A GAME-THEORETIC INTERPRETATION OF ITERATIVE DECODING

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ABSTRACT

Bit interleaved Coded Modulation with iterative decoding is known to provide excellent performance over both Gaussian and fading channels. However a complete analysis of the iterative demodulation is still missing. In this paper, the iterative decoding is analyzed from a game-theoretic point of view in order to explain the good performance of turbo-decoding. It is shown that iterative decoding is a game seeking a solution to an optimization problem obtained from parallel approximations of the maximum likelihood decoding. Surprisingly, the decoder and demapper are not antagonist players. They are involved in a cooperative process in which n selfish players attempt to optimize their own bit-marginals. An interpretation is given in terms of pure Nash Equilibrium and social welfare. The approximate criterion of the sub-optimal problem is the social welfare of the game and is also a performance rating on the distributed optimization process. The convergence is analysed and it is proved that it always exists a convergent iterative sequence leading to a Nash equilibrium of the game. Experimental results are provided in the particular case of BICM decoding.

1. INTRODUCTION

Bit-Interleaved Coded Modulation (BICM) was first suggested by Zehavi in [1] to improve the Trellis Coded Modulation performance over Rayleigh-fading channels. In BICM, the diversity order is increased by using bit-interleavers instead of symbol interleavers. This improvement is achieved at the expense of a reduced minimum Euclidean distance leading to a degradation over non-fading Gaussian channels [1], [2]. This drawback can be overcome by using iterative decoding (BICM-ID) at the receiver.

The iterative decoding scheme used in BICM-ID is very similar to serially concatenated turbo-decoders. Indeed, the serial turbo-decoder makes use of an exchange of information between computationally efficient decoders for each component code. In BICM-ID, the inner decoder is replaced by demapping which is less computationally demanding than a decoding step. Even though this paper focus on iterative decoding for BICM, the results can be applied to the large class of iterative decoders including serial or parallel concatenated turbo-decoders.

The turbo-decoder and more generally iterative decoding was not originally introduced as the solution to an optimization problem, thus rendering the analysis of its convergence and stability very difficult. Among the different attempts to provide an analysis of iterative decoding, the EXIT chart analysis and density evolution have permitted to make significant progress [3] but the results developed within this setting apply only in the case of large block length. Another tool of analysis is the connection of iterative decoding to factor graphs [4] and belief propagation [5]. Convergence results for belief propagation exist but are limited to the case where the corresponding graph is a tree which does not include turbo-codes. A link between iterative decoding and classical optimization algorithms has been made also in [6] where the turbo-decoding is interpreted as a nonlinear block Gauss-Seidel iteration for solving a constrained optimization problem. In [7] and in [8], the turbo-decoding is interpreted in a geometric setting leading to new but incomplete results. The failure to obtain complete results is mainly due to the inability to efficiently describe extrinsic information passing. The relation between the optimal maximum likelihood decoding and iterative decoding is not yet fully understood.

In this paper, we first review the principle of maximum likelihood decoding. We then recall some results of [9] in order to explain how an approximate (and tractable) criterion can be derived from an equivalent and convenient formulation of the maximum likelihood decoding. A new interpretation based on the Hamming distance between binary words is provided. It is proved that iterative decoding can be understood as a game between n players (where n is the length of any codeword at the output of the convolutional encoder). The (tractable) criterion is the social welfare of the game. As we will see, the players are not the decoder and demapper. These two constituent elements are involved in a cooperative process and contribute to the utility function of any player. It turns out that the players are concerned with the optimization of a unique bit-marginal. The convergence toward a Nash Equilibrium (NE) of the game is also studied and we prove that it always exists values of the key parameter β (to be defined in section 4) that guarantee the convergence to a NE of the game from any initialization point. The parameter β is a trade-off between convergence and optimality of the joint optimization process. These results are illustrated for BICM iterative decoding in the simulation part.

2. SYSTEM MODEL

A conventional BICM system [2] is built from a serial concatenation of a convolutional encoder, a bit interleaver and an M-ary bits-to-symbol mapping (where $M = 2^m$) as shown in Figure 1. The sequence of information bits $\mathbf{b}$ of length n_b is first encoded by a convolutional encoder to produce the output encoded bit sequence $\mathbf{c}$ of length n which is then scrambled by a bit interleaver (as opposed to the channel symbols in the symbol interleaved coded sequence) operating on bit indexes. Let $\mathbf{d} = \pi(\mathbf{c})$ denote the interleaved sequence. Then, m consecutive bits of $\mathbf{d}$ are grouped as a symbol. The complex transmitted signal s_k, $1 \leq k \leq n/m$, is then chosen from an M-ary constellation ψ where ψ denotes the mapping scheme. The symbols s_k are passed over a noisy memoryless channel to get the channel outputs y_k. The maximum likelihood sequence detection

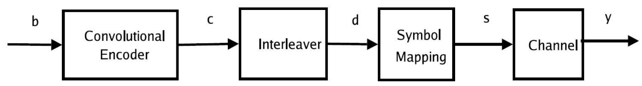

Figure 1: BICM transmission scheme

reads:

$$\widehat{\mathbf{b}}_{MLD} = \arg \max_{\mathbf{b}\in\{0,1\}^{n_b}} p(\mathbf{y} \mid \mathbf{b}) \tag{1}$$

where $p(\mathbf{y} \mid \mathbf{b})$ is the likelihood function which results from concatenating the encoder with the channel. Since there is a one-to-one correspondence between the binary message $\mathbf{b}$ and the interleaved sequence $\mathbf{d}$, eq. (1) is equivalent to searching $\widehat{\mathbf{d}}_{MLD}$ as:

$$\widehat{\mathbf{d}}_{MLD} = \arg \max_{\mathbf{d}\in\{0,1\}^n} p_{ch}(\mathbf{y} \mid \mathbf{d}) I_{co}(\mathbf{d}) \tag{2}$$

where $p_{ch}(\mathbf{y} \mid \mathbf{d})$ is the probability of receiving $\mathbf{y}$ when the sequence transmitted through the channel is the mapping of $\mathbf{d}$ and where $I_{co}(\mathbf{d})$ is the indicator function of the code meaning that $I_{co}(\mathbf{d}) = 1$ if $\mathbf{c} = \pi^{-1}(\mathbf{d})$ is a codeword and 0 elsewhere. Another way to tackle this problem consists in finding the prior PMF on $\mathbf{d}$ which maximizes the *a posteriori* probability of having received $\mathbf{y}$

$$\widehat{\mathbf{p}}_{MLD}(\mathbf{d}) = \arg\max_{\mathbf{p}\in\mathscr{E}_s} \sum_{\mathbf{d}} \mathbf{I}_{co}(\mathbf{d})\mathbf{p}_{ch}(\mathbf{y} \mid \mathbf{d})\mathbf{p}(\mathbf{d}) \tag{3}$$

where $\mathscr{E}_S$ stands for the set of all possible **separable** PMFs on $\mathbf{d}$. By definition, a PMF $\mathbf{p}(\mathbf{d})$ is separable if $\mathbf{p}(\mathbf{d}) = \prod_i p_i(d_i)$ with $p_i(d_i)$ the probability for bit i to be equal to d_i. The optimal solution $\widehat{\mathbf{p}}_{MLD}(\mathbf{d})$ reads

$$\widehat{\mathbf{p}}_{MLD}(\mathbf{d}) = \begin{cases} 1, & \mathbf{d} = \widehat{\mathbf{d}}_{MLD} \\ 0, & otherwise \end{cases} \tag{4}$$

since another weighting (with the constraint $\sum_{\mathbf{d}} p(\mathbf{d}) = 1$) produces a lower likelihood. The formulation in (3) is equivalent to the original problem in (2). The practical implementation of this optimization problem is dismissed due to the presence of a random bit interleaver and to the (large) numerical value of n. In the next section, we present a sub-optimal criterion derived from (3). We will see that the iterative decoding of BICM (and of turbo-like decoding in general) can be obtained from a particular maximization process of this suboptimal criterion. Some aspects of the maximization of the sub-optimal criterion have already been presented in [9]. Some results of [9] are first recalled in section 3 since they are mandatory for a full understanding of this paper. We also provide in section 3 a new interpretation of the suboptimal decoding. New results on the iterative procedure are presented in section 4, they are based on game theory concepts. In section 5, the convergence issue is addressed and the results of the paper are illustrated in section 6. In the rest of the paper, we assume that $\mathbf{p}_{ch}(\mathbf{y} \mid \mathbf{d})$ is a PMF without any extra assumption. The results presented here are thus directly applicable to BICM as well as to serial turbo-codes.

3. SUBOPTIMAL DECODING

The direct computation of the optimal maximum likelihood decoder is non tractable due to the interleaver and to the computational complexity involved in the computation and storage of the 2^n taps of the PMF. A solution regarding the interleaver is to consider separately the two-blocks (mapping and coding) in a particular sense to be defined later. The problem of the computational complexity can be handled by working on the bit-marginals rather than on the PMF of the whole sequence. For that purpose the variable $\mathbf{p}(\mathbf{d})$ is split into the product of the two separable PMFs $\mathbf{l}(\mathbf{d})$ and $\mathbf{q}(\mathbf{d})$. The computation of the bit-marginals is also introduced into the optimal criterion as

$$\begin{aligned}\left(\widehat{\mathbf{l}}_{MLD}(\mathbf{d}), \widehat{\mathbf{q}}_{MLD}(\mathbf{d})\right) = \\ \arg\max_{\mathbf{l},\mathbf{q}\in\mathscr{E}_s} \sum_{d_k} \sum_{\mathbf{d}:d_k} \mathbf{I}_{co}(\mathbf{d})\mathbf{p}_{ch}(\mathbf{y} \mid \mathbf{d})\mathbf{l}(\mathbf{d})\mathbf{q}(\mathbf{d})\end{aligned} \tag{5}$$

The two sums above are exactly the same as the sum over all the words $\mathbf{d}$. The global maximum is again obtained for the optimal choice $\widehat{\mathbf{l}}_{MLD}(\mathbf{d}) = \widehat{\mathbf{q}}_{MLD}(\mathbf{d}) = \widehat{\mathbf{p}}_{MLD}(\mathbf{d})$ as in (4). The formulation in (5) is then equivalent to the original problem since the two solutions $\widehat{\mathbf{l}}_{MLD}(\mathbf{d})$ and $\widehat{\mathbf{q}}_{MLD}(\mathbf{d})$ both select the optimal sequence $\widehat{\mathbf{d}}_{MLD}$ of the maximum likelihood decoding problem. Let $\mathscr{C}_{MLD}$ denote the function (to be maximized) in (5). The direct maximization of $\mathscr{C}_{MLD}$ is not yet tractable. We need to separate the coding part from the mapping and channel part, this can be done by replacing the bit-marginals of the product of two PMFs by the product of the bit-marginals of the two PMFs taken separately. This is of course an approximation leading to $\tilde{\mathscr{C}}_k$ defined as

$$\tilde{\mathscr{C}}_k(\mathbf{l},\mathbf{q}) = \sum_{d_k}\left(\sum_{\mathbf{d}:d_k} \mathbf{I}_{co}(\mathbf{d})\mathbf{q}(\mathbf{d})\right)\left(\sum_{\mathbf{d}:\mathbf{d_k}} \mathbf{p_{ch}}(\mathbf{y} \mid \mathbf{d})\mathbf{l}(\mathbf{d})\right) \tag{6}$$

This approximation deserves some comments. First, the bit-marginals in $\tilde{\mathscr{C}}_k$ are now computable in practice. For example $\sum_{\mathbf{d}:d_k} \mathbf{I}_{co}(\mathbf{d})\mathbf{q}(\mathbf{d})$, $1 \le k \le n$, $d_k \in \{0,1\}$ is exactly the output given by a BCJR algorithm [10]. Next, the function $\tilde{\mathscr{C}}_k$ is dependent of k: the involved quantities are not the same for two different values of k (whereas $\mathscr{C}_{MLD}$ is independent of k). This suggests that $\tilde{\mathscr{C}}_k$ should be used for the **maximization over the** k^{th} **bit-marginal**. A distributed optimization strategy will be discussed in section 4. Last, the functions $\mathscr{C}_{MLD}$ and $\tilde{\mathscr{C}}_k$, $1 \le k \le n$, are equal (meaning there is no approximation) if the two PMFs involved $\mathbf{I}_{co}(\mathbf{d})\mathbf{q}(\mathbf{d})$ and $\mathbf{p}_{ch}(\mathbf{y} \mid \mathbf{d})\mathbf{l}(\mathbf{d}))$ are separable. Notice that for $\mathbf{l}(\mathbf{d}) = \widehat{\mathbf{l}}_{MLD}(\mathbf{d})$ and $\mathbf{q}(\mathbf{d}) = \widehat{\mathbf{q}}_{MLD}(\mathbf{d})$ both PMFs are indeed separable. This is also true for the whole class of "Kronecker" PMFs in which the global optimum is always lying. Since criteria $\tilde{\mathscr{C}}_k$, $1 \le k \le n$, are always non-negative, the joint maximization of $\tilde{\mathscr{C}}_k$ leads to the maximization of $\tilde{\mathscr{C}} = \sum_{k=1}^n \tilde{\mathscr{C}}_k$. We proved in [9] that, in some very specific cases, the global maximum of $\tilde{\mathscr{C}}$ is the maximum likelihood solution. We can go further into the interpretation of $\tilde{\mathscr{C}}$ and the connection with the optimal criterion $\mathscr{C}_{MLD}$. By definition,

$$\begin{aligned}\tilde{\mathscr{C}}(\mathbf{l},\mathbf{q}) &= \textstyle\sum_{k=1}^n \tilde{\mathscr{C}}_k(\mathbf{l},\mathbf{q}) \\ &= \textstyle\sum_{k=1}^n \sum_{\mathbf{d_k}} \sum_{\mathbf{d}:d_k,\mathbf{d}':d_k} \mathbf{I}_{co}(\mathbf{d})\mathbf{q}(\mathbf{d})\mathbf{p_{ch}}(\mathbf{y} \mid \mathbf{d}')\mathbf{l}(\mathbf{d}')\end{aligned} \tag{7}$$

From eq. (7), we can observe that $\tilde{\mathscr{C}}_k$ is the sum of all possible values of $\mathbf{I}_{co}(\mathbf{d})\mathbf{q}(\mathbf{d})\mathbf{p_{ch}}(\mathbf{y} \mid \mathbf{d}')\mathbf{l}(\mathbf{d}')$ restricted to the pairs $(\mathbf{d},\mathbf{d}')$ such that $d_k = d'_k$ (the value of bit k is the same for $\mathbf{d}$ and $\mathbf{d}$'). This means that:

- a pair $(\mathbf{d},\mathbf{d}')$ such that $d_k = d'_k$ and $d_i \ne d'_i$ for all $i \ne k$ is in $\tilde{\mathscr{C}}_k$ but not in $\tilde{\mathscr{C}}_i$, $i \ne k$.
- a pair $(\mathbf{d},\mathbf{d}')$ such that $d_k = d'_k$, $d_j = d'_j$ and $d_i \ne d'_i$ for all $i \notin \{k, j\}$ is in $\tilde{\mathscr{C}}_k$ and $\tilde{\mathscr{C}}_j$ but not in $\tilde{\mathscr{C}}_i$, $i \notin \{k, j\}$.
- a pair $(\mathbf{d},\mathbf{d}')$ such that $d_k = d'_k$ for all $k \in \{1,2,...,n\}$ is in each $\tilde{\mathscr{C}}_k$, $1 \le k \le n$. As a consequence this pair will appear n times in $\tilde{\mathscr{C}}$.

Let $\mathbb{S}_r$ denote the set of pairs $(\mathbf{d},\mathbf{d}')$ such that $d_H(\mathbf{d},\mathbf{d}') = r$ where $d_H(.,.)$ stands for the Hamming distance between two binary words of length n (the Hamming distance between two binary words of equal length is the number of positions at which the corresponding bits are different). The sum in (7) can be organized in the following manner:

$$\tilde{\mathscr{C}}(\mathbf{l},\mathbf{q}) = \sum_{r=0}^{n-1} (n-r) N_r(\mathbf{l},\mathbf{q})$$

where $N_r(\mathbf{l},\mathbf{q}) = \sum_{(\mathbf{d},\mathbf{d}')\in\mathbb{S}_r} \mathbf{I}_{co}(\mathbf{d})\mathbf{q}(\mathbf{d})\mathbf{p_{ch}}(\mathbf{y} \mid \mathbf{d}')\mathbf{l}(\mathbf{d}')$. The set $\mathbb{S}_0$ contains the pairs $(\mathbf{d},\mathbf{d}')$ such that $\mathbf{d} = \mathbf{d}'$, then we obtain

$$\tilde{\mathscr{C}}(\mathbf{l},\mathbf{q}) = n\mathscr{C}_{MLD} + \sum_{r=1}^{n-1} (n-r) N_r(\mathbf{l},\mathbf{q}) \tag{8}$$

Since both $\mathbf{q}$ and $\mathbf{l}$ are PMFs we have $0 \le \sum_{(\mathbf{d},\mathbf{d}')\in\mathbb{S}_0\cup\mathbb{S}_2\cup\ldots\mathbb{S}_{n-1}} \mathbf{q}(\mathbf{d})\mathbf{l}(\mathbf{d}') \le 1$. As a consequence, a global maximum $(\widehat{\mathbf{q}},\widehat{\mathbf{l}})$ is given by $\widehat{\mathbf{q}}(\mathbf{d}) = \delta_{\mathbf{d},\widehat{\mathbf{d}}}$ and $\widehat{\mathbf{l}}(\mathbf{d}') = \delta_{\mathbf{d}',\widehat{\mathbf{d}}'}$ with $(\widehat{\mathbf{d}},\widehat{\mathbf{d}}') = \arg\max_{\mathbf{d},\mathbf{d}'\in\{0,1\}^n} (n - d_H(\mathbf{d},\mathbf{d}'))\mathbf{I}_{co}(\mathbf{d})\mathbf{p}_{ch}(\mathbf{y} \mid \mathbf{d}')$. From (8), we can see the benefit of the decreasing weighting in the criterion since pairs $(\mathbf{d},\mathbf{d}')$ with low Hamming distance are weighted with high factors $(n - d_H(\mathbf{d}, \mathbf{d}'))$. Since $I_{co}(\mathbf{d})$ is an indicator function (with respect to the convolutional encoder), the Hamming distance between $(\mathbf{d}, \mathbf{d}')$ is also the minimum distance between $\mathbf{d}$ and the set of all possible interleaved codewords. Thus, $\widehat{\mathbf{d}}'$ yields the maximum of the channel probability and $\widehat{\mathbf{d}}$ is a codeword with the smallest distance to $\widehat{\mathbf{d}}'$. Taken as such, $\widehat{\mathbf{l}}(\mathbf{d})\widehat{\mathbf{q}}(\mathbf{d})$ is not necessarily a PMF. We will see in section 4 that the iterative solution enforces $\widehat{\mathbf{l}}(\mathbf{d})\widehat{\mathbf{q}}(\mathbf{d})$ to be a PMF. This is in favor of selecting a solution close to the MLD. These facts put together give an explanation for the near optimal performance of the iterative decoding. In this section, we have presented the suboptimal decoding and discussed its connection with the optimal one. We have seen that $\tilde{\mathscr{C}}$ is an accurate approximation of $\mathscr{C}_{MLD}$. In the next section, we focus on the iterative maximization of $\tilde{\mathscr{C}}$.

4. ITERATIVE DECODING AND NASH EQUILIBRIUM

Let $l_k(d_k)$ and $q_k(d_k)$ denote the marginals on any bit d_k then $\mathbf{l}(\mathbf{d}) = \prod_{j=1}^{n} l_j(d_j)$ and $\mathbf{q}(\mathbf{d}) = \prod_{j=1}^{n} q_j(d_j)$. The iterative optimization of the global criterion $\tilde{\mathscr{C}}$ is untractable since the evaluation of the gradient with respect to $q_k(d_k)$ (or $l_k(d_k)$) requires the computation of marginals over two bits: $\sum_{\mathbf{d}:d_k,d_i} \mathbf{I}_{co}(\mathbf{d}) \prod_{j \neq i,k} q_j(d_j)$. The computation of both (1-bit)-marginals and (2-bits)-marginals needs the evaluation of $n + \frac{n(n-1)}{2}$ marginals which would significantly increase the complexity. We have noticed in section 2 that the sub-optimal criterion $\tilde{\mathscr{C}}_k$ is derived from $\mathscr{C}_{MLD}$ when dealing with the k^{th} bit-marginal. We propose here to consider a distributed maximization strategy where $l_k(d_k)$ and $q_k(d_k)$ are chosen in order to maximize $\tilde{\mathscr{C}}_k$ as

$$\left(\widehat{l}_k, \widehat{q}_k\right) = \arg\max_{l_k, q_k \in \mathscr{F}} \tilde{\mathscr{C}}_k(\mathbf{l}, \mathbf{q}) \tag{9}$$

where $\mathscr{F}$ is the set of all possible PMFs on d_k. The solution of (9) is given by

$$\begin{gathered} \widehat{l}_k(d_k)\widehat{q}_k(d_k) = 1 \quad if \\ f_{d_k}(\mathbf{q}, I_{co}) f_{d_k}(\mathbf{1}, p_{ch}(\mathbf{y} \mid \mathbf{d})) > f_{\overline{d}_k}(\mathbf{q}, I_{co}) f_{\overline{d}_k}(\mathbf{1}, p_{ch}(\mathbf{y} \mid \mathbf{d})) \\ \widehat{l}_k(d_k)\widehat{q}_k(d_k) = 0 \quad otherwise \end{gathered} \tag{10}$$

where $f_{d_k}(\mathbf{1}, p_{ch}(\mathbf{y} \mid \mathbf{d})) = \sum_{\mathbf{d}:d_k} \mathbf{p}_{ch}(\mathbf{y} \mid \mathbf{d}) \prod_{j \neq k} l_j(d_j)$, $f_{d_k}(\mathbf{q}, I_{co}) = \sum_{\mathbf{d}:d_k} \mathbf{I}_{co}(\mathbf{d}) \prod_{j \neq k} q_j(d_j)$ and $\overline{d}_k = 1 - d_k$. An iterative process propagating hard estimates (0 or 1) is likely to get stuck in a local minima. A classical solution is to propagate instead soft-estimates (in $[0;1]$) and take hard decisions at the end of the iterative process. For the maximization problem in (9) and motivated by our previous work in [11], possible soft estimates are:

$$\begin{aligned} l_k(d_k) q_k(d_k) &\propto \left(f_{d_k}(\mathbf{q}, I_{co}) f_{d_k}(\mathbf{1}, p_{ch}(\mathbf{y} \mid \mathbf{d})) \right)^{\beta} \\ l_k(\overline{d}_k) q_k(\overline{d}_k) &\propto \left(f_{\overline{d}_k}(\mathbf{q}, I_{co}) f_{\overline{d}_k}(\mathbf{1}, p_{ch}(\mathbf{y} \mid \mathbf{d})) \right)^{\beta} \end{aligned} \tag{11}$$

where β is a positive constant. We can observe that $\beta \to \infty$ (combined with a normalization step) yields the hard estimates (see eq. (11) characterizing the product $l_k q_k$). The individual values of l_k and q_k depend on the scheduling of the successive updates. For $\beta = 1$, the choice of a hybrid Jacobi/Gauss-Seidel scheme yields the classical equation of the iterative decoding of BICM (and of turbo iterative decoding in general) [9]. In turbo-codes literature, the product $l_k(d_k)q_k(d_k)$ is called a posteriori probability (APP) and l_k and q_k are the extrinsics. At this step, there is no justification for choosing a particular value of β. This point will be addressed in sections 5 and 6.
The APP are defined up to a multiplicative constant, we build from (11) the log-likelihood ratios (LLR) $\lambda_{\mathbf{l},k}$ and $\lambda_{\mathbf{q},k}$ as

$$\lambda_{\mathbf{l},k} + \lambda_{\mathbf{q},k} = \log\left(\frac{l_k(d_k) q_k(d_k)}{l_k(\overline{d}_k) q_k(\overline{d}_k)} \right) \tag{12}$$

$$= \beta \log\left(\frac{f_{d_k}(\mathbf{q}, I_{co})}{f_{\overline{d}_k}(\mathbf{q}, I_{co})} \right) + \beta \log\left(\frac{f_{d_k}(\mathbf{1}, p_{ch}(\mathbf{y} \mid \mathbf{d}))}{f_{\overline{d}_k}(\mathbf{1}, p_{ch}(\mathbf{y} \mid \mathbf{d}))} \right) \tag{13}$$

where $\lambda_{\mathbf{l},k} + \lambda_{\mathbf{q},k}$ are the LLRs for the bit in position k, $1 \leq k \leq n$. We also denote $\lambda_{\mathbf{l},-k} + \lambda_{\mathbf{q},-k}$ the LLR vectors for all the bits except the bit in position k. The LLRs in (12-13) are a solution to the maximization of a strictly convex utility function U_k defined in the following manner:

$$\begin{aligned} U_k(\lambda_{\mathbf{l},k}, \lambda_{\mathbf{q},k}, \lambda_{\mathbf{l},-k}, \lambda_{\mathbf{q},-k}) = -\|\lambda_{\mathbf{l},k} + \lambda_{\mathbf{q},k} \\ -\beta \log\left(\frac{f_{d_k}(\mathbf{q}, I_{co})}{f_{\overline{d}_k}(\mathbf{q}, I_{co})} \right) - \beta \log\left(\frac{f_{d_k}(\mathbf{1}, p_{ch}(\mathbf{y}|\mathbf{d}))}{f_{\overline{d}_k}(\mathbf{1}, p_{ch}(\mathbf{y}|\mathbf{d}))} \right) \|^2 \end{aligned} \tag{14}$$

Iterative decoding can thus be understood as a game with n players where each player attempts to maximize (selfishly) its own utility function. We can notice that the decoder and the demapper are involved in a cooperative process since they both contribute to the utility function of all players. More formally, a game $\mathscr{G}$ is defined as a triplet $\mathscr{G} = (K, \{S_i\}_{i \in K}, \{u_i\}_{i \in K})$ where $K = \{1, 2, ..., n\}$ is a finite set of players, $\forall i \in K$, $\{S_i\}$ is the set of strategy of player i and U_i its utility function. In iterative decoding, the set of variables of player i are $s_i = (\lambda_{\mathbf{l},i}, \lambda_{\mathbf{q},i})$ and s_{-i} is the set of variables of the other players. A pure Nash Equilibrium (NE) is defined in the following manner [12].

Definition 1 *A profile $\mathbf{s}^*$ is a (pure) NE for $\mathscr{G}$ if $\forall i \in K$, $\forall s_i' \in S_i$, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*)$. If the inequality holds strictly for all players, then the equilibrium is classified as a strict NE.*

A NE is stable to a single deviation meaning that a player can not increase its utility function by changing unilaterally his strategy. Let $(\lambda_{\mathbf{l}^*}, \lambda_{\mathbf{q}^*})$ denote a pair of LLRs satisfying equations (12-13).

Proposition 1 *Let $(\lambda_{\mathbf{l}^*}, \lambda_{\mathbf{q}^*})$ denote a pair of LLR vectors with elements $\lambda_{\mathbf{l}^*,k}$, $\lambda_{\mathbf{q}^*,k}$ satisfying eq. (12)-(13) $\forall k \in \{0, 1, ..., n\}$. Then, $(\lambda_{\mathbf{l}^*}, \lambda_{\mathbf{q}^*})$ is a NE of the game $\mathscr{G}_{soft}$ with $K = \{0, 1, ..., n\}$, $\{S_i\}_{i \in K} = \mathbb{R}^2$ and $\{U_i\}$ defined as in (14). If the solution of (12)-(13) is unique then $(\lambda_{\mathbf{l}^*}, \lambda_{\mathbf{q}^*})$ is a strict NE of the game $\mathscr{G}_{soft}$.*

Proof: If $\lambda_{\mathbf{l}^*,k}$, $\lambda_{\mathbf{q}^*,k}$ is a solution of (12)-(13) then it is a global maximizer of U_k. Since this is true $\forall k \in \{0, 1, ..., n\}$, $(\lambda_{\mathbf{l}^*}, \lambda_{\mathbf{q}^*})$ is a NE for the game $\mathscr{G}_{soft}$. If the solution of (12)-(13) is unique then $(\lambda_{\mathbf{l}^*}, \lambda_{\mathbf{q}^*})$ is a strict NE.

Proposition 2 *Let $(\lambda_{\mathbf{l}^*}, \lambda_{\mathbf{q}^*})$ denote a pair of LLR vectors with elements $\lambda_{\mathbf{l}^*,k}$, $\lambda_{\mathbf{q}^*,k}$ satisfying eq. (12)-(13) $\forall k \in \{0, 1, ..., n\}$. Then $(\mathbf{l}^*, \mathbf{q}^*)$ is an "induced" equilibrium of the game $\mathscr{G}_{hard}$ with $K = \{0, 1, ..., n\}$, $\{S_i\}_{i \in K} = [0, 1] \times [0, 1]$ and with utility function $\tilde{\mathscr{C}}_i$ defined as in (6) meaning that $\forall i \in K$, $\forall s_i' = (l_i(d_i), q_i(d_i)) \in S_i$,*

$$\tilde{\mathscr{C}}_i(H(s_i^*), s_{-i}^*) \geq \tilde{\mathscr{C}}_i(s_i', s_{-i}^*)$$

where $H(\mathbf{s}_i^*)$ is a hard decision operator that returns the nearest integer for all the elements in $\mathbf{s}_i^*$.

The proof is obvious by considering eq. (10). The equilibrium defined in proposition 2 is not a NE since it does not match definition 1. It is however an equilibrium since a single deviation of user k from $H(l_k^*(d_k)q_k^*(d_k))$ (the hard decision on the APP of the bit in position k) will have a cost (lower value of the utility function of user k) if the other players maintain their strategy $(l_{-i}^*(d_i), q_{-i}^*(d_i))$.

Definition 2 *The social welfare of a game is defined as the sum of the utilities of all players*

$$W = \sum_{i=1}^{n} u_i$$

The social welfare [13] is a measure of efficiency of a society. In the turbo-decoding environment, it can be seen as a performance rating on the efficiency of the optimization process. In this section, two games are considered $\mathcal{G}_{hard}$ and $\mathcal{G}_{soft}$. The social welfare W_{soft} associated with the utility functions U_i has no interest since for any NE $W_{soft} = 0$. At the opposite, the social welfare W_{hard} (associated with the utility functions $\tilde{\mathcal{C}}_i$) coincides with the suboptimal criterion $\tilde{\mathcal{C}}$. The social welfare W_{hard} is thus a performance rating on the efficiency of the optimization process: low scores mean poor (joint)-optimization process whereas high scores mean a near-optimal (joint)-optimisation. Moreover, $\tilde{\mathcal{C}}$ is in connection with the MLD criterion which suggests that the MLD solution has more chance to be obtained for higher scores of W_{hard}. The existence of a NE is an important issue. At that point, we have no guarantee that (12-13) always have a solution. This question has been addressed in [7] in the case of turbo decoding (which includes BICM) by proving the existence of at least one solution to the fixed point problem in (12-13) when $\beta = 1$. Hence, the turbo-decoding algorithm always possesses (at least) one (pure) NE.

5. CONVERGENCE

In BICM and in turbo-decoding in general, the decoding process is initialized by choosing $(\lambda_{\mathbf{l}^{(0)}}, \lambda_{\mathbf{q}^{(0)}}) = \mathbf{0}$. This is equivalent to choosing a uniform distribution for $\mathbf{l}^{(0)}$ and $\mathbf{q}^{(0)}$. Even if a NE always exists there is no guarantee that the iterative process will converge toward a NE when starting from this initialization point. In this section, we consider the usual scheduling of iterative decoding [9], a mixed Jacobi/Gauss-Seidel strategy. In that case, the iterative process solves alternately the two equations below (15-16). The NE is thus obtained as the solution of the system of equations

$$\lambda_{\mathbf{l},k} - \beta \log\left(\frac{f_{d_k}(\mathbf{q}, I_{co})}{f_{\overline{d}_k}(\mathbf{q}, I_{co})}\right) = 0 \quad \forall k \in \{1, n\} \tag{15}$$

$$\lambda_{\mathbf{q},k} - \beta \log\left(\frac{f_{d_k}(\mathbf{l}, p_{ch}(\mathbf{y} \mid \mathbf{d}))}{f_{\overline{d}_k}(\mathbf{l}, p_{ch}(\mathbf{y} \mid \mathbf{d})}\right) = 0 \quad \forall k \in \{1, n\} \tag{16}$$

Obviously, if $(\lambda_{\mathbf{l}^*}, \lambda_{\mathbf{q}^*})$ is a solution of (15-16), it is also solution of (12-13). We use the compact notation $F(\lambda_{\mathbf{l}^*}, \lambda_{\mathbf{q}^*}) = 0$ to denote the system of equations (15-16). To prove the convergence, we calculate the Jacobian ∇F (with respect to $(\lambda_{\mathbf{l}^*}, \lambda_{\mathbf{q}^*})$). Define the measures $p_1(\mathbf{d}) = K_1 I_{co}(\mathbf{d})\mathbf{q}(\mathbf{d})$ and $p_2 = K_2 p_{ch}(\mathbf{y} \mid \mathbf{d})\mathbf{l}(\mathbf{d})$ where K_1 and K_2 are normalization factors. Define the matrices $\mathbf{C}$, $\mathbf{G}$ whose elements are

$$[\mathbf{C}]_{i,j} = p_1[d_j = 1|d_i = 1] - p_1[d_j = 1|d_i = 0]$$
$$[\mathbf{G}]_{i,j} = p_2[d_j = 1|d_i = 1] - p_2[d_j = 1|d_i = 0]$$

The Jacobian ∇F reads [6]:

$$\nabla F = \begin{pmatrix} \mathbf{I} & \beta(\mathbf{I} - \mathbf{C}) \\ \beta(\mathbf{I} - \mathbf{G}) & \mathbf{I} \end{pmatrix} \tag{17}$$

where $\mathbf{I}$ is the identity matrix of size $n \times n$. We proved in [9] that the iterative decoding process is an hybrid Jacobi/Gauss-Seidel method. The Gauss-Seidel method was originally used to solve a linear system of equation $\mathbf{Ax} = \mathbf{b}$. The procedure is known to converge if either $\mathbf{A}$ is symmetric positive-definite, $\mathbf{A}$ is an M-matrix or $\mathbf{A}$ is strictly or irreducibly diagonally dominant. Some of these conditions have been extended to the case of a nonlinear system of equation where $\mathbf{A}$ is replaced by ∇F. In [6], the authors consider the situation where ∇F is an M-matrix. By definition, an M-matrix has non positive off diagonal elements. From (17) and from the definitions of $\mathbf{C}$ and $\mathbf{G}$, it seems very unlikely that ∇F is an M-matrix in most cases. At the opposite, the expression of (17) suggests that ∇F could be a strict diagonally dominant matrix. The extension of the properties of convergence for the Jacobi and Gauss-Seidel method in the case of nonlinear systems of equations with strictly diagonally dominant Jacobian is due to Moré and is published in [14].

Definition 3 *Let* $\mathbf{A}$ *denote a* $n \times n$ *matrix with elements* a_{ij} *in* $\mathbb{R}$. $\mathbf{A}$ *is a strictly diagonal dominant matrix if*

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad \forall i \in \{1, ..., n\}$$

In the literature, many contributions focus on conditions on the indicator function(s) of the constituent encoder(s) or on the channel probability that would guarantee the convergence of the turbo-decoding process ($\beta = 1$) [6, 7]. Up to our knowledge, these conditions are uncheckable in a practical setting. In this paper, the question of the convergence is addressed in a different way:

Proposition 3 *It always exist* $\beta_0 > 0$ *such that* $\forall \beta \leq \beta_0$, $\nabla F(\lambda_{\mathbf{l}}, \lambda_{\mathbf{q}}))$ *is a strictly diagonally dominant matrix for all* $(\lambda_{\mathbf{l}}, \lambda_{\mathbf{q}}) \in \mathbb{R}^n \times \mathbb{R}^n$.

Proof: ∇F is a strictly diagonal dominant matrix if $\beta(\sum_{j=1, j \neq i}^{n} |p[d_j = 1|d_i = 1] - p[d_j = 1|d_i = 0]|) < 1 \ \forall i \in \{1, ..., n\}$ where p stands for either p_1 or p_2. Since p is a PMF, $\sum_{j=1, j \neq i}^{n} |p[d_j = 1|d_i = 1] - p[d_j = 1|d_i = 0]| < n$. Thus choosing $\beta_0 = \frac{1}{n}$ implies that ∇F is a strictly diagonally dominant matrix $\forall \beta \leq \beta_0$.

In the proof above, the bound $\beta_0 = 1/n$ is far from being optimal. The highest possible value of β_0 for a given setting is expected to be (significantly) above $1/n$. This point will be illustrated in the simulation part. We can now draw a conclusion from the results above. The choice of the parameter β is a trade-off between convergence and optimality of the joint-optimization process:

- Small values of β ($\leq \beta_0$) means convergence of the iterative process (this is proved in proposition 3) but leads to small value of the social welfare.
- High values of β means high values of the social welfare $\tilde{\mathcal{C}}$ but the iterative process may not converge. Actually, when β is large, $\mathbf{l}$ and $\mathbf{q}$ are getting closer of a Kronecker PMF. For any NE, $\tilde{\mathcal{C}} = \sum_{k=1}^{n} (l_k(d_k) q_k(d_k))^{\frac{\beta+1}{\beta}}$ (see (11) and (6)) and the value of $\tilde{\mathcal{C}}$ increase when $\mathbf{l}$ and $\mathbf{q}$ are close to a Kronecker PMF.

In the next section, we run several simulations to obtain typical values of β_0 in the particular setting of BICM.

6. SIMULATION

In this section, a classical BICM scheme is used with a $(5,7)$ convolutional code of rate $1/2$. The number of information bits is $n_b = 400$ (a frame). The code bits are passed through a random interleaver, mapped using set partitioning and modulated to 16-QAM symbols. The signal to noise ratio is defined as $\frac{E_b}{N_0}$, where E_b denotes the energy per information bit and N_0 is the noise variance. In this section we consider the "classical " iterative decoding performed using eq. (15-16) with $\beta = 1$ and we make a comparison with the iterative decoding with $\beta \neq 1$. We say that the iterative decoding converges when an agreement is reached between the APP at the output of demapper and decoder. Here, the iterative procedure stops when the norm between these two APP is less than 10^{-4}.

We first consider the frames in which the classical method ($\beta = 1$) fails to converge and search for the highest possible value of β_0 (see proposition (3)) that produces a convergent sequence from the initial point $(\lambda_{\mathbf{l}^{(0)}}, \lambda_{\mathbf{q}^{(0)}}) = \mathbf{0}$. The classical iterative decoding may not converge for low $\frac{E_b}{N_0}$. For $\frac{E_b}{N_0} = 5dB$, among 1000 runs, we identified 171 non-converging sequences. For $\frac{E_b}{N_0} = 6dB$, among 4000 runs, we identified 70 non-converging sequences. For each of them, the value of β has been decreased (by step of -0.05) until convergence in order to obtain an approximate value of β_0 (with a precision of $+/-0.05$). The numerical results are reported in the graphs of figure 2. We can observe that β_0 is significantly larger than $1/n$. The smallest value observed is 0.7 and in most cases $\beta_0 > 0.8$. Unfortunately, convergence does not necessarily means

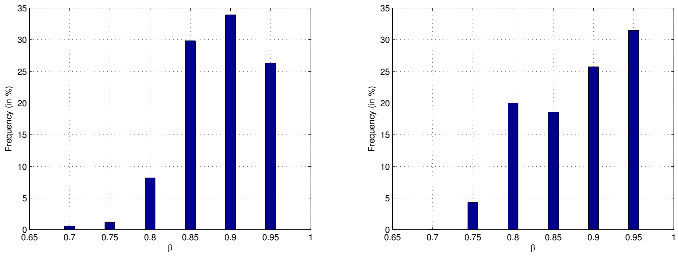

Figure 2: β_0 distribution: (left) $\frac{E_b}{N_0} = 5dB$ - (right) $\frac{E_b}{N_0} = 6dB$

small BER. For instance when $\frac{E_b}{N_0} = 6dB$, the average number of errors observed (message is 400 bits long) in the simulation is 11.84 for the classical decoding and 12.91 when $\beta = \beta_0$.
For $\frac{E_b}{N_0}$ equals to 7dB or 8dB, the classical method is convergent in general. For each frame, the value of β has been increased (by step of $+0.2$) until convergence is lost. The numerical values obtained for β_0 are reported in figure 3. For the $\frac{E_b}{N_0}$ under consideration,

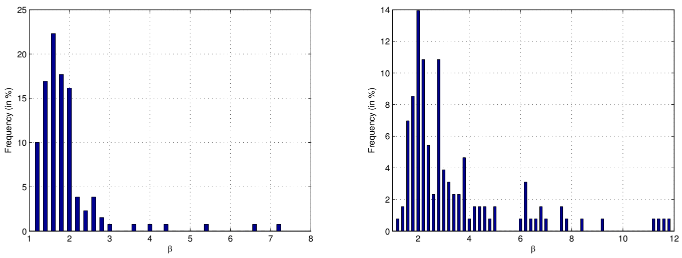

Figure 3: β_0 distribution: (left) $\frac{E_b}{N_0} = 7dB$ - (right) $\frac{E_b}{N_0} = 8dB$

typical values of β_0 are between 1 and 3. We claimed in section 5 that higher values of β lead to higher values of the social welfare. For $\frac{E_b}{N_0} = 7dB$ the average value of $\tilde{\mathscr{C}}$ is 0.86 when $\beta = 1$ and 0.9 when $\beta = \beta_0$. For $\frac{E_b}{N_0} = 8dB$ the average value of $\tilde{\mathscr{C}}$ is 0.89 when $\beta = 1$ and 0.93 when $\beta = \beta_0$. For each of these runs, the value of $\tilde{\mathscr{C}}$ obtained when $\beta = \beta_0$ (> 1) is above the value obtained with $\beta = 1$. In terms of BER, replacing $\beta = 1$ with $\beta = \beta_0$ is not efficient. For example with an $\frac{E_b}{N_0} = 7dB$, the average number of errors observed is 0.21 for the classical decoding and 0.24 when $\beta = \beta_0$. Changing the value of β does not seem to improve the BER, however it may have some incidence on the convergence rate. In particular, it may have the potential for an accelerated convergence at high $\frac{E_b}{N_0}$. This is an open issue for further research.

7. CONCLUSION

The iterative turbo-decoding was not originally introduced as the solution to a well-defined optimization problem. An accurate justification for the near optimal performance of iterative decoding remained incomplete. In this paper, the sub-optimal optimization problem that iterative decoding is seeking to optimize was presented. It was obtained from the maximum likelihood detection problem through parallel approximations. An interpretation in terms of Hamming distance was also provided that gives some clues for understanding the excellent performance of iterative decoding. Furthermore, it was proven that the optimization is in fact similar to a n-player game. Surprisingly, the decoder and demapper are not antagonist players. They are involved in a cooperative process in which n selfish players attempt to optimize their own bit-marginals. The stationary points are the Nash equilibria of the game and the cost function of the sub-optimal optimization problem is the social welfare of the game. We have seen that the social welfare is an indicator of the trade-off obtained between the separate optimization of the bit-marginals and the joint-optimization of the whole sequence. The convergence was also analysed and it is proved that convergence to a Nash equilibrium of the game is always possible but this is at the cost of a smaller social welfare. From the simulation, we have understood that the attainable social welfare is dependent of the signal to noise ratio. We conjecture that the parameter β could serve for increasing the convergence rate at high signal to noise ratio. We are currently investigating in this direction.

ACKNOWLEDGEMENT

The author would like to thank Ziad Naja and Pierre Duhamel for enlightening discussions and for help in developing the software used in this work.

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