CRB FOR ACTIVE RADAR

(1) *Tarek Menni,* (1)(2) *Eric Chaumette,* (1) *Pascal Larzabal,* (1) *Jean-Pierre Barbot*

(1)SATIE, ENS Cachan, CNRS, UniverSud, 61 av President Wilson, F-94230 Cachan, France
(2)ONERA - The French Aerospace Lab, DEMR/TSI, Chemin de la Huniere, F-91120 Palaiseau, France

ABSTRACT

Provided that one keeps in mind the CRB limitations, that is, to become an overly optimistic lower bound when the observation conditions degrades, the CRB is a lower bound of great interest for radar system analysis and design in the asymptotic region. Even though there are many available CRB formulas for target parameters in the open literature, each CRB formula has been derived under particular radar modelling. Therefore, what is missing is a neat CRB formula valid independently of the radar modelling and its underlying approximations. The aim of this paper is to provide a general CRB closed-form valid for all possible diversities (temporal, spatial, code) in case of Gaussian circular deterministic observations, which describe accurately an active radar system.

1. INTRODUCTION

Minimal performance bounds allow for calculation of the best performance that can be achieved in the Mean Square Error (MSE) sense, when estimating parameters of a signal corrupted by noise. Historically the first MSE lower bound for deterministic parameters to be derived was the Cramér-Rao Bound (CRB), which was introduced to investigate fundamental limits of a parameter estimation problem or to assess the relative performance of a specific estimator (efficiency) [15]. It has since become the most popular lower bound due to its simplicity of calculation, the fact that in many cases it can be achieved asymptotically (high SNR and/or large number of snapshots) by Maximum Likelihood Estimators (MLE) [15], and last but not least, its noticeable property of being the lowest bound on the MSE of unbiased estimators. This initial characterization of unbiased estimators has been significantly generalized by Barankin work, who derived the highest lower bound on the MSE (BB) of unbiased estimators, which is generally incomputable analytically [1][14].

Therefore, since then, numerous works detailed in [1][14] have been devoted to deriving computable approximations of the BB and have shown that the CRB and the BB can be regarded as key representatives of two general classes of bounds, respectively the Small-Error bounds and the Large-Error bounds. These works have also shown that in non-linear estimation problems three distinct regions of operation can be observed. In the asymptotic region, the MSE is small and, in many cases, close to the Small-Error bounds. In the *a priori* performance region where the number of independent snapshots and/or the SNR are very low, the observations provide little information and the MSE is close to that obtained from the prior knowledge about the problem. Between these two extremes, there is an additional ambiguity region, also called the transition region. In this region, the MSE of MLEs usually deteriorates rapidly with respect to Small-Error bounds and exhibits a threshold behaviour corresponding to a "performance breakdown". The nature of this phenomenon is specified by a complicated non-smooth behaviour of the likelihood function in the "threshold" area where it tends to generate outliers [15].

Small-Error bound such as the CRB are not able to handle the threshold phenomena, whereas it is revealed by Large-Error bounds that can be used to predict the threshold value. Unfortunately, the computational coast of Large-Error bounds is prohibitive in most applications when the number of unknown parameters increases.

Therefore, provided that one keeps in mind the CRB limitations, that is, to becomes an overly optimistic lower bound when the observation conditions degrades (low SNR and/or low number of snapshots), the CRB is still a lower bound of great interest for radar (or other) system analysis and design in the asymptotic region, as it is simple to calculate and it is usually possible to obtain general closed form expressions. In active radar and sonar, a known waveform is transmitted, and the signals reflected from the targets of interest are used to estimate their parameters. Typically, the received signals are modelled as scaled, delayed, and Doppler-shifted versions of the transmitted signal; see, e.g., [16]. Estimation of the time delay and Doppler shift provides information about the range and radial velocity of the targets. The use of spatial diversity, i.e. antenna arrays, compared with a single sensor, guarantees more accurate range and velocity estimation and allows estimation of the targets direction. Last, but no least, waveform diversity [18] may be used to improve the estimation of all targets parameters. Thus, it is of interest to compute CRB expressions for active radars that are able to take into account all possible diversities. Even though there are many CRB formulas on this topic since numerous works have been done in this field (see references in [15][18]), each CRB formula is specific to some particular radar modelling (narrow band arrays, narrow band transmitted signals, temporally white noise, Doppler effect approximation,....). Therefore, to the best of our knowledge, what is missing is a neat CRB formula valid for any radar signals, independently of underlying approximations. It is the aim of this paper to provide such a general CRB closed-form for Gaussian circular deterministic observation model. Indeed nowadays, the capability of modelling the radar cross section (RCS), or even better the backscattering matrix, of complex target along a given trajectory allows to refine the statistical predictions available from the Swerling amplitude fluctuation models. Therefore, provided relevant backscattering information is available, the most accurate statistical prediction will be obtained from the deterministic observation model (in comparison with he stochastic one which corresponds to Swerling I-II targets).

As the backscattering parameters are generally complex, the first part of the paper (Sec. II) is dedicated to outline a generalization to complex parameters of the Barankin rationale for deriving MSE lower bounds [1] that avoids sophisticated matrix manipulations generally used with complex parameters [5][19]. This approach is worth knowing at it allows simple derivation (constrained CRB) or clarification (regularity conditions) of existing results as detailed in [9]. The second part of the paper (Sec. III and IV) is dedicated not only to outline the derivation of a general CRB formula for

This work has been partly funded by the European Network of excellence NEWCOM++ under the number 216715

band-limited deterministic radar observations (21) that encompasses all previously released CRB expressions, but also to provide a general method optimizing its use.

2. CRAMÉR-RAO BOUND FOR MIXED (REAL AND COMPLEX) PARAMETERS

The notational convention adopted is as follows: a, $\mathbf{a}$, $\mathbf{A}$ indicates respectively a scalar, a vector and a matrix quantity. The matrix/vector conjugate is indicated by a superscript * and the matrix/vector transpose conjugate is indicated by a superscript ^{H}. If $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_P)^T$, then: $\frac{\partial}{\partial \boldsymbol{\theta}} = \left(\frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta_2}, \ldots, \frac{\partial}{\partial \theta_P}\right)^T$. $\odot$ denotes the Hadamard product. $\otimes$ denotes the Kronecker product. $1(\mathbf{x})$ denotes the constant real-valued function equal to 1. $\underline{\mathbf{x}}$ denotes:

$$\begin{cases} \underline{\mathbf{x}} = \mathbf{x} & \text{if } \mathbf{x} \in \mathbb{R}^Q \\ \underline{\mathbf{x}} = \begin{pmatrix} \mathbf{x} \\ \mathbf{x}^* \end{pmatrix} & \text{if } \mathbf{x} \in \mathbb{C}^Q \backslash \mathbb{R}^Q \\ \underline{\begin{pmatrix} \mathbf{x}_c \\ \mathbf{x}_r \end{pmatrix}} = \begin{pmatrix} \mathbf{x}_c \\ \mathbf{x}_c^* \\ \mathbf{x}_r \end{pmatrix} & \text{if } \mathbf{x}_c \in \mathbb{C}^Q \setminus \mathbb{R}^Q,\ \mathbf{x}_r \in \mathbb{R}^{Q'} \end{cases} \tag{1}$$

Regarding the definition of Hermitian product $\langle \mid \rangle$, we adopt the convention used in books of mathematics [4] where a sesquilinear form is a function in two variables on a complex vector space $\mathbb{U}$ which is linear in the first variable and semi-linear in the second. This convention allows to define the Gram matrix $\mathbf{G}\left(\{\mathbf{u}\}_{[1,Q]}, \{\mathbf{c}\}_{[1,P]}\right)$ ($P \times Q$ complex matrix) associated to 2 families of vectors of $\mathbb{U}$, $\{\mathbf{u}\}_{[1,Q]} = \{\mathbf{u}_1, \ldots, \mathbf{u}_Q\}$ and $\{\mathbf{c}\}_{[1,P]} = \{\mathbf{c}_1, \ldots, \mathbf{c}_P\}$ as the one verifying [4]:

$$\left\langle \sum_{q=1}^{Q} x_q \mathbf{u}_q \mid \sum_{p=1}^{P} y_p \mathbf{c}_p \right\rangle = \mathbf{y}^H \mathbf{G}\left(\mathbf{u}_{[1,Q]}, \mathbf{c}_{[1,P]}\right) \mathbf{x} \tag{2}$$

where $\mathbf{x} = (x_1, \ldots, x_Q)^T$, $\mathbf{y} = (y_1, \ldots, y_P)^T$. For notational convenience $\mathbf{G}\left(\{\mathbf{u}\}_{[1,Q]}\right) = \mathbf{G}\left(\{\mathbf{u}\}_{[1,Q]}, \{\mathbf{u}\}_{[1,Q]}\right)$. Beware that most reference signal processing books [15] adopt the opposite convention for sesquilinear form, that is to be semi-linear in the first variable and linear in the second. As a consequence, the equivalent form in "signal processing notation" of any inequality introduced in the present paper is obtained by transposing inequality terms (matrices).

2.1 Differentiability on real or complex field

The sets of complex ($\mathbb{C}$) and real ($\mathbb{R}$) numbers being two fields, the differentiability of a vector of functions $\mathbf{f}(\boldsymbol{\theta}) : (\mathbb{k}')^P \to \mathbb{k}^Q$ where $\mathbb{k}' \equiv \mathbb{C}$ or $\mathbb{R}$ and $\mathbb{k} \equiv \mathbb{C}$ or $\mathbb{R}$ can be characterized by the following property:

$$\mathbf{f}(\boldsymbol{\theta} + d\boldsymbol{\theta}) = \mathbf{f}(\boldsymbol{\theta}) + \frac{\partial \mathbf{f}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^T} d\boldsymbol{\theta} + \mathbf{o}(d\boldsymbol{\theta}), \begin{cases} \mathbf{o}(d\boldsymbol{\theta}) = d\boldsymbol{\theta} \odot \boldsymbol{\varepsilon}(d\boldsymbol{\theta}) \\ \lim\limits_{\|d\boldsymbol{\theta}\| \to 0} \|\boldsymbol{\varepsilon}(d\boldsymbol{\theta})\| = 0 \end{cases} \tag{3}$$

where $\| \ \|$ is the canonical Hermitian (or Euclidian) norm on $(\mathbb{k}')^P$ and $\mathbb{k}^Q$: $\|\mathbf{x}\| = \sqrt{\sum_i |x_i|^2}$. Actually, (3) still holds if $\boldsymbol{\theta}$ have mixed components (complex and real):

$$\begin{cases} \boldsymbol{\theta} = \left(\mathrm{Re}\{\boldsymbol{\theta}_c\}^T, \mathrm{Im}\{\boldsymbol{\theta}_c\}^T, \boldsymbol{\theta}_r^T\right)^T \\ \underline{\boldsymbol{\theta}} = \left(\boldsymbol{\theta}_c^T, (\boldsymbol{\theta}_c^*)^T, \boldsymbol{\theta}_r^T\right)^T \end{cases}, \ \boldsymbol{\theta}_c \in \mathbb{C}^{P_c},\ \boldsymbol{\theta}_r \in \mathbb{R}^{P_r}.$$

Then, any functions of $\boldsymbol{\theta}$ can be written in a dual form:

$$\begin{cases} \mathbf{f}(\boldsymbol{\theta}) & : \mathbb{R}^{2P_c + P_r} \to \mathbb{k}^Q \\ \mathbf{f}(\boldsymbol{\theta}) = \mathbf{h}(\boldsymbol{\theta}_c^1, \boldsymbol{\theta}_c^2, \boldsymbol{\theta}_r)\big|_{(\boldsymbol{\theta}_c, \boldsymbol{\theta}_c^*, \boldsymbol{\theta}_r)} & : \mathbb{C}^{2P_c} \times \mathbb{R}^{P_r} \to \mathbb{k}^Q \end{cases},$$

where $\mathbf{h}(\boldsymbol{\theta}_c^1, \boldsymbol{\theta}_c^2, \boldsymbol{\theta}_r) = \mathbf{f}\left(\frac{\boldsymbol{\theta}_c^1 + \boldsymbol{\theta}_c^2}{2}, \frac{\boldsymbol{\theta}_c^1 - \boldsymbol{\theta}_c^2}{2j}, \boldsymbol{\theta}_r\right)$. Therefore, if $\mathbf{f}(\ )$ and $\mathbf{h}(\ )$ are differentiable (1)(3):

$$\frac{\partial \mathbf{f}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^T} d\boldsymbol{\theta} = \frac{\partial \mathbf{f}(\underline{\boldsymbol{\theta}})}{\partial \underline{\boldsymbol{\theta}}^T} d\underline{\boldsymbol{\theta}} \tag{4}$$

provided that $\boldsymbol{\theta}_c$ and $\boldsymbol{\theta}_c^*$ are formally considered as independent variables for derivation. Note that identity (4) still holds if $P_r = 0$ or $P_c = 0$.

2.2 Unified derivation

Throughout the present paper, unless otherwise stated, $\mathbf{x}$ denotes the random observation vector of dimension N, Ω denotes the observations space and $L^2(\Omega)$ denotes the complex Hilbert space of square integrable functions over Ω. The probability density function (p.d.f.) of $\mathbf{x}$ is denoted $p(\mathbf{x}; \boldsymbol{\theta})$ and depends on a vector of P real parameters $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_P) \in \Theta$, where Θ denotes the parameter space. Additionally, we assume that the observation vector $\mathbf{x}$ corresponds to a parametric observation model involving $P_r \geq 0$ real unknown parameters (delays, directions of arrival, ...) and $P_c \geq 0$ complex unknown parameters (spatial transfer functions components, complex amplitudes, ...) where $2P_c + P_r = P$, leading to a p.d.f. of the dual form:

$$p(\mathbf{x}; \boldsymbol{\theta}),\ \boldsymbol{\theta} = \left(\mathrm{Re}\left\{\boldsymbol{\theta}_c^T\right\}, \mathrm{Im}\left\{\boldsymbol{\theta}_c^T\right\}, \boldsymbol{\theta}_r^T\right)^T \tag{5}$$

$$p(\mathbf{x}; \underline{\boldsymbol{\theta}}),\ \underline{\boldsymbol{\theta}} = \left(\boldsymbol{\theta}_c^T, (\boldsymbol{\theta}_c^*)^T, \boldsymbol{\theta}_r^T\right)^T \tag{6}$$

In the following we will only consider the form (6) since it includes (5) when $P_c = 0$. Let $\underline{\boldsymbol{\theta}}^0$ be a selected value of the parameter $\underline{\boldsymbol{\theta}}$, and $\widehat{\mathbf{g}(\underline{\boldsymbol{\theta}}^0)}(\mathbf{x})$ an estimator of $\mathbf{g}(\underline{\boldsymbol{\theta}}^0)$ where $\mathbf{g}(\underline{\boldsymbol{\theta}}) = (g_1(\underline{\boldsymbol{\theta}}), \ldots, g_{Q_c}(\underline{\boldsymbol{\theta}}), g_{Q_c+1}(\underline{\boldsymbol{\theta}}), \ldots, g_Q(\underline{\boldsymbol{\theta}}))^T$ is a vector of Q functions of $\underline{\boldsymbol{\theta}}$, the first Q_c ones being complex-valued functions, the last $Q_r = Q - Q_c$ being real-valued functions, where $Q_c \in [0, Q]$. Then, the statistical performance of any estimator of $\mathbf{g}(\underline{\boldsymbol{\theta}}^0)$ is fully characterized - including characterization of real and imaginary parts [9] - in the MSE sense, by the computation of:

$$MSE_{\underline{\boldsymbol{\theta}}^0}\left[\boldsymbol{\delta}^T \underline{\widehat{\mathbf{g}(\underline{\boldsymbol{\theta}}^0)}(\mathbf{x})}\right] = \int_\Omega \left|\boldsymbol{\delta}^T \left(\underline{\widehat{\mathbf{g}(\underline{\boldsymbol{\theta}}^0)}(\mathbf{x}) - \mathbf{g}(\underline{\boldsymbol{\theta}}^0)}\right)\right|^2 p(\mathbf{x}; \underline{\boldsymbol{\theta}}^0)\, d\mathbf{x},$$

which is a norm deriving from an Hermitian product $\langle \mid \rangle_{\underline{\boldsymbol{\theta}}^0}$:

$$MSE_{\underline{\boldsymbol{\theta}}^0}\left[\boldsymbol{\delta}^T \underline{\widehat{\mathbf{g}(\underline{\boldsymbol{\theta}}^0)}(\mathbf{x})}\right] = \boldsymbol{\delta}^H \mathbf{G}_{\underline{\boldsymbol{\theta}}^0}\left(\left\{\underline{\widehat{\mathbf{g}(\underline{\boldsymbol{\theta}}^0)}(\mathbf{x}) - \mathbf{g}(\underline{\boldsymbol{\theta}}^0)}\right\}\right) \boldsymbol{\delta} \tag{7}$$

$$\langle g(\mathbf{x}) \mid h(\mathbf{x}) \rangle_{\underline{\boldsymbol{\theta}}} = E_{\underline{\boldsymbol{\theta}}}\left[g(\mathbf{x})\, h^*(\mathbf{x})\right]$$

where:

$$\{\mathbf{h}(\mathbf{x})\} = \{h_1(\mathbf{x}), \ldots, h_Q(\mathbf{x})\} \tag{8}$$

denotes a family of vectors which elements are the vector components. Hence the interest of finding a matrix $\mathbf{B}_{\underline{\boldsymbol{\theta}}^0}$ independent of $\widehat{\mathbf{g}(\underline{\boldsymbol{\theta}}^0)}(\mathbf{x})$ and able to lower bound expression (7). To avoid the trivial solution $\mathbf{B}_{\underline{\boldsymbol{\theta}}^0} = \mathbf{0}$, all that is required is to define a constraint that is not satisfied by the trivial solution, as local unbiasedness for example:

$$E_{\underline{\boldsymbol{\theta}}^0 + d\underline{\boldsymbol{\theta}}}\left[\underline{\widehat{\mathbf{g}(\underline{\boldsymbol{\theta}}^0)}(\mathbf{x})}\right] = \mathbf{g}(\underline{\boldsymbol{\theta}}^0 + d\underline{\boldsymbol{\theta}}) + \mathbf{o}(d\underline{\boldsymbol{\theta}}) \tag{9}$$

meaning that, up to the first order and in the neighbourhood of $\underline{\boldsymbol{\theta}}^0$, $\widehat{g_q(\underline{\boldsymbol{\theta}}^0)}(\mathbf{x})$ remains an unbiased estimator of $g_q(\underline{\boldsymbol{\theta}}^0)$

independently of a - small - variation of $\underline{\boldsymbol{\theta}}$. Actually (9) can be rewritten in two different but equivalent forms:

$$\begin{aligned}
E_{\underline{\boldsymbol{\theta}}^0+d\underline{\boldsymbol{\theta}}}\left[\underline{\widehat{\mathbf{g}\left(\underline{\boldsymbol{\theta}}^0\right)}\left(\mathbf{x}\right)}\right] &= \mathbf{o}\left(d\underline{\boldsymbol{\theta}}\right)+\underline{\mathbf{g}}\left(\underline{\boldsymbol{\theta}}^0\right)+\frac{\partial\underline{\mathbf{g}}\left(\underline{\boldsymbol{\theta}}^0\right)}{\partial\underline{\boldsymbol{\theta}}^T}d\underline{\boldsymbol{\theta}} \\
E_{\underline{\boldsymbol{\theta}}^0+d\underline{\boldsymbol{\theta}}}\left[\underline{\widehat{\mathbf{g}\left(\underline{\boldsymbol{\theta}}^0\right)}\left(\mathbf{x}\right)}\right] &= \mathbf{o}\left(d\underline{\boldsymbol{\theta}}\right)+E_{\underline{\boldsymbol{\theta}}^0}\left[\underline{\widehat{\mathbf{g}\left(\underline{\boldsymbol{\theta}}^0\right)}\left(\mathbf{x}\right)}\right]+ \\
& \qquad E_{\underline{\boldsymbol{\theta}}^0}\left[\underline{\widehat{\mathbf{g}\left(\underline{\boldsymbol{\theta}}^0\right)}\left(\mathbf{x}\right)}\frac{\partial\ln p\left(\mathbf{x};\underline{\boldsymbol{\theta}}^0\right)}{\partial\underline{\boldsymbol{\theta}}^T}\right]d\underline{\boldsymbol{\theta}}
\end{aligned}$$

leading to the constraints (uniqueness of Taylor series):

$$\begin{cases} E_{\underline{\boldsymbol{\theta}}^0}\left[\underline{\widehat{\mathbf{g}\left(\underline{\boldsymbol{\theta}}^0\right)}\left(\mathbf{x}\right)}\frac{\partial\ln p\left(\mathbf{x};\underline{\boldsymbol{\theta}}^0\right)}{\partial\underline{\boldsymbol{\theta}}^T}\right]=\frac{\partial\underline{\mathbf{g}}\left(\underline{\boldsymbol{\theta}}^0\right)}{\partial\underline{\boldsymbol{\theta}}^T} \\ E_{\underline{\boldsymbol{\theta}}^0}\left[\underline{\widehat{\mathbf{g}\left(\underline{\boldsymbol{\theta}}^0\right)}\left(\mathbf{x}\right)}\right]=\underline{\mathbf{g}}\left(\underline{\boldsymbol{\theta}}^0\right)\end{cases}$$

which can be rewritten as:

$$\mathbf{G}_{\underline{\boldsymbol{\theta}}^0}\left(\{\mathbf{u}\}_{[1,Q]},\{\mathbf{c}\}_{[1,P+1]}\right)=\begin{bmatrix}\frac{\partial\underline{\mathbf{g}}^T\left(\underline{\boldsymbol{\theta}}^0\right)}{\partial\underline{\boldsymbol{\theta}}} \\ \mathbf{0}^T\end{bmatrix} \tag{10}$$

where $\{\mathbf{u}\}_{[1,Q]}=\left\{\underline{\widehat{\mathbf{g}\left(\underline{\boldsymbol{\theta}}^0\right)}\left(\mathbf{x}\right)}-\underline{\mathbf{g}}\left(\underline{\boldsymbol{\theta}}^0\right)\right\}$ and $\{\mathbf{c}\}_{[1,P+1]}=\left\{\frac{\partial\ln p\left(\mathbf{x};\underline{\boldsymbol{\theta}}\right)}{\partial\underline{\boldsymbol{\theta}}}^*,1\left(\mathbf{x}\right)\right\}$. As a consequence, the problem of finding a lower bound of $\mathbf{G}_{\underline{\boldsymbol{\theta}}^0}\left(\{\mathbf{u}\}_{[1,Q]}\right)$ in (7) for locally unbiased estimators (9), amounts to the minimization of $\mathbf{G}_{\underline{\boldsymbol{\theta}}^0}\left(\{\mathbf{u}\}_{[1,Q]}\right)$ under the set of linear constraints (10), which solution is a standard algebra result [9]:

$$\min\left\{\mathbf{G}_{\underline{\boldsymbol{\theta}}^0}\left(\{\mathbf{u}\}\right)\right\}=\left(\frac{\partial\underline{\mathbf{g}}\left(\underline{\boldsymbol{\theta}}^0\right)^T}{\partial\underline{\boldsymbol{\theta}}}\right)^H\mathbf{F}_{\underline{\boldsymbol{\theta}}^0}^{-1}\left(\frac{\partial\underline{\mathbf{g}}\left(\underline{\boldsymbol{\theta}}^0\right)^T}{\partial\underline{\boldsymbol{\theta}}}\right) \tag{11}$$

$$\mathbf{F}_{\underline{\boldsymbol{\theta}}^0}=\mathbf{G}_{\underline{\boldsymbol{\theta}}^0}\left(\left\{\frac{\partial\ln p\left(\mathbf{x};\underline{\boldsymbol{\theta}}\right)}{\partial\underline{\boldsymbol{\theta}}}^*\right\}\right)=E_{\underline{\boldsymbol{\theta}}^0}\left[\frac{\partial\ln p\left(\mathbf{x};\underline{\boldsymbol{\theta}}\right)}{\partial\underline{\boldsymbol{\theta}}}\frac{\partial\ln p\left(\mathbf{x};\underline{\boldsymbol{\theta}}\right)}{\partial\underline{\boldsymbol{\theta}}}^H\right]$$

$$\left(\underline{\widehat{\mathbf{g}\left(\underline{\boldsymbol{\theta}}^0\right)}\left(\mathbf{x}\right)}-\underline{\mathbf{g}}\left(\underline{\boldsymbol{\theta}}^0\right)\right)_{eff}^T=\frac{\partial\ln p\left(\mathbf{x};\underline{\boldsymbol{\theta}}\right)}{\partial\underline{\boldsymbol{\theta}}}^H\mathbf{F}_{\underline{\boldsymbol{\theta}}^0}^{-1}\frac{\partial\underline{\mathbf{g}}^T\left(\underline{\boldsymbol{\theta}}^0\right)}{\partial\underline{\boldsymbol{\theta}}}$$

where $\mathbf{F}_{\underline{\boldsymbol{\theta}}^0}^{-1}=\mathbf{CRB}_{\underline{\boldsymbol{\theta}}^0}$ and $\mathbf{F}_{\underline{\boldsymbol{\theta}}^0}$ is the Fisher Information Matrix (FIM).

For sake of completeness, notice is hereby given that the derivation outlined in this section (and detailed in [9]), allows a unique simple derivation, whatever the nature (real or complex) of the unknown parameters that [9]:

- avoids sophisticated matrix manipulations generally used with complex parameters [5][19],
- corrects previously incomplete derivation,
- allows to condense to a few lines previous works [5][12] on FIM singularity and constrained CRB,
- allows to clarify standard regularity conditions so far needlessly too restrictive: usual regularity condition on the differentiability of both $p\left(\mathbf{x};\boldsymbol{\theta}\right)$ and $\mathbf{g}\left(\boldsymbol{\theta}\right)$ at $\boldsymbol{\theta}^0$ can be relaxed to semi-differentiability (left and right differentiability) only, under certain conditions.

3. CRB FOR A SINGLE DETERMINISTIC BAND LIMITED OBSERVATION

In radar, and many other practical problems of interest (sonar, communication, ...), the complex observation vector $\mathbf{x}$ consists of a bandpass signal with bandwidth B $\left(f\in\left[-\frac{B}{2},\frac{B}{2}\right]\right)$, which is the output of an Hilbert filtering leading to an "in-phase" real part associated to a "quadrature" imaginary part [15] *i.e.* a complex circular vector of the form:

$$\mathbf{x}\left(t;\underline{\boldsymbol{\theta}}\right)=\mathbf{s}\left(t;\underline{\boldsymbol{\theta}}_{\mathbf{s}}\right)+\mathbf{n}\left(t;\underline{\boldsymbol{\theta}}_{\mathbf{n}}\right),\quad\underline{\boldsymbol{\theta}}^T=\left(\underline{\boldsymbol{\theta}}_{\mathbf{s}}^T,\underline{\boldsymbol{\theta}}_{\mathbf{n}}^T\right) \tag{12}$$

$$\mathbf{s}\left(t;\underline{\boldsymbol{\theta}}_{\mathbf{s}}\right)=\sum_{m=1}^{M}\mathbf{b}\left(t;\boldsymbol{\varepsilon}_m\right)\sigma_m=\mathbf{B}\left(t;\boldsymbol{\Xi}\right)\boldsymbol{\sigma}$$

$$\underline{\boldsymbol{\theta}}_{\mathbf{s}}^T=\left(\boldsymbol{\sigma}^T,\boldsymbol{\sigma}^H,\boldsymbol{\Xi}^T\right),\boldsymbol{\Xi}^T=\left(\boldsymbol{\varepsilon}_1^T,\ldots,\boldsymbol{\varepsilon}_M^T\right),\boldsymbol{\sigma}=\left(\sigma_1,\ldots,\sigma_M\right)^T$$

where:

- $\mathbf{s}\left(t;\underline{\boldsymbol{\theta}}_{\mathbf{s}}\right)$ is the radar signal of interest consisting of M backscattered signals function of a parametric propagation model $\mathbf{b}\left(t;\boldsymbol{\varepsilon}_m\right)$ of finite duration T depending on K real parameters $\boldsymbol{\varepsilon}_m^T=\left(\varepsilon_{1,m},\ldots,\varepsilon_{K,m}\right)$, and of a complex backscattered amplitude σ_m constant during duration T,
- $\mathbf{n}\left(t;\underline{\boldsymbol{\theta}}_{\mathbf{n}}\right)$ is the nuisance signal consisting of noise plus interference contribution depending on the parameters $\underline{\boldsymbol{\theta}}_{\mathbf{n}}$.

Under the assumption of Gaussian centred nuisance and unknown a priori p.d.f. $p\left(\underline{\boldsymbol{\sigma}}\right)$, (12) belongs to the set of deterministic observation models [15] which p.d.f. at t is:

$$p\left(\mathbf{x};t,\underline{\boldsymbol{\theta}}\right)=\frac{e^{-\left(\mathbf{x}\left(t;\underline{\boldsymbol{\theta}}\right)-\mathbf{s}\left(t;\underline{\boldsymbol{\theta}}_{\mathbf{s}}\right)\right)^H\mathbf{C}_{\mathbf{n}}^{-1}\left(t,\underline{\boldsymbol{\theta}}_{\mathbf{n}}\right)\left(\mathbf{x}\left(t;\underline{\boldsymbol{\theta}}\right)-\mathbf{s}\left(t;\underline{\boldsymbol{\theta}}_{\mathbf{s}}\right)\right)}}{\pi^N\left|\mathbf{C}_{\mathbf{n}}\left(t,\underline{\boldsymbol{\theta}}_{\mathbf{n}}\right)\right|}$$

Additionally, if $\mathbf{n}\left(t;\underline{\boldsymbol{\theta}}_{\mathbf{n}}\right)$ is a wide sense stationary (WSS) band limited process with spectral density matrix $\boldsymbol{\Gamma}\left(f,\underline{\boldsymbol{\theta}}_{\mathbf{n}}\right)$ and autocovariance matrix $\mathbf{R}\left(\tau,\underline{\boldsymbol{\theta}}_{\mathbf{n}}\right)$, then:

$$\boldsymbol{\Gamma}\left(f,\underline{\boldsymbol{\theta}}_{\mathbf{n}}\right)=\int_{-\infty}^{+\infty}\mathbf{R}\left(\tau,\underline{\boldsymbol{\theta}}_{\mathbf{n}}\right)e^{-j2\pi f\tau}d\tau=\sum_{-\infty}^{+\infty}\mathbf{R}\left(\frac{n}{B},\underline{\boldsymbol{\theta}}_{\mathbf{n}}\right)\frac{e^{-j2\pi\frac{f}{B}n}}{B},$$

and using previously released results such as:

- the FIM for temporally white nuisance [2][15],
- the property of the FIM to be invariant to reversible operations on the received signals [21],
- a theorem due to Whittle [3][17, th. 9],

it can be shown [10] that the FIM associated to (12) observed during the finite duration T of $\mathbf{b}\left(t;\boldsymbol{\varepsilon}_m\right)$ is given by:

$$\begin{aligned}
&\mathbf{F}\left(\underline{\boldsymbol{\theta}}\right)=\begin{bmatrix}\mathbf{F}_{\mathbf{s}}\left(\underline{\boldsymbol{\theta}}\right) & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{\mathbf{n}}\left(\underline{\boldsymbol{\theta}}\right)=T\mathbf{A}\left(\underline{\boldsymbol{\theta}}\right)\end{bmatrix} \\
&\mathbf{F}_{\mathbf{s}}\left(\underline{\boldsymbol{\theta}}\right)_{l,k}=2\operatorname{Re}\left\{\left\langle\frac{\partial\mathbf{s}\left(f;\underline{\boldsymbol{\theta}}_{\mathbf{s}}\right)}{\partial\left(\underline{\boldsymbol{\theta}}_{\mathbf{s}}\right)_k}\mid\frac{\partial\mathbf{s}\left(f;\underline{\boldsymbol{\theta}}_{\mathbf{s}}\right)}{\partial\left(\underline{\boldsymbol{\theta}}_{\mathbf{s}}\right)_l}\right\rangle_{\underline{\boldsymbol{\theta}}_{\mathbf{n}}}\right\} \\
&\mathbf{A}\left(\underline{\boldsymbol{\theta}}\right)_{l,k}=\int_{-\frac{B}{2}}^{\frac{B}{2}}tr\left(\boldsymbol{\Gamma}\left(f,\underline{\boldsymbol{\theta}}_{\mathbf{n}}\right)^{-1}\frac{\partial\boldsymbol{\Gamma}\left(f,\underline{\boldsymbol{\theta}}_{\mathbf{n}}\right)}{\partial\left(\underline{\boldsymbol{\theta}}_{\mathbf{n}}\right)_k}\boldsymbol{\Gamma}\left(f,\underline{\boldsymbol{\theta}}_{\mathbf{n}}\right)^{-1}\frac{\partial\boldsymbol{\Gamma}\left(f,\underline{\boldsymbol{\theta}}_{\mathbf{n}}\right)}{\partial\left(\underline{\boldsymbol{\theta}}_{\mathbf{n}}\right)_l}\right)df
\end{aligned} \tag{13}$$

$$\left\langle\mathbf{x}\left(f\right)\mid\mathbf{y}\left(f\right)\right\rangle_{\underline{\boldsymbol{\theta}}_{\mathbf{n}}}=\int_{-\frac{B}{2}}^{\frac{B}{2}}\mathbf{y}\left(f\right)^H\boldsymbol{\Gamma}\left(f,\underline{\boldsymbol{\theta}}_{\mathbf{n}}\right)^{-1}\mathbf{x}\left(f\right)df \tag{14}$$

As we are mainly interested in the estimation of the parameters of the M signals backscattered by the M targets, we will only focus in the following on $\mathbf{CRB}_{\underline{\boldsymbol{\theta}}_{\mathbf{s}}|\underline{\boldsymbol{\theta}}}\left(\underline{\boldsymbol{\theta}}\right)=\mathbf{F}_{\mathbf{s}}^{-1}\left(\underline{\boldsymbol{\theta}}\right)$ where:

$$\mathbf{F}_{\mathbf{s}}\left(\underline{\boldsymbol{\theta}}\right)_{l,k}=2\operatorname{Re}\left\{\left\langle\frac{\partial\mathbf{s}\left(f;\underline{\boldsymbol{\theta}}_{\mathbf{s}}\right)}{\partial\left(\underline{\boldsymbol{\theta}}_{\mathbf{s}}\right)_k}\mid\frac{\partial\mathbf{s}\left(f;\underline{\boldsymbol{\theta}}_{\mathbf{s}}\right)}{\partial\left(\underline{\boldsymbol{\theta}}_{\mathbf{s}}\right)_l}\right\rangle_{\underline{\boldsymbol{\theta}}_{\mathbf{n}}}\right\}$$

For sake of legibility in the following, the dependency of vectors and matrices of $L^2\left(\Omega\right)$, e.g. $\mathbf{s}\left(f;\underline{\boldsymbol{\theta}}_{\mathbf{s}}\right)$, $\mathbf{b}\left(f;\boldsymbol{\varepsilon}_m\right)$, $\mathbf{B}\left(f;\boldsymbol{\Xi}\right)$ ..., on frequency f will be omitted wherever this omission is unambiguous. Additionally, let notation $\{\mathbf{A}\left(f\right)\}$ be the generalization of (8) denoting the family of column vectors of matrix $\mathbf{A}\left(f\right)$:

$$\{\mathbf{A}\left(f\right)\}=\{\left[\mathbf{a}_1\left(f\right)\ldots\mathbf{a}_Q\left(f\right)\right]\}=\{\mathbf{a}_1\left(f\right),\ldots,\mathbf{a}_Q\left(f\right)\} \tag{15}$$

Then, if $\mathbf{A}(f) = [\mathbf{a}_1(f)\dots\mathbf{a}_Q(f)]$ and $\mathbf{C}(f) = [\mathbf{c}_1(f)\dots\mathbf{c}_P(f)]$, $\mathbf{G}_{\boldsymbol{\theta}_\mathbf{n}}(\{\mathbf{A}\},\{\mathbf{C}\})$ is the $P\times Q$ complex matrix defined by:

$$\mathbf{G}_{\boldsymbol{\theta}_\mathbf{n}}(\{\mathbf{A}\},\{\mathbf{C}\})_{p,q} = \langle \mathbf{a}_q \mid \mathbf{c}_p\rangle_{\underline{\boldsymbol{\theta}}_\mathbf{n}} = \langle \mathbf{a}_q(f) \mid \mathbf{c}_p(f)\rangle_{\underline{\boldsymbol{\theta}}_\mathbf{n}} \quad (16)$$

Let $\Pi_{\{\mathbf{B}(\boldsymbol{\Xi})\}}$ denote the orthonormal projector on *span* $\{\mathbf{B}(f;\boldsymbol{\Xi})\}$, i.e. the span of the vector columns of matrix $\mathbf{B}(f;\boldsymbol{\Xi})$:

$$\begin{aligned}\Pi_{\{\mathbf{B}(\boldsymbol{\Xi})\}}(\mathbf{a}) &= \Pi_{\{\mathbf{B}(f;\boldsymbol{\Xi})\}}(\mathbf{a}(f)) \qquad (17)\\ &= \mathbf{B}(f;\boldsymbol{\Xi})\,\mathbf{G}^{-1}_{\boldsymbol{\theta}_\mathbf{n}}(\{\mathbf{B}(\boldsymbol{\Xi})\})\,\mathbf{G}_{\boldsymbol{\theta}_\mathbf{n}}(\mathbf{a},\{\mathbf{B}(\boldsymbol{\Xi})\})\end{aligned}$$

and let $\Pi^{\perp}_{\{\mathbf{B}(\boldsymbol{\Xi})\}}$ denote the orthonorrmal projector on the orthogonal complement of *span* $\{\mathbf{B}(f;\boldsymbol{\Xi})\}$:

$$\Pi_{\{\mathbf{B}(\boldsymbol{\Xi})\}}(\mathbf{a}) + \Pi^{\perp}_{\{\mathbf{B}(\boldsymbol{\Xi})\}}(\mathbf{a}) = \mathbf{a} \quad (18)$$

Let us define:

$$\Pi_{\{\mathbf{B}(\boldsymbol{\Xi})\}}(\{\mathbf{A}\}) = \left\{\Pi_{\{\mathbf{B}(\boldsymbol{\Xi})\}}(\mathbf{a}_1),\dots,\Pi_{\{\mathbf{B}(\boldsymbol{\Xi})\}}(\mathbf{a}_Q)\right\} \quad (19)$$

then:

$$\mathbf{G}_{\boldsymbol{\theta}_\mathbf{n}}\left(\Pi^{\perp}_{\{\mathbf{B}(\boldsymbol{\Xi})\}}(\{\mathbf{A}\}),\Pi^{\perp}_{\{\mathbf{B}(\boldsymbol{\Xi})\}}(\{\mathbf{C}\})\right)_{p,q} = \mathbf{G}_{\boldsymbol{\theta}_\mathbf{n}}\left(\Pi^{\perp}_{\{\mathbf{B}(\boldsymbol{\Xi})\}}(\mathbf{a}_q),\Pi^{\perp}_{\{\mathbf{B}(\boldsymbol{\Xi})\}}(\mathbf{c}_p)\right)$$

$$\begin{aligned}\mathbf{G}_{\boldsymbol{\theta}_\mathbf{n}}\left(\Pi^{\perp}_{\{\mathbf{B}(\boldsymbol{\Xi})\}}(\mathbf{a}_q),\Pi^{\perp}_{\{\mathbf{B}(\boldsymbol{\Xi})\}}(\mathbf{c}_p)\right) &= \mathbf{G}_{\boldsymbol{\theta}_\mathbf{n}}(\mathbf{a}_q,\mathbf{c}_p)\\ &-\mathbf{G}^H_{\boldsymbol{\theta}_\mathbf{n}}(\mathbf{c}_p,\{\mathbf{B}(\boldsymbol{\Xi})\})\,\mathbf{G}^{-1}_{\boldsymbol{\theta}_\mathbf{n}}(\{\mathbf{B}(\boldsymbol{\Xi})\})\,\mathbf{G}_{\boldsymbol{\theta}_\mathbf{n}}(\mathbf{a}_q,\{\mathbf{B}(\boldsymbol{\Xi})\})\end{aligned}$$

Finally, using notation (15-19), some simple derivative computations lead to:

$$\begin{aligned}\mathbf{F_s}(\boldsymbol{\theta}) &= \begin{bmatrix}\mathbf{F}_{\boldsymbol{\sigma},\boldsymbol{\sigma}^*} & \mathbf{0} & \mathbf{F}_{\boldsymbol{\sigma},\boldsymbol{\Xi}}\\ \mathbf{0} & \mathbf{F}_{\boldsymbol{\sigma}^*,\boldsymbol{\sigma}} & \mathbf{F}_{\boldsymbol{\sigma}^*,\boldsymbol{\Xi}}\\ \mathbf{F}_{\boldsymbol{\Xi},\boldsymbol{\sigma}^*} & \mathbf{F}_{\boldsymbol{\Xi},\boldsymbol{\sigma}} & \mathbf{F}_{\boldsymbol{\Xi},\boldsymbol{\Xi}}\end{bmatrix}\\ \mathbf{F}_{\boldsymbol{\sigma},\boldsymbol{\sigma}^*} &= \mathbf{F}^*_{\boldsymbol{\sigma}^*,\boldsymbol{\sigma}} = \mathbf{G}_{\boldsymbol{\theta}_\mathbf{n}}(\{\mathbf{B}(\boldsymbol{\Xi})\},\{\mathbf{B}(\boldsymbol{\Xi})\})\\ \mathbf{F}_{\boldsymbol{\Xi},\boldsymbol{\sigma}^*} &= \mathbf{F}^H_{\boldsymbol{\sigma},\boldsymbol{\Xi}} = \mathbf{G}_{\boldsymbol{\theta}_\mathbf{n}}\left(\{\mathbf{B}(\boldsymbol{\Xi})\},\left\{\frac{\partial \mathbf{s}(\boldsymbol{\theta}_\mathbf{s})}{\partial \boldsymbol{\Xi}^T}\right\}\right)\\ \mathbf{F}_{\boldsymbol{\Xi},\boldsymbol{\sigma}} &= \mathbf{F}^H_{\boldsymbol{\sigma}^*,\boldsymbol{\Xi}} = \mathbf{F}^*_{\boldsymbol{\Xi},\boldsymbol{\sigma}^*}\end{aligned}$$

Then, by resorting to the inverse of a partitioned matrix, one obtains that [10]:

$$\mathbf{BCR}^{-1}_{\Xi}(\boldsymbol{\theta}) = 2\,\mathrm{Re}\left\{\mathbf{G}_{\boldsymbol{\theta}_\mathbf{n}}\left(\Pi^{\perp}_{\{\mathbf{B}(\boldsymbol{\Xi})\}}\left(\left\{\frac{\partial \mathbf{s}}{\partial \boldsymbol{\Xi}^T}\right\}\right)\right)\right\} \quad (20)$$

Additionally, after tedious, though straightforward computation detailed in [10], (20) can be rewritten as:

$$\mathbf{BCR}^{-1}_{\Xi}(\boldsymbol{\theta}) = 2\,\mathrm{Re}\left\{\mathbf{H}_{\Xi}(\boldsymbol{\theta})\odot(\boldsymbol{\Sigma}_s\otimes \mathbf{1}_{K\times K})\right\} \quad (21)$$

$$\mathbf{H}_{\Xi}(\boldsymbol{\theta}) = \begin{bmatrix}\mathbf{H}(\boldsymbol{\theta})_{1,1} & \dots & \mathbf{H}(\boldsymbol{\theta})_{1,M}\\ \vdots & \ddots & \vdots\\ \mathbf{H}(\boldsymbol{\theta})_{M,1} & \dots & \mathbf{H}(\boldsymbol{\theta})_{M,M}\end{bmatrix}, \quad \boldsymbol{\Sigma}_s = \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^H\right)^T$$

$$\mathbf{H}(\boldsymbol{\theta})_{m_1,m_2} = \mathbf{G}_{\boldsymbol{\theta}_\mathbf{n}}\left(\Pi^{\perp}_{\{\mathbf{B}(\boldsymbol{\Xi})\}}\left(\left\{\frac{\partial \mathbf{b}(\boldsymbol{\varepsilon}_{m_2})}{\partial \boldsymbol{\varepsilon}^T}\right\}\right), \Pi^{\perp}_{\{\mathbf{B}(\boldsymbol{\Xi})\}}\left(\left\{\frac{\partial \mathbf{b}(\boldsymbol{\varepsilon}_{m_1})}{\partial \boldsymbol{\varepsilon}^T}\right\}\right)\right)$$

where $\mathbf{1}_{P\times P}$ is a $P\times P$ matrix of ones.

4. CRB FOR L DETERMINISTIC BAND LIMITED OBSERVATIONS

Results released in the previous section can be extended to the observation of M targets during L independent band limited observations with band B^l:

$$\mathbf{x}^l\left(t;\underline{\boldsymbol{\theta}}^l\right) = \mathbf{s}\left(t;\underline{\boldsymbol{\theta}}^l_\mathbf{s}\right) + \mathbf{n}^l\left(t;\underline{\boldsymbol{\theta}}^l_\mathbf{n}\right), \quad \mathbf{s}\left(t;\underline{\boldsymbol{\theta}}^l_\mathbf{s}\right) = \mathbf{B}^l\left(t;\boldsymbol{\Xi}^l\right)\boldsymbol{\sigma}^l \quad (22)$$

where:

• $\left(\boldsymbol{\Xi}^l\right)^T = \left(\left(\boldsymbol{\varepsilon}^l_1\right)^T,\dots,\left(\boldsymbol{\varepsilon}^l_M\right)^T\right)$ and $\boldsymbol{\varepsilon}^l_m$ is the vector of parameters of dimension P^l for the l^{th} observation model and the m^{th} target,

• $\boldsymbol{\sigma}^l = \left(\sigma^l_1,\dots,\sigma^l_M\right)^T$ is the vector of complex amplitudes of the M targets for the l^{th} observation model,

• $\mathbf{B}^l\left(t;\boldsymbol{\Xi}^l\right) = \left[\mathbf{b}^l\left(t;\boldsymbol{\varepsilon}^l_1\right)\dots\mathbf{b}^l\left(t;\boldsymbol{\varepsilon}^l_M\right)\right]$ and $\mathbf{b}^l\left(t;\boldsymbol{\varepsilon}^l_m\right)$ is a vector of N parametric propagation model of finite duration T^l, depending on a vector of P^l real parameters $\boldsymbol{\varepsilon}^l_m$,

• $\mathbf{n}^l\left(t;\underline{\boldsymbol{\theta}}^l_\mathbf{n}\right)$ are Gaussian complex circular independent noises with spectral density matrix $\boldsymbol{\Gamma}^l\left(f;\underline{\boldsymbol{\theta}}^l_\mathbf{n}\right), f\in\left[-\frac{B^l}{2},\frac{B^l}{2}\right]$.

Let $\underline{\mathbf{O}}^T = \left(\underline{\mathbf{O}}^T_\mathbf{s},\underline{\mathbf{O}}^T_\mathbf{n}\right)$, $\underline{\mathbf{O}}^T_\mathbf{s} = \left(\left(\underline{\boldsymbol{\theta}}^1_\mathbf{s}\right)^T,\dots,\left(\underline{\boldsymbol{\theta}}^L_\mathbf{s}\right)^T\right)$ and $\underline{\mathbf{O}}^T_\mathbf{n} = \left(\left(\underline{\boldsymbol{\theta}}^1_\mathbf{n}\right)^T,\dots,\left(\underline{\boldsymbol{\theta}}^L_\mathbf{n}\right)^T\right)$. Then:

$$\mathbf{F}(\underline{\mathbf{O}}) = \begin{bmatrix}\mathbf{F_s}(\underline{\mathbf{O}}) & \mathbf{0}\\ \mathbf{0} & \mathbf{F_n}(\underline{\mathbf{O}})\end{bmatrix}, \mathbf{F_s}(\underline{\mathbf{O}}) = \begin{bmatrix}\mathbf{F_s}\left(\underline{\boldsymbol{\theta}}^1\right) & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \ddots & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{F_s}\left(\underline{\boldsymbol{\theta}}^L\right)\end{bmatrix} \quad (23)$$

where $\mathbf{F_s}\left(\underline{\boldsymbol{\theta}}^l\right)$ is given by (13) with:

$$\langle \mathbf{x}(f) \mid \mathbf{y}(f)\rangle_{\underline{\boldsymbol{\theta}}^l_\mathbf{n}} = \int_{-\frac{B^l}{2}}^{\frac{B^l}{2}} \mathbf{y}(f)^H\,\boldsymbol{\Gamma}^l\left(f,\underline{\boldsymbol{\theta}}^l_\mathbf{n}\right)^{-1}\mathbf{x}(f)\,df.$$

Expression (23) provides all the terms requested to compute any particular CRB associated to L observations.
As an example, in most applications and reference papers [19] or textbooks [6][11][15], a more restrictive model is generally considered where the vector of N parametric functions $\mathbf{b}^l\left(t;\boldsymbol{\varepsilon}^l_m\right)$ and the noise spectral density matrix $\boldsymbol{\Gamma}^l\left(f;\underline{\boldsymbol{\theta}}^l_\mathbf{n}\right)$ are invariant during the L observations:

$$\mathbf{b}^l\left(t;\boldsymbol{\varepsilon}^l_m\right) = \mathbf{b}(t;\boldsymbol{\varepsilon}_m), \quad \boldsymbol{\Gamma}^l\left(f;\underline{\boldsymbol{\theta}}^l_\mathbf{n}\right) = \boldsymbol{\Gamma}(f;\underline{\boldsymbol{\theta}}_\mathbf{n}). \quad (24)$$

Then, there are two ways of computing the resulting $\mathbf{BCR}^{-1}_{\Xi}(\boldsymbol{\theta})$. The first one (standard) consists in considering that the L observations are quasi i.i.d. ($\boldsymbol{\sigma}^l$ excepted). As a consequence, since the FIM are covariance matrices (11) associated to each observation (13), they can be added and:

$$\begin{aligned}\mathbf{BCR}^{-1}_{\Xi}(\boldsymbol{\theta}) &= 2L\,\mathrm{Re}\left\{\mathbf{H}_{\Xi}(\boldsymbol{\theta})\odot(\boldsymbol{\Sigma}_s\otimes\mathbf{1}_{K\times K})\right\} \quad (25)\\ \boldsymbol{\Sigma}_s &= \frac{1}{L}\sum_{l=1}^{L}\boldsymbol{\sigma}^l\left(\boldsymbol{\sigma}^l\right)^H\end{aligned}$$

The second one, more general, consists in exploiting relationship between parameters during the L observations. Indeed, the usual invariance hypotheses (24) are a set of parameter equality constraints:

$$\boldsymbol{\Xi}^l - \boldsymbol{\Xi}^1 = \mathbf{0},\ \underline{\boldsymbol{\theta}}^l_\mathbf{n} - \underline{\boldsymbol{\theta}}^1_\mathbf{n} = \mathbf{0},\ l\in[1,L] \quad \Leftrightarrow \quad \mathbf{f}(\underline{\mathbf{O}}) = \mathbf{0}$$

Therefore the FIM (25) associated to the set of L observations is also a constrained FIM [5][13][9] leading to a constrained CRB given by:

$$\mathbf{BCR}_{\Xi}(\boldsymbol{\theta}) = \mathbf{U}^*_{\underline{\mathbf{O}}}\left(\mathbf{U}^T_{\underline{\mathbf{O}}}\mathbf{F_s}(\underline{\mathbf{O}})\,\mathbf{U}^*_{\underline{\mathbf{O}}}\right)^{-1}\mathbf{U}^T_{\underline{\mathbf{O}}}$$

where $\mathbf{U}_{\underline{\Theta}}$ is a basis of $\ker\left\{\frac{\partial \mathbf{f}(\underline{\mathbf{O}})}{\partial \underline{\Theta}^T}\right\}$. This second approach highlights the capability of computing, at least numerically, from $\mathbf{F_s}(\underline{\mathbf{O}})$ (23) any specific CRB associated to particular hypotheses linking the parameters of the L observations.

5. MISCELLANEOUS COMMENTS

First of all, to the best of our knowledge, (21) and (25) have never been released neither in papers [2][19] nor text books [7][15][18]. They encompasses all previously released results on this topic, including:

• the standard narrow band case at a single frequency f_0, where (22) becomes [15][19]:

$$\mathbf{x}^l\left(\underline{\boldsymbol{\theta}}^l\right) = \mathbf{B}(f_0;\boldsymbol{\Xi})\,\boldsymbol{\sigma}^l + \mathbf{n}^l(\underline{\boldsymbol{\theta}}_{\mathbf{n}}),\ \boldsymbol{\Gamma}^l(f,\underline{\boldsymbol{\theta}}_{\mathbf{n}}) = \mathbf{R}(\underline{\boldsymbol{\theta}}_{\mathbf{n}})\,\delta(f-f_0)$$

$$\langle \mathbf{x} \mid \mathbf{y}\rangle_{\underline{\boldsymbol{\theta}}_{\mathbf{n}}} = \mathbf{y}(f_0)^H\,\mathbf{R}(\underline{\boldsymbol{\theta}}_{\mathbf{n}})^{-1}\,\mathbf{x}(f_0)$$

and (21) leads to [15][19]:

$$\mathbf{H}(\boldsymbol{\theta})_{m_1,m_2} = \frac{\partial \mathbf{b}(f_0;\boldsymbol{\varepsilon}_{m_1})}{\partial \boldsymbol{\varepsilon}^T}^H \Pi^{\perp}_{\{\mathbf{B}(\boldsymbol{\Xi})\}} \frac{\partial \mathbf{b}(f_0;\boldsymbol{\varepsilon}_{m_2})}{\partial \boldsymbol{\varepsilon}^T}$$

$$\Pi^{\perp}_{\{\mathbf{B}(\boldsymbol{\Xi})\}} = \mathbf{I} - \mathbf{B}(f_0;\boldsymbol{\Xi})\left(\mathbf{B}(f_0;\boldsymbol{\Xi})^H\,\mathbf{B}(f_0;\boldsymbol{\Xi})\right)^{-1}\mathbf{B}(f_0;\boldsymbol{\Xi})^H$$

• the temporally white nuisance case [2][18] for narrow band arrays, where:

$$\langle \mathbf{x}(f) \mid \mathbf{y}(f)\rangle_{\underline{\boldsymbol{\theta}}_{\mathbf{n}}} = B\int_{-\frac{T}{2}}^{\frac{T}{2}} \mathbf{y}(t)^H\,\mathbf{R}(\underline{\boldsymbol{\theta}}_{\mathbf{n}})^{-1}\,\mathbf{x}(t)\,dt = \sum_{i=-\frac{BT}{2}}^{\frac{BT}{2}} \mathbf{y}\left(\tfrac{i}{B}\right)^H \mathbf{R}(\underline{\boldsymbol{\theta}}_{\mathbf{n}})^{-1}\,\mathbf{x}\left(\tfrac{i}{B}\right)$$

Moreover, it is also possible to consider that $\mathbf{b}(t;\boldsymbol{\varepsilon}_m)$ results from a combination of a set of P elementary signals $\mathbf{e}_p(t;\boldsymbol{\varepsilon}_m)$:

$$\mathbf{b}(t;\boldsymbol{\varepsilon}_m) = \sum_{p=1}^{P} \mathbf{e}_p(t;\boldsymbol{\varepsilon}_m)$$

Then, according to the definition of each $\mathbf{e}_p(t;\boldsymbol{\varepsilon}_m)$ and its localization in time, (21) and (25) implicitly take into account the multiple impulsions case necessary to properly take into account both Doppler effect and waveform diversity, including modulated pulse and OFDM[18].

Last let us recall that, for each source [20]:

• the highest (worst) CRB is obtained when the sources amplitudes are fully correlated ($rank(\boldsymbol{\Sigma}_s) = 1$),

• the lowest (best) CRB is obtained when the sources amplitudes are uncorrelated ($\boldsymbol{\Sigma}_s$ is diagonal).

6. CONCLUSION

We have provided in this paper a general CRB expression (21)(23)(25) for band-limited radar signals. This expression allows to take into account all the possible diversities (temporal, spatial, code) in a single formalism independent of the underlying radar scene modelling (narrow or wide band arrays, narrow or wide band transmitted signals, noise plus interference features, ...). It is key feature when one wants to assess the benefits of new trends in waveform design (like use of OFDM signals [18]) on high resolution capabilities of active radar in comparison with transmission of standard linear FM modulated pulse [10].

REFERENCES

[1] E. Chaumette, J. Galy, A. Quinlan, P. Larzabal, "A New Barankin Bound Approximation for the Prediction of the Threshold Region Performance of Maximum-Likelihood Estimators", IEEE Trans. on SP, vol. 56, no. 11, pp. 5319-5333, Nov. 2008

[2] A. Dogandzic and A. Nehorai,"Cramér–Rao Bounds for Estimating Range, Velocity, and Direction with an Active Array", IEEE Trans. on SP, vol. 49, pp. 1122-1137 , Jun. 2001

[3] B. Friedlander, "On the Cramer-Rao Bound for Time Delay and Doppler Estimation", IEEE Trans. on IT, vol. 30, no. 3, pp. 575-580, May 1984.

[4] R.A. Horn, C.R. Johnson, *Matrix Analysis.* Cambridge University Press, 1999

[5] A.K. Jagannatham, B.D. Rao, "Cramér–Rao Lower Bound for Constrained Complex Parameters", IEEE SP letters, vol. 11, no. 11, november 2004

[6] S.M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory.* Englewood Cliffs, NJ: Prentice-Hall, 1993.

[7] N Levanon, E. Mozeson, *Radar Signals*, Wiley-Interscience 2004

[8] T.L. Marzetta, "A simple derivation of the constrained multiple parameter Cramér–Rao bound", IEEE Trans. on SP, vol. 41, no. 6, pp. 2247–2249, June 1993.

[9] T. Menni, E. Chaumette, P. Larzabal and J. P. Barbot, "New results on Deterministic Cramér-Rao bounds for real and complex parameters", submitted to IEEE Trans. on SP, jan 2011

[10] T. Menni, E. Chaumette, P. Larzabal, "CRB for Radar Signals: application to waveform design", in preparation for submission to IEEE Trans. on AES

[11] L. Scharf, *Statistical Signal Processing*, Addison-Wiley, 1991

[12] P. Stoica and T. L. Marzetta, "Parameter estimation problems with singular information matrices", IEEE Trans. on SP, vol. 49, no1, pp. 87-90, Jan. 2001

[13] P. Stoica and B. C. Ng, "On the Cramér–Rao bound under parametric constraints", IEEE SP Letters, vol. 5, no. 7, pp. 177-179, July 1998

[14] K. Todros and J. Tabrikian, "General Classes of Performance Lower Bounds for Parameter Estimation-Part I: Non-Bayesian Bounds for Unbiased Estimators", IEEE Trans. on IT, vol. 56, no. 10, october 2010

[15] H.L. Van Trees, *Optimum Array Processing*, New-York, Wiley-Interscience, 2002

[16] H. L. Van Trees, Detection, *Estimation and Modulation Theory. Part. III.* New York: Wiley, 1971.

[17] P. Whittle, "The analysis of multiple stationary time series," J. Royal Statist. Sot., vol. 15, pp. 125-139, 1953

[18] M.C. Wicks, E.L. Mokole, S.D. Blunt, R.S. Schneible, V.J. Amuso , *Principles of Waveform Diversity and Design*, SciTech Publishing 2010

[19] S. F. Yau and Y. Bresler, "A Compact Cram&-Rao Bound Expression for Parametric Estimation of Superimposed Signals", IEEE Trans. on SP, vol. 40, no. 5, pp 1226-1230, may 1992

[20] S. F. Yau and Y. Bresler, "Worst Case Cramér-Rao Bounds for Parametric Estimation of Superimposed Signals", IEEE Trans. on SP, vol. 40. no. 12, pp 2973-2986, december 1992

[21] M. Wax, "The joint estimation of differential delay, Doppler, and phase." IEEE Trans. on IT, vol. 28, no. 5, pp. 817-820, Sept. 1982.