

# ADDITIVE DISCRETE LINEAR CANONICAL TRANSFORM AND OTHER ADDITIVE DISCRETE OPERATIONS

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## ABSTRACT

In this paper, we derive the discrete linear canonical transform (DLCT) that has the additivity property. It is the discrete counterpart of the continuous linear canonical transform (LCT). The LCT is a generalization of the Fourier transform (FT) and the fractional Fourier transform (FRFT) and is suitable for signal analysis. The discrete counterparts of the FT and the FRFT have already been derived. However, since the DLCT has four parameters  $\{a, b, c, d\}$ , it is hard to derive the DLCT that has the additivity property. In this paper, we use bilinear mapping together with the discrete time Fourier transform to derive the additive DLCT successfully. We can also use the similar method to derive the discrete 2-D non-separable LCT, the discrete fractional delay, the discrete fractional scaling, the discrete fractional differentiation, and the discrete geometric twisting operations that have the additivity property successfully.

## 1. INTRODUCTION

The continuous linear canonical transform (LCT) is [1]:

$$O_{LCT}^{(a,b,c,d)}[f(t)] = \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty} e^{j\frac{d}{2b}u^2 - j\frac{1}{b}ut + j\frac{a}{2b}t^2} f(t) dt \quad \text{when } b \neq 0, \quad (1)$$

$$O_{LCT}^{(a,b,c,d)}[f(t)] = \sqrt{d} e^{j\frac{cd}{2}u^2} f(du) \quad \text{when } b = 0. \quad (2)$$

The continuous LCT has four parameters  $\{a, b, c, d\}$  and the constraint that  $ad - bc = 1$  should be satisfied. It has the additivity property as follows:

$$O_{LCT}^{(a_2,b_2,c_2,d_2)} \left\{ O_{LCT}^{(a_1,b_1,c_1,d_1)} [f(t)] \right\} = O_{LCT}^{(e,f,g,h)} [f(t)],$$

where  $\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ . (3)

That is, the combination of two LCTs can be represented by the  $2 \times 2$  matrix operation as in (3). The LCT is useful in time-frequency analysis, filter design, communication, acoustics, optics, wave propagation analysis, signal sampling, phase retrieval, and image processing [1-6].

The LCT is a generalization of many operations. When  $\{a, b, c, d\} = \{0, 1, -1, 0\}$ , it becomes the Fourier transform (FT). When  $\{a, b, c, d\} = \{\cos\alpha, \sin\alpha, -\sin\alpha, \cos\alpha\}$ , the LCT reduces to the fractional Fourier transform (FRFT) [3][6]

$$O_{FRFT}^{\alpha} [f(t)] = \sqrt{\frac{1-j\cot\alpha}{2\pi}} \int_{-\infty}^{\infty} e^{j\frac{1}{2}u^2 \cot\alpha - ju t \csc\alpha + \frac{j}{2}t^2 \cot\alpha} f(t) dt. \quad (4)$$

The FRFT has the additivity property as follows:

$$O_{FRFT}^{\beta} \left\{ O_{FRFT}^{\alpha} [f(t)] \right\} = O_{FRFT}^{\alpha+\beta} [f(t)].$$

When  $\{a, b, c, d\} = \{1, \lambda z, 0, 1\}$ , the LCT becomes the Fresnel transform. Furthermore, when  $\{a, b, c, d\} = \{\sigma, 0, 0, \sigma^{-1}\}$ , from (2), the LCT becomes the scaling operation:

$$O_{LCT}^{(\sigma,0,0,\sigma^{-1})} [f(t)] = \sqrt{\sigma^{-1}} f(u/\sigma). \quad (5)$$

The discrete version of the FT is the well-known discrete Fourier transform (DFT) [7]. Furthermore, based on eigenvector decomposition, the discrete version of the FRFT, i.e., the discrete fractional Fourier transform (DFRFT), was derived in [8][9]. The DFRFT proposed in [8][9] has the additivity property:

$$\mathbf{F}^{\beta} \mathbf{F}^{\alpha} = \mathbf{F}^{\alpha+\beta} \quad (6)$$

and is useful for discrete signal analysis, digital filter design, and image processing.

As the above description, the discrete counterparts of both the FT and the FRFT have been known. However, the discrete LCT that has the additivity property has not been derived yet.

Note that, the continuous LCT has four parameters  $\{a, b, c, d\}$ . Therefore, it is rather hard to convert it into the discrete form that also has the additivity property as in (3). Furthermore, since the eigenfunctions of the LCT varies with  $\{a, b, c, d\}$ , it is improper to use eigenvector decomposition to derive the discrete LCT.

In this paper, we use the method of bilinear mapping to derive the discrete LCT with the additivity property. See Sections 2 and 3. Furthermore, in Section 4, we use the similar way to derive the 2-D non-separable discrete LCT, the discrete fractional scaling, the discrete fractional delay, and the discrete fractional differentiation / integration operations with additivity properties successfully.

## 2. DERIVING THE ADDITIVE LINEAR CANONICAL TRANSFORM

The continuous LCT in (1)(2) has the additivity property. We can convert it into the discrete linear canonical transform that also has the additivity property, i.e., the **additive discrete linear canonical transform (ADLCT)**. The proc-

ess of the ADLCT is as follows. Suppose that  $x[n]$  is the input sequence. Then,

**(Step 1)** First, we perform the discrete-time Fourier transform (DTFT) for  $x[n]$ :

$$X(\omega) = \sum_n x[n] e^{-j\omega n}, \quad \text{where } -\pi < \omega < \pi. \quad (7)$$

**(Step 2)** Then, we convert  $X(\omega)$  into  $X_1(\omega)$ :

$$X_1(\omega) = \sqrt{\phi'(\omega)} X(\phi(\omega)), \quad (8)$$

where  $\phi(\omega)$  should be a one-to-one mapping operation,  $-\infty < \omega < \infty$ , and  $-\pi < \phi(\omega) < \pi$ . Specially, if we choose

$$\phi(\omega) = 2 \operatorname{atan}\left(\frac{\omega}{2}\right), \quad (9)$$

where  $\operatorname{atan}$  means the arctangent, then (8) becomes the bilinear transform (BT) [10] as:

$$X_1(\omega) = BT[X(\omega)] = \sqrt{\frac{4}{\omega^2 + 4}} X(2 \operatorname{atan}(\omega/2)). \quad (10)$$

**(Step 3)** Then, we perform the LCT with parameters  $\{d, -c, -b, a\}$  for  $X_1(\omega)$ :

$$\begin{aligned} X_2(\rho) &= O_{LCT}^{(d, -c, -b, a)} [X_1(\omega)] \\ &= \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty} e^{-j\frac{a}{2c}\rho^2 + j\frac{1}{c}\rho\omega - j\frac{d}{2c}\omega^2} X_1(\omega) d\omega. \end{aligned} \quad (11)$$

Note that, in this step, we use the parameters  $\{d, -c, -b, a\}$  instead of  $\{a, b, c, d\}$ . It due to the fact that Steps 1 and 5 are analogous to the FT and the IFT and

$$O_{LCT}^{(a, b, c, d)} [x(t)] = IFT \left( O_{LCT}^{(d, -c, -b, a)} \{ FT [x(t)] \} \right). \quad (12)$$

**(Step 4)** Then, we convert  $X_2(\rho)$  into  $Y(\rho)$ , where

$$Y(\rho) = \sqrt{\frac{1}{\phi'(\phi^{-1}(\rho))}} X_2(\phi^{-1}(\rho)), \quad (13)$$

$-\pi < \rho < \pi$  and  $-\infty < \phi^{-1}(\rho) < \infty$ . Note that (13) is the inverse operation of (8). Specially, if we choose  $\phi(\omega)$  as in (9), then (13) becomes the inverse bilinear transform (IBT):

$$Y(\omega) = IBT[X_2(\omega)] = \left| \sec\left(\frac{\omega}{2}\right) \right| X_2\left(2 \tan\left(\frac{\omega}{2}\right)\right). \quad (14)$$

**(Step 5)** Then we perform the inverse discrete-time Fourier transform (IDTFT) for  $Y(\rho)$ :

$$y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(\omega) e^{j\omega n} d\omega. \quad (15)$$

The output  $y[n]$  is the **ADLCT** of  $x[n]$ . We denote it by:

$$y[n] = O_{ADLCT}^{(a, b, c, d)} (x[k]). \quad (16)$$

To prove that the proposed ADLCT is additive, we can express the operation in Steps 1-5 as:

$$O_{ADLCT}^{(a, b, c, d)} (x[n]) = IDTFT \left( IBT \left\{ O_{LCT}^{(d, -c, -b, a)} \left[ BT \left( DTFT \{ x[n] \} \right) \right] \right\} \right). \quad (17)$$

Since the IDTFT is the inverse operation of the DTFT and the IBT is the inverse operation of the BT:

$$IDTFT \left( DTFT \{ z[n] \} \right) = z[n],$$

$$IBT \{ BT \{ z[n] \} \} = z[n],$$

therefore,

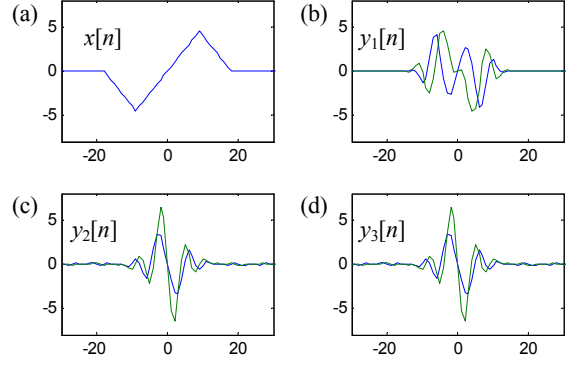


Figure 1 – (a) The input  $x[n]$ . (b)  $y_1[n]$  is the ADLCT with parameters  $\mathbf{A}_1$  (defined in (20)) for  $x[n]$ . (c)  $y_2[n]$  is the ADLCT with parameters  $\mathbf{A}_2$  (defined in (21)) for  $y_1[n]$ . (d)  $y_3[n]$  is the ADLCT with parameters  $\mathbf{A}_3 = \mathbf{A}_1 \mathbf{A}_2$  for  $x[n]$ . Note that  $y_3[n] = y_2[n]$ , which show that the ADLCT indeed has the additivity property.

$$\begin{aligned} O_{ADLCT}^{(a_2, b_2, c_2, d_2)} \left\{ O_{ADLCT}^{(a_1, b_1, c_1, d_1)} (x[n]) \right\} &= IDTFT \left( IBT \left\{ O_{LCT}^{(d_2, -c_2, -b_2, a_2)} \right. \right. \\ &\left. \left. \left[ BT \left( DTFT \left\{ IDTFT \left[ IBT \left( O_{LCT}^{(d_1, -c_1, -b_1, a_1)} BT \left\{ DTFT \{ x[n] \} \right\} \right] \right) \right] \right\} \right] \right\} \right) \\ &= IDTFT \left( IBT \left\{ O_{LCT}^{(d_2, -c_2, -b_2, a_2)} \left[ O_{LCT}^{(d_1, -c_1, -b_1, a_1)} \left( BT \left\{ DTFT \{ x[n] \} \right\} \right) \right] \right\} \right) \\ &= IDTFT \left( IBT \left\{ O_{LCT}^{(d_3, -c_3, -b_3, a_3)} \left[ BT \left( DTFT \{ x[n] \} \right) \right] \right\} \right) \\ &= O_{ADLCT}^{(a_3, b_3, c_3, d_3)} (x[n]). \end{aligned} \quad (18)$$

From the additivity property of the LCT in (3), the values of  $a_3, b_3, c_3$ , and  $d_3$  in (17) are

$$\begin{aligned} \begin{bmatrix} d_3 & -c_3 \\ -b_3 & a_3 \end{bmatrix} &= \begin{bmatrix} d_2 & -c_2 \\ -b_2 & a_2 \end{bmatrix} \begin{bmatrix} d_1 & -c_1 \\ -b_1 & a_1 \end{bmatrix}, \\ \mathbf{F}^{-1} \begin{bmatrix} d_3 & -c_3 \\ -b_3 & a_3 \end{bmatrix} \mathbf{F} &= \mathbf{F}^{-1} \begin{bmatrix} d_2 & -c_2 \\ -b_2 & a_2 \end{bmatrix} \mathbf{F} \mathbf{F}^{-1} \begin{bmatrix} d_1 & -c_1 \\ -b_1 & a_1 \end{bmatrix} \mathbf{F}, \\ \text{where } \mathbf{F} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \\ \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} &= \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}. \end{aligned} \quad (19)$$

Therefore, as the original continuous LCT, the proposed ADLCT also has the additivity property.

### 3. SIMULATIONS, PROPERTIES, AND APPLICATIONS

In Fig. 1, we perform some simulations to show the additivity property of the proposed ADLCT. The input signal is

$$\begin{aligned} x[n] &= n/2 \quad \text{for } -9 \leq n \leq 9, \quad x[n] = -(n-18)/2 \quad \text{for } 10 \leq n \leq 17, \\ x[n] &= -(n+18)/2 \quad \text{for } -17 \leq n \leq -10, \\ x[n] &= 0 \quad \text{otherwise.} \end{aligned}$$

We plot it in Fig. 1(a). Then, we perform the ADLCT with parameters  $\{a_1, b_1, c_1, d_1\}$  for  $x[n]$ , where

$$\mathbf{A}_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.8 \\ -0.5 & 1.2 \end{bmatrix} \quad (20)$$

and obtain the output  $y_1[n]$ , see Fig. 1(b). Then we further perform the ADLCT with parameters  $\{a_2, b_2, c_2, d_2\}$  for  $y_1[n]$ , where

$$\mathbf{A}_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} 1 & 0.8 \\ -1 & 0.2 \end{bmatrix}, \quad (21)$$

and obtain the output  $y_2[n]$  (plotted in Fig. 1(c)).

Then, in Fig. 1(d), we perform the ADLCT with parameters  $\{a_3, b_3, c_3, d_3\}$  for the original signal  $x[n]$ , where

$$\begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} = \mathbf{A}_2 \mathbf{A}_1 = \begin{bmatrix} 0.1 & 1.76 \\ -0.6 & -0.56 \end{bmatrix}. \quad (22)$$

Note that, from Figs. 1(c) and 1(d),  $y_3[n]$  is fully equivalent to  $y_2[n]$ , which show that the proposed ADLCT indeed has the additivity property.

We list some properties of the proposed ADLCT as follows. We use  $x[n]$  and  $y[n]$  to denote the input and output of the ADLCT.

(a) If  $x[n]$  is even, then  $y[n]$  is also even.

Similarly, if  $x[n]$  is odd, then  $y[n]$  is also odd.

(b) If  $y[n] = O_{ADLCT}^{(a,b,c,d)}(x[n])$ , then  $y[-n] = O_{ADLCT}^{(a,b,c,d)}(x[-n])$ .

(c) If  $x[n]$  is real and  $y[n] = O_{ADLCT}^{(0,\sigma,-\sigma^{-1},0)}(x[n])$  (Note that, in this case, the DLCT is analogous to the scaled Fourier transform), then

$$y[n] = y^*[-n].$$

Similarly, if  $x[n] = x^*[-n]$  and  $y[n] = O_{ADLCT}^{(0,\sigma,-\sigma^{-1},0)}(x[n])$ , then  $y[n]$  is real.

(d) Suppose that  $e(t)$  is an eigenvector of the continuous LCT:

$$O_{LCT}^{(a,b,c,d)}[e(t)] = \lambda e(t),$$

then

$$E[n] = IDTFT\{IBT[e(t)]\}$$

is the eigenvector of the ADLCT and its eigenvalue is also  $\lambda$ :

$$O_{ADLCT}^{(a,b,c,d)}(E[n]) = \lambda E[n]. \quad (23)$$

(e) Parseval's theorem (i.e., the energy preservation theorem) can also be applied to the ADLCT. That is,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |y[n]|^2, \quad (24)$$

where  $y[n]$  is the ADLCT of  $x[n]$ . More generally,

$$\sum_{n=-\infty}^{\infty} x[n]x_1^*[n] = \sum_{n=-\infty}^{\infty} y[n]y_1^*[n],$$

where  $y_1[n]$  is the ADLCT of  $x_1[n]$ .

Then, we show an example of using the proposed ADLCT for the application of noise removing. In Fig. 2(a), we show the noise interfered signal  $x[n] = \text{Gaussian function} + \text{noise}$ . Then, we perform the ADLCT with parameters  $\{0.6, 1, -0.4, 1\}$  for  $x[n]$  and show the result in Fig. 2(b). We can see that, in Fig. 2(b), the signal part and the noise part are separated. Then, we set

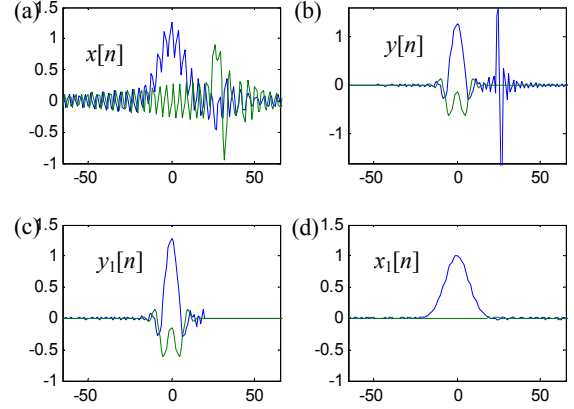


Figure 2 – Application of the ADLCT for filtering the noise. (a) A noise-interfered signal  $x[n]$ . (b) The ADLCT of  $x[n]$ . (c) Truncating the noise part. (d) The reconstructed signal.

$$y_1[n] = y[n] \text{ for } n \leq 19, \quad y_1[n] = 0 \text{ for } n \geq 20, \quad (25)$$

as in Fig. 2(c). Then, we perform the inverse ADLCT for  $y_1[n]$  and obtain the reconstructed signal  $x_1[n]$ . Note that, in Fig. 2(d), the noise in  $x[n]$  is almost removed. The normalized mean square error (NMSE) between  $x_1[n]$  and the original signal is only

$$NMSE = 0.38\%,$$

which shows that the proposed ADLCT is useful for filter design. Moreover, other applications of the continuous FRFT and the continuous LCT, such as communication, random process analysis, time-frequency analysis, image encryption, space-variant pattern recognition, beam shaping, and radar system analysis [1-6] are also the potential applications of the proposed ADLCT.

#### 4. OTHER ADDITIVE DISCRETE OPERATIONS

In fact, in addition to deriving the ADLCT, the concept in Section 2 can also be used for converting other continuous additive operations into the discrete additive operations.

##### 4-1 Two-Dimensional Additive Discrete Non-Separable Linear Canonical Transform

The 2-D non-separable linear canonical transform (2-D NSLCT) is the 2-D counterpart of the 1-D LCT. Its definition is [10][11]:

$$O_{NSLCT}^{(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})}(g(x,y)) = \frac{1}{2\pi\sqrt{-\det(\mathbf{B})}} \exp\left(\frac{j}{2} \mathbf{w} \mathbf{D} \mathbf{B}^{-1} \mathbf{w}^T\right) \int \exp(-j\mathbf{x} \mathbf{B}^{-1} \mathbf{w}^T) \exp\left(\frac{j}{2} \mathbf{x} \mathbf{B}^{-1} \mathbf{A} \mathbf{x}^T\right) g(\mathbf{x}) d\mathbf{x}, \quad (26)$$

where  $\mathbf{x} = [x, y]$ ,  $\mathbf{w} = [\omega, h]$ ,

$\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$  are all  $2 \times 2$  matrices.

The 2-D NSLCT is even more general than the 1-D LCT. It is useful in optical analysis, radar system analysis, and signal processing. It has the additivity property as:

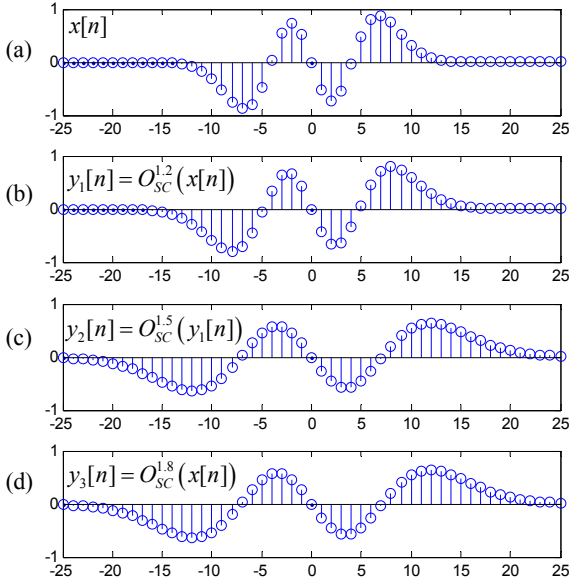


Figure 3 – Simulations for the **proposed additive discrete fractional scaling operation** (see subsection 4-2). (a)  $x[n]$ , (b)  $y_1[n]$  is the discrete scaling of  $x[n]$  ( $\sigma = 1.2$ ), (c)  $y_2[n]$  is the discrete scaling of  $y_1[n]$  ( $\sigma = 1.5$ ), (d)  $y_3[n]$  is the discrete scaling of  $x[n]$  ( $\sigma = 1.8$ ). Note that  $y_3[n] = y_2[n]$ , which show that the proposed additive discrete fractional scaling operation indeed has the additivity property.

$$O_{NSLCT}^{(A_2, B_2, C_2, D_2)} \left\{ O_{NSLCT}^{(A_1, B_1, C_1, D_1)} (x[n]) \right\} = O_{NSLCT}^{(A_3, B_3, C_3, D_3)} (x[n]),$$

$$\text{where } \begin{bmatrix} \mathbf{A}_3 & \mathbf{B}_3 \\ \mathbf{C}_3 & \mathbf{D}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_2 & \mathbf{B}_2 \\ \mathbf{C}_2 & \mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D}_1 \end{bmatrix}. \quad (27)$$

The 2-D NSLCT has 16 parameters and is very complicated. However, we can use the following process to convert it into the discrete version and the additivity property is preserved. We call it the two-dimensional additive discrete non-separable linear canonical transform (**2-D ANSDLCT**):

**(Step 1)** Perform the 2-D DTFT for the input  $x[m, n]$

$$X(\omega, h) = \sum_m \sum_n x[m, n] e^{-j\omega m} e^{-jhn},$$

$$\text{where } -\pi < \omega < \pi, \quad -\pi < h < \pi.$$

**(Step 2)** Perform the 2-D bilinear transform for  $X(\omega, h)$ :

$$X_1(\omega, h) = BT \left[ X(\omega, h) \right] = \sqrt{\frac{4}{\omega^2 + 4} \frac{4}{h^2 + 4}} X(2 \tan(\omega/2), 2 \tan(h/2)), \quad (28)$$

$$\text{where } -\infty < \omega < \infty \text{ and } -\infty < h < \infty.$$

**(Step 3)** Perform the 2-D NSLCT for  $X_1(\omega, h)$ , but the parameters are changed into  $\{\mathbf{D}, -\mathbf{C}, -\mathbf{B}, \mathbf{A}\}$ :

$$X_2(u, v) = O_{NSLCT}^{(\mathbf{D}, -\mathbf{C}, -\mathbf{B}, \mathbf{A})} (X_1(\omega, h)).$$

The reason why the parameters are changed into  $\{\mathbf{D}, -\mathbf{C}, -\mathbf{B}, \mathbf{A}\}$  can be seen from (12).

**(Step 4)**  $Y(u, v) = IBT(X_2(u, v))$

$$= \left| \sec\left(\frac{u}{2}\right) \sec\left(\frac{v}{2}\right) \right| X_2(2 \tan(u/2), 2 \tan(v/2)), \quad (29)$$

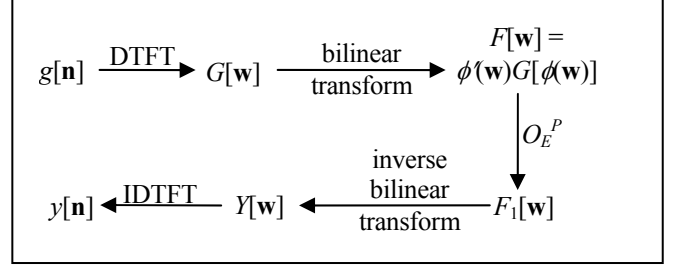


Figure 4 – The method for converting any additive continuous operation into the discrete operation that has the additivity property.

where  $-\pi < u < \pi, \quad -\pi < v < \pi.$

$$\text{(Step 5)} \quad y[m, n] = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} Y(u, v) e^{jum} e^{jvn} dudv.$$

Then,  $y[m, n]$  is the 2-D ANSDLCT of  $x[m, n]$ . We denote it as

$$y[m, n] = O_{ANSDLCT}^{(A, B, C, D)} (x[m, n]).$$

From the process similar to that in (17)-(19), we can prove that the ANSDLCT defined above also has the additivity property as follows:

$$O_{ANSDLCT}^{(A_2, B_2, C_2, D_2)} \left\{ O_{ANSDLCT}^{(A_1, B_1, C_1, D_1)} (x[n]) \right\} = O_{ANSDLCT}^{(A_3, B_3, C_3, D_3)} (x[n]), \quad (30)$$

where  $\mathbf{A}_3, \mathbf{B}_3, \mathbf{C}_3,$  and  $\mathbf{D}_3$  are defined the same as those in (27).

#### 4-2 Additive Discrete Fractional Scaling

Remember that the scaling operation is a special case of the continuous LCT with parameters  $\{\sigma, 0, 0, \sigma^{-1}\}$  (see (5)). Therefore, we can define the **additive discrete fractional scaling operation** as the special case of the ADLCT with parameters  $\{\sigma, 0, 0, \sigma^{-1}\}$ :

$$O_{SC}^{\sigma} (x[n]) = O_{ADLCT}^{(\sigma, 0, 0, \sigma^{-1})} (x[n]). \quad (31)$$

We can use (31) to scale a signal with arbitrary dilation ratio. Moreover, the proposed discrete scaling operation satisfies the following additivity property:

$$O_{SC}^{\sigma_2} \left\{ O_{SC}^{\sigma_1} (x[n]) \right\} = O_{SC}^{\sigma_2 \sigma_1} (x[n]). \quad (32)$$

For example, for the discrete signal  $x[n]$  in Fig. 3(a), we can use (31) to perform the discrete fractional scaling  $x[n]$  with dilation ratio  $\sigma = 1.2$ . The result  $y_1[n]$  is shown in Fig. 3(b). Note that the width of  $y_1[n]$  is near to 1.2 times of that of  $x[n]$ . Then, we further scale  $y_1[n]$  with dilation ratio  $\sigma = 1.5$  and show the result  $y_2[n]$  in Fig. 3(c). In Fig. 3(d), we perform the discrete fractional scaling for  $x[n]$  with  $\sigma = 1.8$  and the result is denoted by  $y_3[n]$ . From Figs. 3(c) and 3(d), we can see that  $y_2[n]$  is equivalent to  $y_3[n]$ , i.e.,

$$O_{SC}^{1.5} \left\{ O_{SC}^{1.2} (x[n]) \right\} = O_{SC}^{1.8} (x[n]), \quad (33)$$

which shows the discrete fractional scaling operation in (31) indeed has the additivity property. The proposed discrete fractional scaling operation is useful for signal analysis and image processing.

### 4-3 Additive Discrete Geometric Twisting Operations

Similarly, the geometric twisting operation is a special case of the continuous 2-D NSLCT with parameters  $\{\mathbf{A}, \mathbf{0}, \mathbf{0}, (\mathbf{A}^T)^{-1}\}$  [10][11]. Thus, we can also define the **additive discrete geometric twisting operations** as the 2-D ANSDLCT with parameters  $\{\mathbf{A}, \mathbf{0}, \mathbf{0}, (\mathbf{A}^T)^{-1}\}$

$$O_{twisting}^{\mathbf{A}}(x[m, n]) = O_{ANSDLCT}^{(\mathbf{A}, \mathbf{0}, \mathbf{0}, (\mathbf{A}^T)^{-1})}(x[m, n]). \quad (34)$$

From the additivity properties of the 2-D ANSDLCT, it is no hard to prove that the operation in (34) has the additivity property.

### 4-4 Additive Discrete Fractional Delay

We can also use the process similar to that in (7)-(16) to define the discrete additive fractional delay, except for that Step 3 is changed into

$$X_2(\rho) = e^{jn_0\rho} X_1(\rho), \quad (35)$$

where  $n_0$  can be non-integer. Then the output  $y[n]$  is near to  $x[n-n_0]$ . Moreover, using the similar way as that in (17)-(19), we can prove that the discrete fractional delay operation defined by the above method is additive.

### 4-5 Additive Discrete Fractional Differentiation and Integration

The continuous differentiation and integration with the fractional orders (i.e., fractional calculus) have been widely discussed in literature [12]. To define the discrete version of the fractional differentiation and the fractional integration operations, we can also follow the process in (7)-(16), but (11) is modified as

$$X_2(\rho) = (j\rho)^\alpha X_1(\rho). \quad (36)$$

When  $\alpha > 0$ , it becomes the discrete fractional differentiation operation with order  $\alpha$  ( $\alpha$  can be non-integer). When  $\alpha < 0$ , it becomes the discrete fractional integration operation with order  $-\alpha$ .

### 4-6 The General Method to Define the Discrete Operation with the Additivity Property

Suppose that  $O_C^P$  is a continuous operation that has the additivity operation and  $P$  is its parameters:

$$O_C^{P_2} \{O_C^{P_1} [f(\mathbf{x})]\} = O_C^{P_2 \bullet P_1} [f(\mathbf{x})], \quad (37)$$

where  $\bullet$  is some operation. First, we define the operation  $O_E^P$  as:

$$O_E^P [f(\mathbf{x})] = FT \left( O_C^P \{IFT[f(\mathbf{x})]\} \right).$$

Then, we can use the method as in Fig. 4 to convert it into a discrete operation (denoted by  $O_D^P$ ). The derived discrete operation  $O_D^P$  will have the following additivity property:

$$O_D^{P_2} \{O_D^{P_1} [g[\mathbf{n}]]\} = O_D^{P_2 \bullet P_1} [g[\mathbf{n}]]. \quad (38)$$

Using the above method, we are able to convert **any** continuous operation that has the additivity property into its discrete counterpart with the additivity property. Moreover,

the derived discrete operation  $O_D^P$  will have the properties similar to those of the original continuous operation  $O_C^P$ .

## 5. CONCLUSION

In this paper, we introduce a novel method to derive the discrete version of the LCT that has the additivity property, i.e., the ADLCT. The derived ADLCT works similar to its continuous counterpart and is useful for filtering the noise and fractional scaling.

Moreover, the proposed method can also be used for deriving the discrete versions of the 2-D NSLCT, the discrete fractional delay operation, the discrete geometric twisting operation, and the discrete fractional differentiation / integration operations with additivity properties successfully.

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