A NEW STRUCTURE FOR THE DESIGN OF VARIABLE FRACTIONAL-DELAY FIR FILTERS

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ABSTRACT

In this paper, a new structure is introduced for the design of variable fractional-delay (VFD) FIR filters. Based on Taylor series expansion of the desired frequency response, an effective structure can be derived. Design example shows that the performance of the proposed system is better or the required number of independent coefficients is less than the existing structures.

1. INTRODUCTION

Variable fractional-delay (VFD) filters belong to a branch of variable digital filters which are applied in applications in which the frequency characteristics need to be adjustable online without redesigning a new filter. Due to their wide applications in signal processing and communication systems, the design of VFD filters has received considerable attention in the past decade. Since the Farrow structure is proposed in 1988 [1], several works have been announced [2-14], especially, basing on FIR-based structure [3-5] [7] [9] [13-14]. Conventionally, the transfer function for designing a VFD FIR filter is represented by

$$H(z,p)=\sum_{n=0}^{N}\sum_{m=0}^{M}h_c(n,m)p^m z^{-n}=\sum_{m=0}^{M}G_{c,m}(z)p^m \tag{1}$$

where p is the adjustable parameter, N is set to be even in the paper and the subfilters

$$G_{c,m}(z)=\sum_{n=0}^{N}h_c(n,m)z^{-n}\,. \tag{2}$$

The whole system in (1) can be implemented by the structure shown in Fig. 1(a) [1]. In this paper, the integer M is assumed to be odd for simplicity. For improving the complexity, Deng has recently proposed a hybrid structure [14] and the transfer function of low-complexity system can be represented by

$$H(z,p)=z^{-K}\sum_{m=0}^{\hat{M}}G_{d,2m}(z)p^{2m}+O(z)\sum_{m=0}^{\hat{M}}G_{d,2m+1}(z)p^{2m+1} \tag{3}$$

where $\hat{M}=(M-1)/2$,

$$G_{d,2m}(z)=\sum_{n=0}^{N_I}h_d(n,2m)z^{-n}\text{ , Type I subfilter, } N_I\text{ : even,} \tag{4a}$$

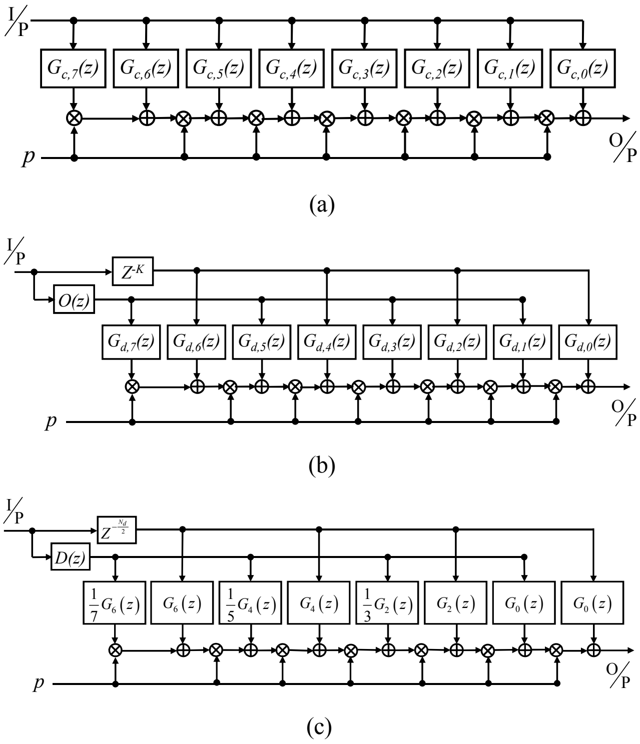

Figure 1 – The structures for VFD FIR filters. (M=7) (a) Conventional method. [1] (b) Deng's method. [14] (c) Proposed method.

$$O(z)=\sum_{n=0}^{N_o}h_o(n)z^{-n}\text{ , Type II one-band prefilter, } N_o\text{ : odd,} \tag{4b}$$

$$G_{d,2m+1}(z)=\sum_{n=0}^{N_{IV}}h_d(n,2m+1)z^{-n}\text{ , Type IV subfilter, } N_{IV}\text{ : odd} \tag{4c}$$

and K must satisfy

$$K+\frac{N_I}{2}=\frac{N_0+N_{IV}}{2}\,. \tag{5}$$

The corresponding structure of (3) is shown in Fig. 1(b).
In this paper, a new structure will be proposed for the design of VFD FIR filters. The structure is derived based on Taylor series expansion of $e^{-j\omega p}$, and the system design will be de-

veloped in Section 2. By the given example, it can be shown that the accuracy of the proposed method is much better than the conventional method under the cost of larger delay. Comparing the proposed method with Deng's method [14], the demonstration also shows that the required number of independent coefficients for the proposed structure is less than that in the Deng's structure when the accuracy of the proposed method is comparable with Deng's method.

2. PROBLEM FORMULATION AND DESIGN EXAMPLE

For designing a VFD digital filter, the desired frequency response is given by

$$H_d(\omega,p)=e^{-j(I+p)\omega},\quad |\omega|\le\omega_p,\quad -0.5\le p\le 0.5 \tag{6}$$

where I is a prescribed integer. Applying Taylor series expansion,

$$\begin{aligned}
&H_d(\omega,p)\\
&=e^{-jI\omega}\sum_{m=0}^{\infty}\frac{(-j\omega p)^m}{m!}\\
&\approx e^{-jI\omega}\sum_{m=0}^{M}\frac{(-j\omega p)^m}{m!}\\
&=e^{-jI\omega}\left(\sum_{m=0}^{\hat{M}}\frac{(-j\omega p)^{2m}}{(2m)!}+\sum_{m=0}^{\hat{M}}\frac{(-j\omega p)^{2m+1}}{(2m+1)!}\right)\\
&=e^{-jI\omega}\left(1+\frac{-\omega^2}{2!}p^2+\frac{\omega^4}{4!}p^4+\cdots+\frac{(-1)^{\hat{M}}\omega^{2\hat{M}}}{(2\hat{M})!}p^{2\hat{M}}\right)\\
&+e^{-jI\omega}(-j\omega p)\left(1+\frac{1}{3}\frac{-\omega^2}{2!}p^2+\frac{1}{5}\frac{\omega^4}{4!}p^4+\cdots+\frac{1}{M}\frac{(-1)^{\hat{M}}\omega^{2\hat{M}}}{(2\hat{M})!}p^{2\hat{M}}\right).
\end{aligned} \tag{7}$$

Hence, the proposed transfer function in this paper is represented by

$$\begin{aligned}
&H(z,p)\\
&=z^{-\frac{N_d}{2}}\left(G_0(z)+G_2(z)p^2+G_4(z)p^4+\cdots+G_{2\hat{M}}(z)p^{2\hat{M}}\right)\\
&+D(z)\left(G_0(z)p+\frac{1}{3}G_2(z)p^3+\frac{1}{5}G_4(z)p^5+\cdots+\frac{1}{M}G_{2\hat{M}}(z)p^{M}\right)
\end{aligned} \tag{8}$$

and the system structure is shown in Fig. 1(c). In (8), the transfer functions of $D(z)$ and $G_{2m}(z)$ are given as below:

$$D(z)=\sum_{n=0}^{N_d}d(n)z^{-n}\text{, Type III prefilter(differentiator), } N_d\text{ : even ,} \tag{9a}$$

$$G_{2m}(z)=\sum_{n=0}^{N_g}g(n,m)z^{-n}\text{, Type I subfilter, } N_g\text{ : even .} \tag{9b}$$

It can be observed that $D(z)$ is designed as a differentiator with magnitude $-\omega$.

The frequency response of (8) can be represented by

$$\begin{aligned}
&H(e^{j\omega},p)\\
&=e^{-j\left(\frac{N_d}{2}+\frac{N_g}{2}\right)\omega}\left(\sum_{m=0}^{\hat{M}}\hat{G}_{2m}(\omega)p^{2m}+j\hat{D}(\omega)\sum_{m=0}^{\hat{M}}\frac{1}{2m+1}\hat{G}_{2m}(\omega)p^{2m+1}\right)\\
&=e^{-j\left(\frac{N_d}{2}+\frac{N_g}{2}\right)\omega}\hat{H}(\omega,p)
\end{aligned} \tag{10}$$

where

$$\hat{D}(\omega)=\sum_{n=1}^{N_d/2}\hat{d}(n)\sin(n\omega), \tag{11a}$$

$$\hat{G}_{2m}(\omega)=\sum_{n=0}^{N_g/2}\hat{g}(n,m)\cos(n\omega), \tag{11b}$$

$$\hat{H}(\omega,p)=\sum_{m=0}^{\hat{M}}\hat{G}_{2m}(\omega)p^{2m}+j\hat{D}(\omega)\sum_{m=0}^{\hat{M}}\frac{1}{2m+1}\hat{G}_{2m}(\omega)p^{2m+1} \tag{11c}$$

and

$$\hat{d}(n)=2d\left(\frac{N_d}{2}-n\right),\quad 1\le n\le\frac{N_d}{2}, \tag{11d}$$

$$\hat{g}(n,m)=\begin{cases}g\left(\frac{N_g}{2},m\right), & n=0,\\ 2g\left(\frac{N_g}{2}-n,m\right), & 1\le n\le\frac{N_g}{2}.\end{cases} \tag{11e}$$

By (6) and (10), the integer delay I in (6) can be set as $I=\frac{N_d}{2}+\frac{N_g}{2}$.

2.1 Design of the prefilter

By defining

$$\mathbf{d}=\left[\hat{d}(1),\ \hat{d}(2),\cdots,\hat{d}\left(\frac{N_d}{2}\right)\right]^T \tag{12a}$$

$$\mathbf{s}(\omega)=\left[\sin(\omega),\ \sin(2\omega),\cdots,\sin\left(\frac{N_d}{2}\omega\right)\right]^T \tag{12b}$$

where T denotes a transpose operator, (11a) can be represented in a vector product form as

$$\hat{D}(\omega)=\mathbf{d}^T\mathbf{s}(\omega). \tag{13}$$

Hence, the objective error function for designing the prefilter $D(z)$ can be represented by

$$e(\mathbf{d})=\int_0^{\omega_p}\left(-\omega-\mathbf{d}^T\mathbf{s}(\omega)\right)^2d\omega=s_d+\mathbf{r}_d^T\mathbf{d}+\mathbf{d}^T\mathbf{Q}_d\mathbf{d} \tag{14}$$

where

$$s_d=\int_0^{\omega_p}\omega^2d\omega=\frac{\omega_p^3}{3}, \tag{15a}$$

$$\mathbf{r}_d = 2\int_0^{\omega_p} \omega \mathbf{s}(\omega) d\omega, \tag{15b}$$

$$\mathbf{Q}_d = \int_0^{\omega_p} \mathbf{s}(\omega)\mathbf{s}^T(\omega) d\omega, \tag{15c}$$

and the solution is

$$\mathbf{d} = -\frac{1}{2}\mathbf{Q}_d^{-1}\mathbf{r}_d \tag{16}$$

2.2 Design of the subfilters $G_{2m}(z)$

By (7) and (8), the frequency response of $G_0(z)$ is inherently required to meet $e^{-j\frac{N_g}{2}\omega}$, so its coefficients can be set as

$$g(n,0) = \delta\left(n - \frac{N_g}{2}\right) \tag{17}$$

and (11c) can be rewritten as

$$\begin{aligned}\hat{H}(\omega,p) = 1 + \sum_{m=1}^{\hat{M}}\sum_{n=0}^{N_g/2} \hat{g}(n,m)p^{2m}\cos(n\omega) + jp\hat{D}(\omega) \\ + j\hat{D}(\omega)\sum_{m=1}^{\hat{M}}\sum_{n=0}^{N_g/2}\frac{1}{2m+1}\hat{g}(n,m)p^{2m+1}\cos(n\omega).\end{aligned} \tag{18}$$

Let

$$\mathbf{g} = \left[\hat{g}(0,1),\cdots,\hat{g}(\tfrac{N_g}{2},1),\cdots,\hat{g}(0,\hat{M}),\cdots,\hat{g}(\tfrac{N_g}{2},\hat{M})\right]^T, \tag{19a}$$

$$\mathbf{c}_r(\omega,p) = \left[p^2,\cdots,p^2\cos(\tfrac{N_g}{2}\omega),\cdots,p^{2\hat{M}},\cdots,p^{2\hat{M}}\cos(\tfrac{N_g}{2}\omega)\right]^T, \tag{19b}$$

$$\mathbf{c}_i(\omega,p) = \left[\frac{1}{3}p^3,\cdots,\frac{1}{3}p^3\cos(\tfrac{N_g}{2}\omega),\cdots,\frac{1}{M}p^M,\cdots,\frac{1}{M}p^M\cos(\tfrac{N_g}{2}\omega)\right]^T, \tag{19c}$$

(18) can be represented as

$$\hat{H}(\omega,p) = 1 + \mathbf{g}^T\mathbf{c}_r(\omega,p) + jp\hat{D}(\omega) + j\hat{D}(\omega)\mathbf{g}^T\mathbf{c}_i(\omega,p) \tag{20}$$

where $\hat{D}(\omega)$ is assumed to be derived by the technique in Subsection 2.1. Hence, the objective error function for designing the subfilters $G_{2m}(z)$ can be represented by

$$\begin{aligned}&e(\mathbf{g}) \\ &= \int_{-0.5}^{0.5}\int_0^{\omega_p}\left|e^{-j\omega p} - \hat{H}(\omega,p)\right|^2 d\omega dp \\ &= \int_{-0.5}^{0.5}\int_0^{\omega_p}\Big\{\left[\cos(\omega p) - 1 - \mathbf{g}^T\mathbf{c}_r(\omega,p)\right]^2 \\ &\qquad + \left[-\sin(\omega p) - p\hat{D}(\omega) - \hat{D}(\omega)\mathbf{g}^T\mathbf{c}_i(\omega,p)\right]^2\Big\} d\omega dp \\ &= s + \mathbf{r}^T\mathbf{g} + \mathbf{g}^T\mathbf{Q}\mathbf{g}\end{aligned} \tag{21}$$

where

$$\begin{aligned}s = \int_{-0.5}^{0.5}\int_0^{\omega_p}\left(\cos(\omega p) - 1\right)^2 d\omega dp \\ + \int_{-0.5}^{0.5}\int_0^{\omega_p}\left(\sin(\omega p) + p\hat{D}(\omega)\right)^2 d\omega dp,\end{aligned} \tag{22a}$$

$$\begin{aligned}\mathbf{r} = -2\int_{-0.5}^{0.5}\int_0^{\omega_p}\left(\cos(\omega p) - 1\right)\mathbf{c}_r(\omega,p) d\omega dp \\ + 2\int_{-0.5}^{0.5}\int_0^{\omega_p}\left(\sin(\omega p) + p\hat{D}(\omega)\right)\hat{D}(\omega)\mathbf{c}_i(\omega,p) d\omega dp,\end{aligned} \tag{22b}$$

$$\begin{aligned}\mathbf{Q} = \int_{-0.5}^{0.5}\int_0^{\omega_p}\mathbf{c}_r(\omega,p)\mathbf{c}_r^T(\omega,p) d\omega dp \\ + \int_{-0.5}^{0.5}\int_0^{\omega_p}\hat{D}^2(\omega)\mathbf{c}_i(\omega,p)\mathbf{c}_i^T(\omega,p) d\omega dp\end{aligned}, \tag{22c}$$

and the desired solution is

$$\mathbf{g} = -\frac{1}{2}\mathbf{Q}^{-1}\mathbf{r}. \tag{23}$$

2.3 Design example

To evaluate the performance of the presented methods, the normalized root-mean-squared error of variable frequency response, the maximum absolute error of variable frequency response and the maximum absolute group-delay error are defined by

$$\varepsilon_{rms} = \left[\frac{\int_{-0.5}^{0.5}\int_0^{\omega_p}\left|H_d(\omega,p) - H(e^{j\omega},p)\right|^2 d\omega dp}{\int_{-0.5}^{0.5}\int_0^{\omega_p}\left|H_d(\omega,p)\right|^2 d\omega dp}\right]^{1/2} \times 100\% \tag{24a}$$

$$\varepsilon_m = \max\left\{\left|H_d(\omega,p) - H(e^{j\omega},p)\right|,\ 0 \le \omega \le \omega_p,\ -0.5 \le p \le 0.5\right\} \tag{24b}$$

$$\varepsilon_\tau = \max\left\{\left|\tau_d(\omega,p) - \tau(\omega,p)\right|,\ 0 \le \omega \le \omega_p,\ -0.5 \le p \le 0.5\right\}, \tag{24c}$$

respectively, where $\tau_d(\omega,p)$ and $\tau(\omega,p)$ denote the desired group-delay response and the actual group-delay response, respectively. To compute the errors in (24), the frequency ω and the parameter p are uniformly sampled at step sizes $\omega_p/200$ and $1/60$, respectively.

Notice that the required numbers of independent coefficients for the conventional method [1] [7], Deng's method [14] and the proposed method are

Conventional method : $\left(\frac{N}{2}+1\right)\frac{M-1}{2} + \frac{N}{2}\frac{M+1}{2}$,

Deng's method : $\frac{N_o+1}{2} + \left(\frac{N_I}{2}+1\right)\frac{M-1}{2} + \left(\frac{N_{IV}+1}{2}\right)\frac{M+1}{2}$,

Proposed method : $\frac{N_d}{2} + \left(\frac{N_g}{2}+1\right)\frac{M-1}{2} + \frac{M-1}{2}$.

For system delay,

Conventional method : $\frac{N}{2}$,

Deng's method : $\frac{N_o+N_{IV}}{2}$,

Proposed method : $\frac{N_d+N_g}{2}$.

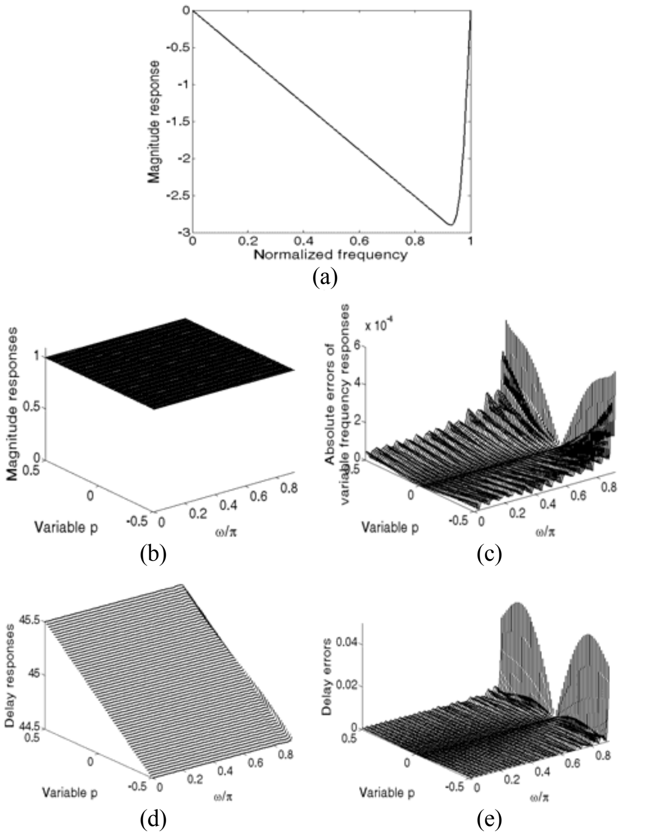

Figure 2 – Design of a VFD FIR filter by using the proposed method. (a) Magnitude response of the prefilter $D(z)$. (b) Magnitude of variable frequency response. (c) Absolute error of variable frequency response. (d) Group-delay response. (e) Absolute error of group-delay response.

Example 1: In this example, a VFD FIR filter is designed with $\omega_p = 0.92\pi$ and $M = 7$. For the proposed method, $N_d = 62$ and $N_g = 28$ are used. Fig. 2(a) presents the magnitude response of the prefilter $D(z)$, the magnitude and absolute error of variable frequency response are shown in Fig. 2(b) and (c), respectively, while Fig. 2(d) and (e) depict the group-delay response and the group-delay error, respectively. For comparison, $N = 50$ is used in the conventional method, and $N_o = 61$, $N_I = 30$, $N_{IV} = 29$ are adopted in Deng's method. In Table 1, the related results are tabulated and it can be noted that the accuracy of the proposed system is much better than that of the conventional method under the cost of larger delay. Also, it has been shown that the number of independent coefficients for the proposed method is less than that in Deng's method under comparable performance.

3. CONCLUSION

A new structure has been proposed for the design of VFD FIR filters in this paper. The structure is derived basing on Taylor series expansion of $e^{-j\omega p}$. Comparing with the existing structures, for example the conventional Farrow structure [1] and the hybrid structure proposed by Deng [14], the performance of the proposed structure is much better than that of the former under the cost of larger delay, and the number of independent coefficients is less than that of Deng's method under comparable performance.

Table 1. Comparisons for the design of a VFD FIR filter with $\omega_p = 0.92\pi$ and $M = 7$.

Method	Conventional method[1][13]	Deng's method [14]	Proposed method
Filter orders	$N = 50$	$N_o = 61$, $N_I = 30$, $N_{IV} = 29$	$N_d = 62$, $N_g = 28$
Number of independent coefficients	178	139	76
Delay	25	45	45
$\varepsilon_{rms}(\%)$	0.01304431	0.00501232	0.00523281
$\varepsilon_m\left(\times 10^{-4}\right)$	22.489788	6.70100328	5.35265579
ε_τ	0.11499281	0.05199766	0.04596809

Remark: For the conventional method, the VFD FIR filter is designed by the technique in Section I of [13].

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