k(u_0, u_{a_0}), k(u_0, u_{a_1}), \ldots, k(u_0, u_{a_{n-1}})]^T \in \mathbb{R}^{n-1}$, where $(\cdot)^T$ stands for transpose. An estimate of $d_n$ is then given as $d_n = v_n(u_0) = k_\theta^T h_{\theta}, \quad n \in \mathbb{N}$. The point here is how to design the dictionary and how to update the coefficients $h_{j,n}$.

In [6], a coherence-based dictionary-designing scheme has been proposed. Suppose for simplicity that the kernel has a unit Hilbert norm, i.e., $k(x, x) = 1$, $\forall x \in \mathcal{U}$; the Gaussian kernel satisfies this condition. With the initial dictionary $\{k(\cdot, a_0)\}$ indicated by $f_{n-1} := \{0\}$, the scheme inserts $k(\cdot, a_n)$, $n \in \mathbb{N}$, into the dictionary if the following condition is satisfied:

$$\max_{j \in f_{n-1}} |k(u_0, a_j)| \leq \delta, \quad n \in \mathbb{N},$$

where $\delta > 0$ is the threshold that governs the degree of sparsity and the coherence of the dictionary. Letting $\zeta > 0$ and $\zeta > 0$ be kernel parameters of Gaussian kernels, the same dictionary size is

\[ \text{1The term "dictionary" stands for the set of basis vectors that are used to construct a nonlinear estimator.} \]
obtained by setting their corresponding thresholds \( \delta_\rho > 0 \) and \( \delta_\eta > 0 \) such that \( \delta_\rho^{CS} = \delta_\eta^{CS} \); this idea is used to determine the values of \( \delta \) in Section 4.

With the initial \( h_0 = h_{0,0} := 0 \), the step size \( \eta \in (0,2) \), and the regularization parameter \( \rho > 0 \), the update rule of KNLMS is given as follows.

(i) If (2) is unsatisfied, \( J_n := J_{n-1} \), and

\[
h_{n+1} = h_n + \eta \frac{d_n - k_n^T h_n}{\|k_n\|^2 + \rho} k_n, \tag{3}
\]

(ii) If (2) is satisfied, \( J_n := J_{n-1} \cup \{n\} \), and

\[
h_{n+1} = h_n + \eta \frac{d_n - k_n^T h_n}{\|k_n\|^2 - \rho} k_n,
\]

where \( k_n := [k_n^T, \mathbf{x}(u_0, u_n)]^T \) and \( h_n := [h_n^T, 0]^T \).

Although its affine projection version is proposed in [6], we solely consider the NLMS version to present our idea as simply as possible.

3. MULTI-KERNEL NLMS ALGORITHMS

We present multi-kernel NLMS algorithms based on two different sparsification techniques. Let \( K_n := \mathcal{W} \times \mathcal{Y} \rightarrow \mathbb{R}, \) \( \mathcal{M} := \{1,2,\cdots,M\} \), be a set of \( M \) distinct kernels to be employed.

3.1 Algorithm with Coherence-based Sparsification

With \( J_{n-1} := \{ \omega_1, \omega_2, \cdots, \omega_{n-1} \} \subset \{0,1,\cdots,n-1\} \) indicating a dictionary, our estimator takes the following form:

\[
\psi_n(u) := \sum_{m \in \mathcal{M}} \sum_{j \in J_{n-1}} h_{m,n}^{(m)} k_m(u, u_j), \quad u \in \mathcal{W},
\]

where \( h_{m}^{(m)} \in \mathbb{R}, m \in \mathcal{M}, j \in J_{n-1} \). The estimates \( \widehat{h}_n := \psi_n(u_n) \) of \( h_n \) can be rewritten in a vector form as follows:

\[
\psi_n(u_n) = \sum_{m \in \mathcal{M}} h_{m,n}^T k_m(u_n),
\]

where \( h_{m,n} := [h_{m,n}^{(m)} h_{m,n}^{(m)} \cdots h_{m,n}^{(m)}]^T \in \mathbb{R}^{r_m \times 1} \) and \( \kappa_{m,n} := [\kappa_{m}(u_0, u_{0,n}), \kappa_{m}(u_0, u_{1,n}), \cdots, \kappa_{m}(u_{n-1,n}, u_{n-1,n})]^T \in \mathbb{R}^{r_m \times 1} \). Moreover, it can be written in a matrix form as

\[
\psi_n(u_n) = \text{tr}(H_n^T K_n),
\]

where \( H_n := [h_{1,n} h_{2,n} \cdots h_{M,n}] \), \( K_n := [k_{1,n} k_{2,n} \cdots k_{M,n}] \), and \( \text{tr}(\cdot) \) stands for the trace of matrix.

The dictionary is constructed on the following criterion:

\[
\|K_n\|_{\max} := \max_{m \in \mathcal{M}} \max_{j \in J_{n-1}} \|k_m(u_n, u_j)\| \leq \delta_{CS}, \quad n \in \mathbb{N},
\]

where \( \delta_{CS} > 0 \) is the threshold. In the same way as in [6, Proposition 2], it can readily be proved that the dictionary size under (8) is finite for any sequence \( \{u_n\}_{n=0}^{\infty} \subset \mathcal{Y} \) due to the compactness of \( \mathcal{Y} \).

Let \( H_0 := [h_0^{(1)}, h_0^{(2)}, \cdots, h_0^{(M)}] := 0_M \) be the initial row vector, where \( 0_M \in \mathbb{R}^M \) denotes the length \( M \) zero vector. With the step size \( \eta_{CS} \in [0,2] \) and the regularization parameter \( \rho_{CS} > 0 \), the update rule is given as follows.

(i) If (8) is unsatisfied, \( J_n^{CS} := J_n^{CS} \) and

\[
H_{n+1} = H_n + \eta_{CS} \frac{d_n - \text{tr}(H_n^T K_n)}{\|K_n\|_{\max}^2 + \rho_{CS}} K_n.
\]

(ii) If (8) is satisfied, \( J_n^{CS} := J_{n-1} \cup \{n\} \) and

\[
H_{n+1} = H_n + \eta_{CS} \frac{d_n - \text{tr}(K_n^T H_n)}{\|K_n\|_{\max}^2 + \rho_{CS}} K_n,
\]

where \( H_n := [H_n^T, 0_M]^T \) and \( K_n := [K_n^T, 0_M]^T \) with \( K_n := [\kappa_1(u_0, u_n), \kappa_2(u_0, u_n), \cdots, \kappa_M(u_0, u_n)]^T \).

We name the above algorithm the multi-kernel NLMS algorithm with coherence-based sparsification (MKNLMS-CS). For \( M = 1 \), the MKNLMS-CS algorithm coincides with the KNLM algorithm, and hence it is a generalization of KNLMS.

3.2 Algorithm with Block Soft-thresholding

In \( \mathbb{R}^{p \times q} \) for any \( p, q \in \mathbb{N} \setminus \{0\} \), define an inner product as \( \langle A, B \rangle := \text{tr}(A^T B) \) and its induced norm as \( \|A\| := \sqrt{\langle A, A \rangle} = \|A\|_F \). \( \mathcal{A} \subseteq \mathbb{R}^{p \times q} \). If in particular \( q = 1 \), the inner product and norm are reduced respectively to the standard inner product and the Euclidean norm. With \( J_{n-1}^{RT} := \{0_0, 0_1, \cdots, 0_{n-1}\} \subset \{0,1,\cdots,n-1\} \) indicating a dictionary, our estimator produces an estimate of \( d_n \) in the following form:

\[
\psi_n(u_n) = (H_n, K_n),
\]

where \( H_n \) and \( K_n \) are defined as in Section 3.1.

Define the metric distance \( d(H, \Pi_n) := \min_{G \in \Pi_n} \|H - G\| \) between \( H \in \mathbb{R}^{(r_n+1) \times M} \) and

\[
\Pi_n := \{X \in \mathbb{R}^{(r_n+1) \times M} : \langle K_n, X \rangle = d_n\},
\]

where \( K_n := [K_n^T, K_n]^T \in \mathbb{R}^{(r_n+1) \times M} \) is the same as defined in Section 3.1. Our cost function is then given as follows:

\[
\Theta_n(H) := \frac{1}{2} d^2(H, \Pi_n) + \lambda \sum_{i=1}^{r_n+1} w_i \|h_i\|, \quad n \in \mathbb{N},
\]

(13) \( =: \Theta_n^{(1)}(H) \rightarrow \Theta_n^{(2)}(H) \)

\[
H = [h_1^T h_2^T \cdots h_{r_n+1}^T]^T \in \mathbb{R}^{(r_n+1) \times M}.
\]

(14)

Here \( \lambda > 0 \) and

\[
w_i := \left\{ \begin{array}{ll} \varepsilon, & \text{if } \|h_i\| > \tau, \\ 1, & \text{otherwise}, \end{array} \right. \]

\[
i = 1,2,\cdots,r_n+1,
\]

(15)

with \( \varepsilon > 0 \) and \( \tau > 0 \). The squared distance term \( \Theta^{(1)}_n \) in (13) contributes to reducing the empirical risks. Meanwhile, the (weighted) block \( l_1 \) regularization term \( \Theta^{(2)}_n \) promotes the sparsity of the estimator at the block level, where the (block) sparsity is controlled by the parameters \( \lambda, \varepsilon, \) and \( \tau \). The block \( l_1 \) norm has been used, as an alternative to the nowadays popular \( l_1 \) regularization, for an estimator involving a block structure, such as in the grouped Lasso estimator [16], the block-sparse signal recovery [17], and the multiple kernel learning [18]. If the block size is \( M = 1 \), it coincides with the (weighted) \( l_1 \) norm.

Since the block \( l_1 \) norm is nonsmooth, the stochastic gradient approach cannot be applied to \( \Theta_n \). Noticing however that \( \Theta_n \) is the
sum of smooth and nonsmooth convex functions, we can apply the adaptive proximal forward-backward splitting method [12].

We define the augmented matrix $H_k := [H_1^T \ 0_M]^T \in \mathbb{R}^{(r_n+1) \times M}$. The dictionary is initialized as $H_0^B := 0$, which makes $K_0 = k_0^T := [k_1(u_0, u_0), k_2(u_0, u_0), \ldots, k_M(u_0, u_0)]$ and $H_0 = 0_M^T$. The proposed algorithm is given as follows:

$$H_{n+1} := T \left[ \text{prox}_{\mu \theta_n^B} \left( \bar{H}_n - \mu \nabla \theta_n^B(H_n) \right) \right], \quad n \in \mathbb{N}. \quad (16)$$

where $\mu \in (0, 2)$ is the step size parameter and $\nabla \theta_n^B$ denotes the gradient of $\theta_n^B$. The algorithm, including the operators $T$ and $\text{prox}_{\mu \theta_n^B}$, is elaborated below in three steps.

**Step 1:** Compute

$$\bar{H}_n := H_n - \mu \theta_n^B(H_n) = H_n + \mu (P_{H}(H_n) - H_n), \quad (17)$$

where

$$P_{H}(H_n) := \arg \min_{G \in \mathbb{H}} \|H - G\|, \quad H \in \mathbb{R}^{(r_n+1) \times M}, \quad (18)$$

denotes the projection of $H$ onto $\Pi_n$, and it is readily verified that

$$P_{H}(H_n) = H_n + \frac{d_n - \langle K_n, \bar{H}_n \rangle}{\|K_n\|^2} K_n. \quad (19)$$

**Step 2:** Compute

$$\bar{H}_n := \text{prox}_{\mu \theta_n^B}(\bar{H}_n), \quad (20)$$

where

$$\text{prox}_{\mu \theta_n^B} : \mathbb{R}^{(r_n+1) \times M} \rightarrow \mathbb{R}^{(r_n+1) \times M},$$

is called the proximity operator of $\theta_n^B$ of index $\mu$ [19]. The $i$th row of $\bar{H}_n^B$ is computed as [20]

$$(\bar{H}_n^B)_{2i} = \max \left\{ 1 - \frac{\lambda \|w_{i+1}\|}{\|H_{n+1}\|}, 0 \right\}, \quad (i) \in \mathbb{R}^{1 \times M}, \quad (21)$$

where $\lambda$ denotes the $i$th row of the operator. The operator in (22) is essentially equivalent to the operator exploited in [21, Eq. (2.4)]; we call it weighted block soft-thresholding. The weighted block soft-thresholding operator eliminates the blocks (i.e., the row vectors) $(\bar{H}_n^B)_{1i}$, having small norms, since such blocks make no significant contribution in estimating $d_n$. Accordingly the parameters $\varepsilon$ and $\tau$ in (15) should be reasonably small so that the blocks $(\bar{H}_n^B)_{1i}$, having sufficiently large norms are not seriously affected.

**Step 3:** Compute

$$H_{n+1} := T(\bar{H}_n) \in \mathbb{R}^{r_n \times M}, \quad (23)$$

where $T$ is an operator that eliminates all the zero row vectors, and $r_n \in \mathbb{N}$ denotes the number of nonzero row vectors of $\bar{H}_n^B$. Namely, $H_{n+1}$ consists of all the nonzero row vectors of $\bar{H}_n^B$. The set $\mathcal{J}_{n+1} := \mathcal{J}_{n+1} \cup \{n\}$ is obtained by eliminating from $\mathcal{J}_{n+1}$ all the indices that correspond to the zero row vectors of $\bar{H}_n^B$.

We name the algorithm in (16) the multi-kernel NLMS algorithm with block soft-thresholding (MKNLMS-BT).

### 3.3 Computational Complexity and Remarks

We discuss the computational complexity and memory requirements of the proposed algorithms. Suppose that $M$ Gaussian kernels with different kernel parameters $\kappa_m, m = 1, \ldots, M$, are employed, as the overall complexity depends on the kernels. The complexity and memory requirements of the proposed and KNLMS algorithms are summarized in Table 1. Note that, given a $u_j, j \in \mathcal{F}_C$, the computation of $\kappa_m(u_j, u_k)$ for all $m = 1, \ldots, M$ requires no more than $L + M$ multiplications. This is because (i) once $\|u_j - u_k\|^2$ is computed, it can be used for evaluating all $\kappa_m(u_j, u_k)$, and (ii) $\exp(-\theta_m) = 1, \ldots, M$, can be calculated and stored prior to adaptation.

As will be seen in the following section, the use of multiple kernels allows us to achieve the same amount of error as the single kernel case with a smaller value of $r_n$. Let $\beta \in (0, 1)$ denote the ratio of the $r_n$ value of MKNLMS-CS, or MKNLMS-BT, to that of KNLMS. Then, MKNLMS-CS, or MKNLMS-BT, has lower complexity than KNLMS provided that $L > (4BM^2 - 3)/(1 - \beta)$, or $L > (6BM^2 - 3)/(1 - \beta)$. Here the complexity for an exponential calculation is counted as that for a multiplication. Moreover the memory requirements of both MKNLMS-CS and MKNLMS-BT are lower than that of KNLMS provided that $L > (6BM^2 - 3)/(1 - \beta)$. We emphasize that the efficiency of MKNLMS-BT in computation and memory requirements is due to the block soft-thresholding operator. Some remarks on the proposed algorithms are given below.

**Remark 1**

(a) A kernel adaptive filter is characterized as an element of RKHS associated with a kernel employed. In the multi-kernel approach, however, we have multiple RKHSs associated with the kernels. How can we characterize our estimators? Indeed the estimator of MKNLMS-CS in (5) is characterized as an element of the Cartesian product of the $M$ RKHSs as

$$\text{argmin} \left\{ \sum_{i=1}^{J} \sum_{j=1}^{M} \psi_i(w_j) A + \sum_{i=1}^{J} \psi_i(w_j) B \right\} \in \mathbb{R}^{1 \times M},$$

where $\psi_i$ is the inner product in the product space defined as the sum of the inner products in each RKHS; the same applies to MKNLMS-BT. It has been shown in [20] that a positive definite kernel can be defined in the product space and that the reproducing property of RKHS is handed down to the product space as well as the representer theorem. The proposed approach is more general than the one trying to design an optimal kernel as a convex combination of multiple kernels [9–11].

(b) When the data are nonstationary, kernel functions that make significant contribution in estimation tend to change from time
to time. In such a scenario, the MKNLMS-CS algorithm discards wasted kernel functions from the dictionary. In contrast, the MKNLMS-CS algorithm keeps all the kernel functions during the whole adaptation process once they are inserted into the dictionary.

(c) From a theoretical point of view, it would be more appropriate to present the MKNLMS-CS algorithm in the $M$-fold Cartesian product of the $\ell_2$ space [20]. We take the present form as it is more convenient for implementation. The algorithm enjoys the monotone approximation property: if $H_0 := \text{argmin}_{H \in \mathcal{H}} \| H - H^* \|$, then $\| H_n - H^* \| < \| H_0 - H^* \|$, for all $H \in \Omega$. Each datum $d_n$ is predicted with $u_n := [d_{n-1}, d_{n-2}]^T$, $n \in \mathbb{N}$, in an online fashion (i.e., $L = 2$).

In all the simulations, we fix the step size of KNLSM, MKNLMS-CS, and MKNLMS-BT to $\eta = \eta_{\text{CS}} = \mu = 9.0 \times 10^{-2}$, and the regularization parameter of KNLSM and MKNLMS-CS to $\rho = 3.0 \times 10^{-2}$ and $\rho_{\text{CS}} = 3.0 \times 10^{-2}$, respectively. For the proposed algorithms, we adopt Gaussian kernels with different kernel parameters $\alpha_1, \alpha_2, \ldots, \alpha_M$. For KNLSM, we also adopt a Gaussian kernel.

Figure 1(a) depicts the learning curves of MKNLMS-CS for $M = 2$, $\alpha_1 = 1$, $\alpha_2 = 4$, and KNLSM for $\alpha = 3.73$; this choice of $\alpha$ is also used in [6]. The thresholds for KNLSM and MKNLMS-CS are set respectively to $\delta = 0.24$ and $\delta_{\text{CS}} = 0.68$ so that the average values of $r_n$ of both algorithms are approximately 12. MSE is calculated by taking an arithmetic average over 200 experiments. For reference, the learning curve of NLMS for the step size 0.012 is also plotted.

Figure 1(b) depicts the learning curves of MKNLMS-CS for $M = 2$, $\alpha_1 = 1$, $\alpha_2 = 10$, and KNLSM for $\alpha = 1.3, 10$. The thresholds for KNLSM is set to $\delta = 0.8, 0.55, 0.13$ so that the average values of $r_n$ of both algorithms are approximately 20. For the proposed algorithm, we set the parameters to $\lambda = 5.0 \times 10^{-2}$, $\epsilon = 1.0 \times 10^{-2}$, $\tau = 1.5 \times 10^{-2}$; the same values of $\lambda$ and $\epsilon$ are used in all the experiments in this paper.

Figure 2 plots MSE against the average value of $r_n$ for KNLSM with $\alpha = 3.73$, and MKNLMS-CS and MKNLMS-CS-BT for $M = 2$, $\alpha_1 = 1$, $\alpha_2 = 4$. The marked points are obtained by changing $\delta$, $\delta_{\text{CS}}$, and $\tau$ as follows. $\delta$ is varied from 0.15 to 0.2 in increments of 0.01 and from 0.2 to 0.7 in increments of 0.02. $\delta_{\text{CS}}$ is varied from 0.3 to 0.9 in increments of 0.02. $\tau$ is varied from $0.7 \times 10^{-2}$ to $3.7 \times 10^{-2}$ in increments of 0.1. The MSE values are obtained by averaging over the last 2,000 samples of 10,000 samples for 200 experiments.

Figure 3 plots the MSEs of KNLSM, MKNLMS-CS, and MKNLMS-CS-BT for different choices of kernel parameters. For KNLSM, we change $\alpha$ within the range of $[0.5, 10]$. For MKNLMS-CS and MKNLMS-CS-BT, we fix $\alpha_1 = 1.0$ for $M = 2$, and change $\alpha_2$ within the range of $[0.5, 10]$. For comparison, we also plot the MSE of NLMS for the step size 0.012. The value of $\delta$, $\delta_{\text{CS}}$, or $\tau$ is chosen for each $\alpha$ or $\alpha_2$ so that the average value of $r_n$ becomes 20 approximately. The MSE values are calculated in the same way as in Fig. 2.

4. NUMERICAL EXAMPLES

We compare the performance of the proposed algorithms with KNLSM [6] in online prediction of highly nonlinear time series generated by $d_n := [0.8 - 0.5 \exp(-d_{n-1}^2)]d_{n-1} - [0.3 + 0.9 \exp(-d_{n-1}^2)]d_{n-2} + 0.1 \sin(d_{n-1})\pi$ with $d_{-2} := d_{-1} := 0.1$; this is a benchmark problem described in [6]. The signals $d_n$ are corrupted by a zero-mean Gaussian noise with variance 0.01. Each datum $d_n$ is predicted with $u_n := [d_{n-1}, d_{n-2}]^T$, $n \in \mathbb{N}$, in an online fashion (i.e., $L = 2$). The simulation results obtained suggest the following advantages of the proposed algorithms.

1. Given a value of MSE, the multi-kernel approach could reduce the average value of $r_n$ compared to KNLSM. This implies the computational efficiency of the proposed approach when $L$ becomes large (see Section 3.3).

2. Given an average value of $r_n$, the multi-kernel approach could improve the MSE performance compared to KNLSM. The improvement is considerable particularly when the average value of $r_n$ is low. For instance, the difference in decibel between the MSEs of KNLSM and MKNLMS-CS averaged over the last 2,000 samples in Fig. 1(a) is approximately 1.8 [dB].

3. Compared to the single kernel approach with a ‘good’ kernel parameter, the multi-kernel approach could achieve comparable (or better) performance without specifying a good kernel parameter. In other words, the proposed algorithm is insensitive to the choice of kernel parameters (hence reducing the efforts for the kernel-parameter selection). This implies that the proposed approach is advantageous when data are nonstationary and a good kernel parameter changes from time to time. In such scenarios, in addition to the expected advantages in performance, MKNLMS-CS will bring a particular advantage in computational complexity and memory requirements (see Remark 1(b)).
The potential advantages of the proposed algorithm (MKNLMS-BT) exploits the weighted block soft-thresholding operator. The present study has exemplified that the use of multiple kernels leads to efficient adaptive filtering for nonlinear systems. We repeat finally that there are other possible approaches, to be investigated, to exploiting multiple kernels in adaptive filtering.

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REFERENCES


