

DESIGN OF VARIABLE FRACTIONAL ORDER DIFFERENTIATOR USING EXPANSION OF HYPERBOLIC FUNCTION

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ABSTRACT

In this paper, the design of variable fractional order differentiator (VFOD) using expansion of hyperbolic function is presented. First, the ideal frequency response is decomposed into the sum of hyperbolic cosine and sine functions. Then, the power series expansions of hyperbolic functions are used to implement VFOD. The proposed VFOD requires less storage requirement of filter coefficients and implementation complexity than the conventional Farrow structure at cost of longer filter delay. Finally, the numerical examples are demonstrated to show the effectiveness of the proposed design approaches.

1. INTRODUCTION

In recent years, fractional calculus has been received great attentions in many engineering applications and science including image processing, automatic control, electromagnetic theory and electrical networks [1]-[9]. The integer order derivative $\frac{d^q f(x)}{dx^q}$ of a function $f(x)$ has been generalized to the fractional order derivative $\frac{d^p f(x)}{dx^p}$, where q is an integer and p is a fractional number. One of important research topics of digital signal processing in fractional calculus is to design a fractional order differentiator (FOD) such that the fractional order derivative of a digital signal can be obtained by using FOD. The ideal frequency response of fractional order differentiator is given by

$$H_d(\omega, p) = (j\omega)^p e^{-j\omega D} \quad (1)$$

where D is a prescribed integer delay and p is a fractional number in the range $[-0.5, 0.5]$. Thus, the FOD design problem is how to find a filter such that its actual frequency response fits the ideal response $H_d(\omega, p)$ as well as possible. When the order p is fixed, it is called the fixed fractional order differentiator (FFOD) design [4]-[6]. If the order p is adjustable, it is called the variable fractional order differentiator (VFOD) design. This problem is a research branch of variable filter design. So far, several methods have been proposed to solve the VFOD design problem such as the weighted least squares method [7][8] and series expansion method [9]. Each method has its unique features.

In the conventional VFOD design, the Taylor series expansion of filter coefficient is used to decompose the design of fractional order differentiator into some fixed sub-filters designs. Then, the WLS method is applied to design sub-filters. One of the main advantages of this approach is that the designed variable fractional order differentiator can be implemented efficiently by using Farrow structure [7]. Except Taylor series expansion, there exist various power series expansion methods in the literature [10]. Thus, it is interesting to use these expansion methods to design variable fractional order differentiators. The purpose of this paper is to study the hyperbolic function based expansion design method and to compare this method with conventional Taylor series expansion method.

This paper is organized as follows. In section II, the conventional WLS design of variable fractional order differentiator using Taylor series expansion is first described briefly. In section III, the ideal frequency response is decomposed into the sum of hyperbolic cosine and sine functions. Then, the power series expansions of hyperbolic functions are used to obtain two implementation structures of variable fractional order differentiator which only contains two kinds of sub-filters. Finally, some numerical examples are demonstrated to show the effectiveness of the proposed design methods and conclusions are made.

2. CONVENTIONAL DESIGN

In this section, the conventional WLS method will be reviewed briefly. The transfer function of the variable FIR filter used to approximate $H_d(\omega, p)$ is chosen as follows:

$$H_1(z, p) = \sum_{n=0}^N h_n(p) z^{-n} \quad (2)$$

Using the Taylor series expansion, filter coefficient $h_n(p)$ can be expressed as power series form:

$$h_n(p) = \sum_{m=0}^{\infty} a_{nm} p^m \quad (3)$$

Because p is a fractional number, $h_n(p)$ can be truncated into the polynomial function in p of degree M below:

$$h_n(p) \approx \sum_{m=0}^M a_{nm} p^m \quad (4)$$

Since coefficient $h_n(p)$ is real valued, the frequency response $H_1(e^{j\omega}, p)$ is conjugate symmetric, i.e.,

$$H_1(e^{-j\omega}, p) = H_1(e^{j\omega}, p)^* \quad (5)$$

where $*$ denotes the complex conjugate. Substituting (4) into (2), the transfer function can be rewritten as

$$\begin{aligned} H_1(z, p) &= \sum_{m=0}^M \sum_{n=0}^N a_{nm} z^{-n} p^m \\ &= \sum_{m=0}^M A_m(z) p^m \end{aligned} \quad (6)$$

where $A_m(z) = \sum_{n=0}^N a_{nm} z^{-n}$. Thus, the filter $H_1(z, p)$ can

be implemented by the structure I shown in Fig.1(a) once the sub-filters $A_m(z)$ have been designed. This structure is the well-known Farrow structure in the literature. Because the sub-filters $A_m(z)$ are all fixed, we can adjust the parameter p to change the order of the differentiator. Now, the design problem becomes how to find a_{nm} such that the frequency response $H_1(e^{j\omega}, p)$ fits $H_d(\omega, p)$ as well as possible. In the following, the weighted least squares method is described. Using the lexicographic ordering to map two-dimensional (2-D) sequences into one-dimensional (1-D) sequences, the frequency response $H_1(e^{j\omega}, p)$ can be rewritten as

$$H_1(e^{j\omega}, p) = \mathbf{a}^T \mathbf{c}(\omega, p) \quad (7)$$

where superscript T denotes transpose and two vectors are defined by

$$\begin{aligned} \mathbf{a} &= [a_{00} \quad a_{01} \quad \cdots \quad a_{NM}]^T \\ \mathbf{c}(\omega, p) &= [e^{-j\omega 0} p^0 \quad e^{-j\omega 0} p^1 \quad \cdots \quad e^{-j\omega N} p^M]^T \end{aligned} \quad (8)$$

In WLS method, the coefficient vector \mathbf{a} is obtained by minimizing the weighted least squares error below:

$$J_1(\mathbf{a}) = \int_{-0.5}^{0.5} \int_{\Omega} W(\omega, p) |H_1(e^{j\omega}, p) - H_d(\omega, p)|^2 d\omega dp \quad (9)$$

where $W(\omega, p)$ is a nonnegative weighting function, and region $\Omega = \Omega^+ \cup \Omega^-$ with interested frequency bands given by $\Omega^+ = [\omega_1, \omega_2]$ and $\Omega^- = [-\omega_2, -\omega_1]$. Substituting (7) into (9) and using conjugate symmetry in (5), we get

$$\begin{aligned} J_1(\mathbf{a}) &= \int_{-0.5}^{0.5} \int_{\Omega} W(\omega, p) |\mathbf{a}^T \mathbf{c}(\omega, p) - H_d(\omega, p)|^2 d\omega dp \\ &= \mathbf{a}^T \Phi \mathbf{a} - 2\mathbf{a}^T \boldsymbol{\varphi} + r \end{aligned} \quad (10)$$

where matrix Φ vector $\boldsymbol{\varphi}$ and scalar r are given by

$$\begin{aligned} \Phi &= 2 \int_{-0.5}^{0.5} \int_{\omega_1}^{\omega_2} W(\omega, p) \text{Re}[\mathbf{c}(\omega, p) \mathbf{c}(\omega, p)^H] d\omega dp \\ \boldsymbol{\varphi} &= 2 \int_{-0.5}^{0.5} \int_{\omega_1}^{\omega_2} W(\omega, p) \text{Re}[\mathbf{c}(\omega, p)^* H_d(\omega, p)] d\omega dp \\ r &= 2 \int_{-0.5}^{0.5} \int_{\omega_1}^{\omega_2} W(\omega, p) |H_d(\omega, p)|^2 d\omega dp \end{aligned} \quad (11)$$

where superscript H denotes the Hermitian and $\text{Re}[\cdot]$ stands for real part of a complex number. Because $J_1(\mathbf{a})$ is a

quadratic function of \mathbf{a} , the optimal solution is unique and can be obtained by solving simultaneous linear equation:

$$\Phi \mathbf{a} = \boldsymbol{\varphi} \quad (12)$$

Since matrix Φ is a positive-definite, real and symmetric matrix, the simultaneous linear equation can be solved by a computationally efficient method, like Cholesky decomposition. Now, let us study one example below.

Example 1: In this example, the performance of Farrow structure is studied. The design parameters are chosen as $N = 40$, $M = 5$, $D = 20$, $W(\omega, p) = 1$, $\omega_1 = 0.05\pi$ and $\omega_2 = 0.95\pi$. Fig.2(a)(b) show the magnitude response and absolute error of frequency response of the designed variable fractional order differentiator. To evaluate the performance, the normalized root mean squares (NRMS) error is defined by

$$E_k = \left(\frac{\int_{\omega_1}^{\omega_2} \int_{-0.5}^{0.5} |H_k(e^{j\omega}, p) - H_d(\omega, p)|^2 dp d\omega}{\int_{\omega_1}^{\omega_2} \int_{-0.5}^{0.5} |H_d(\omega, p)|^2 dp d\omega} \right)^{\frac{1}{2}} \times 100\% \quad (13)$$

Obviously, the smaller NRMS error E_k , the better performance of the design method. In this paper, the double integrals in Eq.(13) are computed by using numerical rectangular integration method in which the step size of ω is $\frac{\pi}{300}$ and the step size of p is $\frac{1}{40}$. In this example, the NRMS error E_1 is 0.6028%. Because the sub-filters $A_k(z)$ are not linear phase filters, the number of filter coefficients needed to be stored in memory is $(N+1)(M+1) = 246$. This number is often large, so it is an interesting research topic to reduce the memory requirement of filter coefficient storage in the Farrow structure of Fig.1(a). In this paper, the expansion method of hyperbolic function will be used to achieve this purpose. Moreover, the number of multiplications to implement all sub-filters of the Farrow structure I in Fig.1(a) is $(N+1)(M+1) = 246$ for this example.

3. PROPOSED DESIGN METHOD

In this section, the hyperbolic function is first reviewed. Then, the power series expansions of hyperbolic functions are used to design variable fractional order differentiator. Two implementation structures are developed which only contain two kinds of sub-filters, so the memory requirement of filter coefficient storage can be reduced.

3.1 Hyperbolic Function

The hyperbolic cosine and sine functions are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad (14a)$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad (14b)$$

Clearly, $\cosh x$ is an even function, and $\sinh x$ is an odd function. It is easy to show that the following equality is valid:

$$e^x = \cosh x + \sinh x \quad (15)$$

This means that the exponential function is the sum of hyperbolic cosine and sine functions. Once $\cosh x$ and $\sinh x$ are expanded into the infinite series, the exponential function e^x is expanded. In what follows, two structures are studied:

3.2 Structure II

From the book [10], we have the power series expansion of hyperbolic function:

$$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \quad (16a)$$

$$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \quad (16b)$$

Substituting (16) into (15), we have

$$e^x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + x \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+1)!} \quad (17)$$

Because the frequency response $(j\omega)^p$ is rewritten as the following form:

$$\begin{aligned} (j\omega)^p &= e^{\ln(j\omega)^p} \\ &= e^{p \ln(j\omega)} \end{aligned} \quad (18)$$

Using (18) and the variable substitution $x = p \ln(j\omega)$, equation (17) reduces to

$$\begin{aligned} (j\omega)^p &= \sum_{k=0}^{\infty} \frac{[\ln(j\omega)]^{2k} p^{2k}}{(2k)!} \\ &+ \ln(j\omega) p \sum_{k=0}^{\infty} \frac{[\ln(j\omega)]^{2k} p^{2k}}{(2k+1)!} \end{aligned} \quad (19)$$

Because the p is a fractional number, the above series expansion can be truncated into the following form

$$\begin{aligned} (j\omega)^p &\approx \sum_{k=0}^L \frac{[\ln(j\omega)]^{2k} p^{2k}}{(2k)!} \\ &+ \ln(j\omega) p \sum_{k=0}^L \frac{[\ln(j\omega)]^{2k} p^{2k}}{(2k+1)!} \end{aligned} \quad (20)$$

In the above, the truncation orders of both series expansions are chosen as L . Choosing $D = (L+1)R$ and multiplying both sides of (20) by the factor $e^{-j\omega D}$, we obtain the following expression:

$$\begin{aligned} H_d(\omega, p) &= (j\omega)^p e^{-j\omega D} \\ &\approx \sum_{k=0}^L \frac{\{[\ln(j\omega)]^2 e^{-j\omega R}\}^k (e^{-j\omega R})^{L+1-k}}{(2k)!} p^{2k} \\ &+ p [\ln(j\omega) e^{-j\omega R}] \sum_{k=0}^L \frac{\{[\ln(j\omega)]^2 e^{-j\omega R}\}^k (e^{-j\omega R})^{L-k}}{(2k)!(2k+1)} p^{2k} \end{aligned} \quad (21)$$

If two sub-filters $B(z)$ and $F(z)$ are designed to satisfy the following two approximation conditions:

$$B(e^{j\omega}) \approx [\ln(j\omega)]^2 e^{-j\omega R} \quad (22a)$$

$$F(e^{j\omega}) \approx \ln(j\omega) e^{-j\omega R} \quad (22b)$$

then the frequency response of the following filter

$$\begin{aligned} H_2(z, p) &= \sum_{k=0}^L \frac{B(z)^k z^{-R(L+1-k)}}{(2k)!} p^{2k} \\ &+ pF(z) \sum_{k=0}^L \frac{B(z)^k z^{-R(L-k)}}{(2k)!(2k+1)} p^{2k} \end{aligned} \quad (23)$$

will approximate the ideal response $H_d(\omega, p)$ well. Moreover, the above filter can be implemented efficiently by using the structure II in Fig.1(b) where we choose $L = 2$. In this structure, the sub-filters corresponding to p^{2k} and p^{2k+1} share the same filter $B(z)^k$. Compared the structures in Fig.1(a)(b), three observations are made below:

(1) Storage requirement: There are $M+1$ sub-filters $A_k(z)$ needed to be designed and implemented in structure I, but there are only two sub-filters $B(z)$ and $F(z)$ needed to be designed and realized in structure II. So, the memory storage of filter coefficients in structure II is less than that of conventional structure I.

(2) Filter delay: The delay D of structure I is often chosen as half order of sub-filters $A_k(z)$ in (6), that is, $D = \frac{N}{2}$. And, the delay of structure II is $D = (L+1)R = (L+1) \times \frac{N}{2}$ if delay of sub-filters $B(z)$ and $F(z)$ is chosen as $R = \frac{N}{2}$. So, the delay of structure II is greater than the one of structure I.

(3) Implementation complexity: The structure I needs to implement $M+1$ sub-filters, but the structure II only needs to implement $L+1$ sub-filters. In this paper, the L is chosen half of M , so the implementation complexity of structure II is less than that of structure I. This is because the sub-filters corresponding to p^{2k} and p^{2k+1} share the same filter $B(z)^k$ in the structure II.

Now, the remaining problem is how to design two sub-filters $B(z)$ and $F(z)$ in structure II. The transfer functions of two sub-filter are chosen as

$$B(z) = \sum_{n=0}^N b(n) z^{-n} \quad (24a)$$

$$F(z) = \sum_{n=0}^N f(n) z^{-n} \quad (24b)$$

And, the filter coefficients are determined by minimizing the following cost function:

$$J_2(\mathbf{b}) = \int_{\omega_1}^{\omega_2} |B(e^{j\omega}) - [\ln(j\omega)]^2 e^{-j\omega R}|^2 d\omega \quad (25a)$$

$$J_3(\mathbf{f}) = \int_{\omega_1}^{\omega_2} |F(e^{j\omega}) - \ln(j\omega) e^{-j\omega R}|^2 d\omega \quad (25b)$$

where the coefficient vectors $\mathbf{b} = [b(0) \ b(1) \ \dots \ b(N)]^T$ and $\mathbf{f} = [f(0) \ f(1) \ \dots \ f(N)]^T$. Because this is a standard least squares FIR filter design problem, its optimal solu-

tion can be obtained easily [11]. Moreover, the delay R is often chosen as half order of sub-filter, so we select $R = \frac{N}{2}$ in the design of structure II. Now, let us study one example below.

Example 2: In this example, the performance of structure II is studied. The design parameters are chosen as $N = 40$, $L = 2$, $R = 20$, $D = (L+1)R = 60$, $\omega_1 = 0.05\pi$ and $\omega_2 = 0.95\pi$. Fig.3(a)(b) show the magnitude response and absolute error of frequency response of the designed variable fractional order differentiator. In this example, the NRMS error E_2 is 0.5192%. The number of filter coefficients of sub-filters $B(z)$ and $F(z)$ needed to be stored in memory is $2(N+1) = 82$. Moreover, the number of multiplications to implement all sub-filters of the structure II in Fig.1(b) is $(N+1)(L+1) = 123$ for this example.

3.3 Structure III

In equation (20), the truncation orders of both series expansions are equal to L . In fact, these two truncation orders may differ. Now, one of the other choices is studied below:

$$(j\omega)^p \approx \sum_{k=0}^L \frac{[\ln(j\omega)]^{2k} p^{2k}}{(2k)!} + \ln(j\omega) p \sum_{k=0}^{L-1} \frac{[\ln(j\omega)]^{2k} p^{2k}}{(2k+1)!} \quad (26)$$

Clearly, the truncation orders of two series expansions are L and $L-1$. Choosing $D = LR$ and multiplying both sides by the factor $e^{-j\omega D}$, we obtain the following expression:

$$H_d(\omega, p) = (j\omega)^p e^{-j\omega D} \approx \sum_{k=0}^L \frac{\{[\ln(j\omega)]^2 e^{-j\omega R}\}^k (e^{-j\omega R})^{L-k}}{(2k)!} p^{2k} + p [\ln(j\omega) e^{-j\omega R}] \sum_{k=0}^{L-1} \frac{\{[\ln(j\omega)]^2 e^{-j\omega R}\}^k (e^{-j\omega R})^{L-1-k}}{(2k)!(2k+1)} p^{2k} \quad (27)$$

Using the approximations of two sub-filters $B(z)$ and $F(z)$ in equation (22), then the frequency response of the following filter

$$H_3(z, p) = \sum_{k=0}^L \frac{B(z)^k z^{-R(L-k)}}{(2k)!} p^{2k} + p F(z) \sum_{k=0}^{L-1} \frac{B(z)^k z^{-R(L-1-k)}}{(2k)!(2k+1)} p^{2k} \quad (28)$$

will approximate the ideal response $H_d(\omega, p)$ well. The above filter can be implemented efficiently by using the structure III shown in Fig.1(c) where we choose $L = 3$. In this structure, the sub-filters corresponding to p^{2k} and p^{2k+1} also share the same filter $B(z)^k$. The comparisons of structure III with conventional Farrow structure in Fig.1(a)

are similar to those of structure II, so they are omitted here. The designs of sub-filters $B(z)$ and $F(z)$ in structure III are also same as those in structure II. Now, let us study one design example of structure III below.

Example 3: In this example, the performance of structure III is studied. The design parameters are chosen as $N = 40$, $L = 3$, $R = 20$, $D = LR = 60$, $\omega_1 = 0.05\pi$ and $\omega_2 = 0.95\pi$. In this example, the NRMS error E_3 is 0.5120%. The number of filter coefficients of sub-filters $B(z)$ and $F(z)$ needed to be stored in memory is $2(N+1) = 82$. Moreover, the number of multiplications to implement all sub-filters of the structure III in Fig.1(c) is $(N+1)(L+1) = 164$ for this example.

Compared the results in examples 1-3, it is clear that the proposed structures II and III require less memory to store filter coefficients and less number of multiplications to implement VFOD than conventional Farrow structure I. However, the proposed structures need a longer filter delay D .

4. CONCLUSIONS

In this paper, a series expansion of hyperbolic function has been presented to implement variable fractional order differentiator. As a result, the proposed VFOD requires less storage requirement of filter coefficients and implementation complexity than conventional Farrow structure at expense of longer filter delay. Some numerical examples are demonstrated to show the effectiveness of the proposed structure. However, only one-dimensional fractional order differentiator is studied in this paper. Thus, it is interesting to extend the proposed method to design two-dimensional fractional order differentiator in the future.

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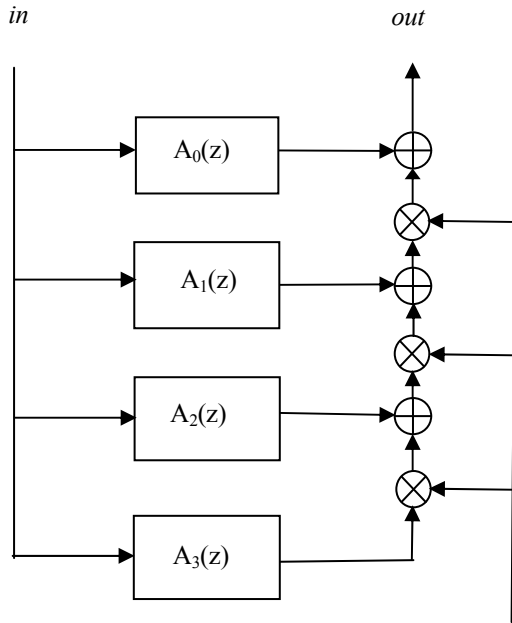


Fig.1 (a) Structure I.

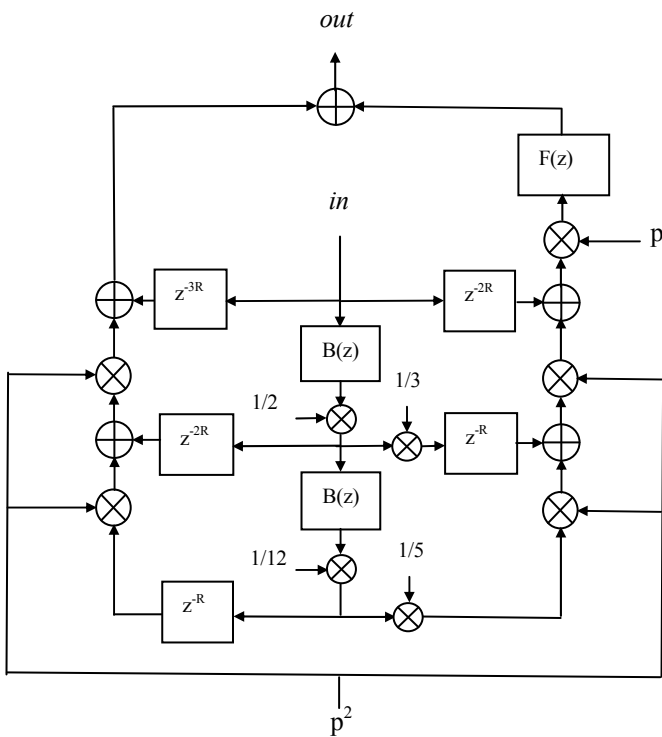


Fig.1 (b) Structure II.

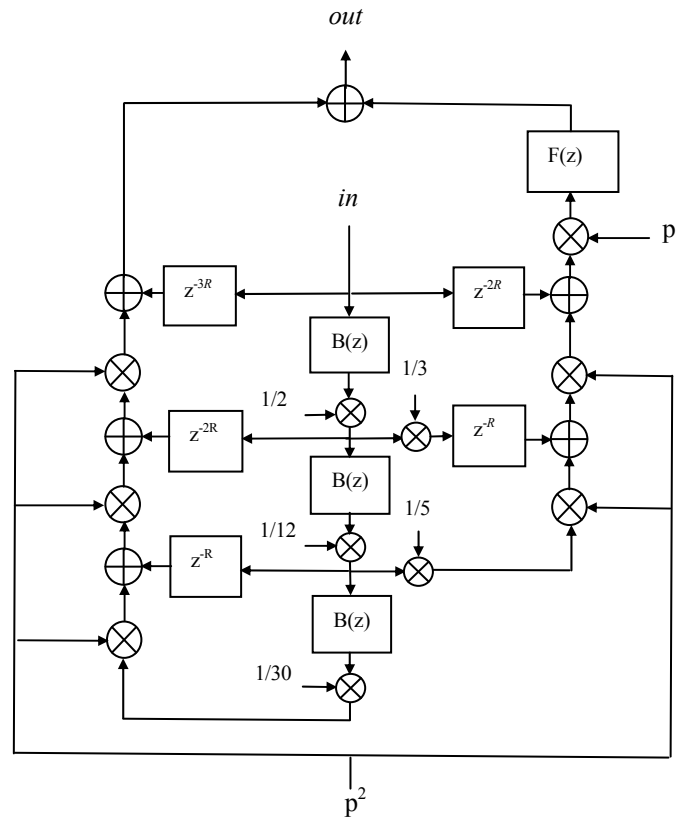


Fig.1(c) Structure III.

Fig.1 The structures for implementing the variable fractional order differentiator (a) Structure I with $M=3$. (b) Structure II with $L=2$. (c) Structure III with $L=3$.

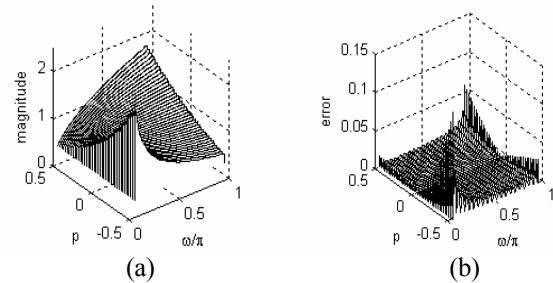


Fig.2 The designed results of structure I. (a) Magnitude response $|H_1(e^{j\omega}, p)|$ (b) Error $|H_1(e^{j\omega}, p) - H_d(\omega, p)|$ of frequency response.

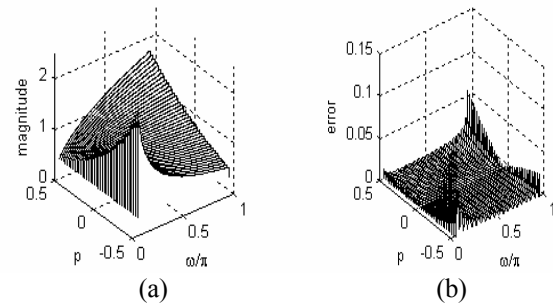


Fig.3 The designed results of structure II. (a) Magnitude response $|H_2(e^{j\omega}, p)|$ (b) Error $|H_2(e^{j\omega}, p) - H_d(\omega, p)|$ of frequency response.