THE GENERALIZATION OF DISCRETE STOCKWELL TRANSFORMS

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ABSTRACT

This paper provides a constructive framework to unify and expand various existing discrete Stockwell transforms. Under the developed framework, we are able to flexibly adjust the time/frequency sampling resolution as well as the amount of information redundancy tailored to a specific signal and certain computational consideration. The discrete Stockwell transforms designed under the framework are invertible and reserve the absolutely referenced phase.

1. INTRODUCTION

Being a hybrid of the short-time Fourier transform and the wavelet transforms, the Stockwell transforms (the ST, [1]) provides a time-frequency representation of a signal with a frequency-dependent resolution. Due to its easy-interpretation, multi-resolution analysis and the ability of maintaining the meaningful local phase information, the ST has established successes in many areas including geophysics [2] and biomedicine [3, 4].

One main drawback of the ST is the amount of information redundancy in its resulting time-frequency representation. That causes large computing consumption and limits its use in dealing with large size of data. To improve its computational efficiency, the discrete orthonormal Stockwell transform (DOST) is proposed [5, 6]. The DOST is based on a set of orthonormal basis functions that localize the Fourier spectrum of the signal. It samples the time-frequency representation given by the ST with zero information redundancy and retains the advantageous phase properties of the ST. The development of the DOST releases the potential of the ST for more practical applications.

However, due to its non-redundancy, the DOST provides a rather coarse time-frequency representation with its frequency resolution proportionally scaled to the logarithm of the frequency. Such a representation may not be always easy to interpret and be sufficient to reveal all the details within a specific signal. Certain amount of information redundancy producing a finer representation is sometimes preferable when analyzing a signal.

Thus, the main objective of this paper is to provide a constructive framework that not only embraces the conventional discrete ST and the DOST, but also allows us to design a discrete ST that flexibly samples in the time-frequency domain and easily adjusts the amount of information redundancy tailored to a specific signal. Thus we consider it as the generalization of the discrete STs (GDSTs).

2. REVIEW OF THE ST, DST AND DOST

2.1 The Stockwell Transform

The continuous Stockwell transform (ST) of a signal $h(t)$ is defined as:

$$s(\tau,f)=\int_{-\infty}^{\infty}h(t)\frac{|f|}{\sqrt{2\pi}}e^{-(\tau-t)^2f^2/2}e^{-i2\pi ft}dt \quad (1)$$

where f is the frequency variable, t is the time variable and τ is the time translation. Note that the width of the Gaussian window function is proportionally to the inverse of the frequency. With multi-resolution, the Stockwell spectrum $s(\tau,f)$ reveals how frequency components in the signal vary over time.

Since the integration of the Stockwell spectrum with respect to time yields the Fourier spectrum of the signal, i.e.,

$$H(f)=\int_{-\infty}^{\infty}s(\tau,f)d\tau \quad (2)$$

the inverse Stockwell transform can be given by

$$h(t)=\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}s(\tau,f)d\tau\right\}e^{i2\pi ft}df \quad (3)$$

In addition, the ST can be expressed in the Fourier domain

$$s(\tau,f)=\int_{-\infty}^{\infty}H(\xi+f)\,e^{-\frac{(2\pi\xi)^2}{2f^2}}e^{i2\pi\xi\tau}d\xi \quad (4)$$

where $H(f)$ is the Fourier spectrum of $h(t)$.

2.2 The Discrete Stockwell Transform

Let $h[l]=h(l\cdot T), l=0,1,\cdots,N-1$, be the samples of the continuous signal $h(t)$, where T is the sampling interval. Its discrete Fourier transform is given by:

$$H[m]=\sum_{l=0}^{N-1}h[l]e^{-i2\pi ml/N} \quad (5)$$

where the discrete frequency index $m=0,1,\cdots,N-1$.

Discretization of Eq. (4) leads to the discrete ST (DST):

$$s[k,n]=\sum_{m=0}^{N-1}e^{-\frac{2\pi^2m^2}{n^2}}H[m+n]e^{\frac{i2\pi mk}{N}} \quad (6)$$

where k is the index for time translation and n is the index for frequency shift. Function $\exp(-2\pi^2m^2/n^2)$ is the Gaussian window in the frequency domain. Implementation of the DST based on Eq. (4) can utilize the Fast Fourier transform (FFT) and hence lead to fast computation.

Similarly, discretizing Eq. (3) yields the inverse DST

$$h[l]=\frac{1}{N}\sum_{n=0}^{N-1}\left\{\frac{1}{N}\sum_{k=0}^{N-1}s[k,n]\right\}e^{i2\pi nl/N} \quad (7)$$

For a signal of length N, the DST produces N^2 number of coefficients in the time-frequency domain.

2.3 The Discrete Orthonormal ST

The DOST is an orthonormal version of the DST, producing N point time-frequency representation for a signal of length N. Thus, the DOST reduces the information redundancy of

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the DST to zero and leads to the maximum efficiency of a representation.

The DOST can be defined as an inner product between a time series $h[l]$ and a set of orthonormal basis tions $g_{\langle \upsilon,\beta,\tau\rangle}[l]$:

$$g_{\langle \upsilon,\beta,\tau\rangle}[l] = \frac{1}{\sqrt{N\beta}} \sum_{f=\upsilon-\frac{\beta}{2}}^{\upsilon+\frac{\beta}{2}-1} e^{i2\pi lf/N} e^{-i2\pi\tau f/\beta} e^{-i\pi\tau} \quad (8)$$

where three parameters $\langle \upsilon,\beta,\tau\rangle$ sample the time-frequency domain: υ is the frequency label representing the υ-th sampled frequency band, β is the width of the frequency band, and τ is the time label indicating the τ-th time sampling interval corresponding to the υ-th frequency sampling band in the time-frequency domain (i.e,. $\tau = \{0,1,2,\cdots,\beta-1\}$).

Here, to ensure the orthogonality of the basis functions, υ, β and τ must satistify the following conditions:

- $\tau = \{0,1,2,\cdots,\beta-1\}$.
- υ and β must be selected such that each of frequency sample is used once and only once

One popular choice of sampling the time-frequency domain is the octave sampling [5], in which the frequency sampling bandwidth β doubles for each increasing frequency. Denote the discrete Fourier frequency point $n = n(\upsilon) = 3\beta/2$ be the centre of the υ -th frequency sampling band and the discrete time point $k = \tau N/\beta$ be the left end point of the τ-th time sampling interval. Then the $\langle k,n\rangle$ are the sampling points in the time-frequency domain and $s[k,n]$ is the DOST coefficient corresponds to the point $\langle k,n\rangle$. The coefficients of the DOST at the rest discrete points in the time-frequency domain can be obtained via interpolation.

By the Parseveal Theorem, the DOST has the two equivalent expressions:

$$s[k,n] = \langle h, g_{\langle k,n\rangle}\rangle = \sum_{m=0}^{N-1} \overline{g_{\langle k,n\rangle}(m)} h(m) = \frac{1}{N}\langle H, G_{\langle k,n\rangle}\rangle \quad (9)$$

where the Fourier spectrum of the orthonormal DOST basis function $g_{\langle k,n\rangle}$ is a rectangular function with a phase modulation:

$$G_{\langle k,n\rangle}(f) = \sqrt{\frac{N}{\beta}} \Pi_{[\beta,2\beta-1]}(f) e^{\frac{-i2\pi(f-n)k}{N}} \quad (10)$$

Here

$$\Pi_{\Omega}(f) = \begin{cases} 1 & f \in \Omega + \mathrm{pN} \\ 0 & f \notin \Omega + \mathrm{pN} \end{cases}$$

where $\mathrm{p} = 0, \pm 1, \pm 2, \cdots, \pm\infty$. Hence, similar to the DST, the DOST can be calculated in the Fourier domain:

$$s[k,n] = \sum_{l=0}^{N-1} \frac{1}{\sqrt{N\beta}} \Pi_{[\beta,2\beta-1]}(l) e^{\frac{i2\pi(l-n)k}{N}} H[l] = \sum_{m=0}^{N-1} \frac{1}{\sqrt{N\beta}} \Pi_{[-\beta/2,\beta/2-1]}(m) e^{\frac{i2\pi mk}{N}} H[m+n] \quad (11)$$

Note that due to the periodicity of the functions within the summation, the sum range for m from 0 to N-1 is the same as that from -n to $N-n-1$. Substituting $k = \tau N/\beta$ and $n = 3\beta/2$ into Eq. (11) yields

$$\begin{aligned} s[k,n] &= s[\tau,\upsilon] \\ &= \sum_{m=-n}^{N-n-1} \frac{1}{\sqrt{N\beta}} \Pi_{[-\beta/2,\beta/2-1]}(m) e^{\frac{i2\pi mk}{N}} H[m+n] \\ &= \sqrt{\frac{\beta}{N}} \frac{1}{\beta} \sum_{m=-\beta/2}^{\beta/2-1} e^{\frac{i2\pi m\tau}{\beta}} H\left[m+\tfrac{3}{2}\beta\right] \\ &= e^{-i\pi\tau} \sqrt{\frac{\beta}{N}} \frac{1}{\beta} \sum_{m=0}^{\beta-1} e^{\frac{i2\pi m\tau}{\beta}} H[m+\beta] \end{aligned} \quad (12)$$

Equation (12) enables us to utilize an β-point DFT to improve the computational speed, especially when β is the power of 2.

3. GENERALIZATION OF THE DSTS

3.1 The Framework of the DST

For a signal of length N, the numerical implementation of the forward and inverse DST can be summarized as below:

Forward DST:

- Apply an N-point DFT to calculate the Fourier spectrum of the signal H[m];
- Multiple $H[m+n]$ with the Gaussian window function $W[m] = e^{-2\pi^2 m^2/n^2}$;
- For each fixed frequency shift $n = 0,1,\cdots,N-1$, apply an N-point inverse DFT to $W[m]H[m+n]$ in order to calculate the DST coefficients $s[k,n]$, where $k = 0,1,\cdots,N-1$;

Inverse DST:

- Apply an N-point DFT to $s[k,n]$with respect to time index k to obtain the windowed Fourier spectrum $W[m]H[m+n]$. Note that $W[0]$ =1 yields $W[0]H[n] = H[n]$, the n-th Fourier coefficient of the signal;
- Apply an N-point inverse DFT *to* $H[n]$ to recover the original signal $h[l]$

Value at the centre of the frequency window function, i.e., $\mathrm{W}[0] = 1 \neq 0$, guarantees the invertibility of the DST.

Note that there is a DST coefficient calculated for every pair of $\langle k,n\rangle$in the time-frequency domain. Therefore, we define the frequency sampling resolution (FSR) of the DST equal to 1 and the time sampling resolution (TSR) of the DST equal to 1 as well.

Hence, we can introduce the following key parameters that essentially control the numerical computation of the various versions of the DST:

- FSR: 1
- TSR: 1
- Frequency window width equal to the central frequency n;

3.2 The Framework of the DOST

The forward and inverse DOST can be implemented in a similar framework to those of the DST, but have a number of significant differences as summarized below:

Forward DOST:

- Apply an N-point DFT to calculate the Fourier spectrum of the signal $H[m]$;
- Multiple $H[m+n]$ with the rectangular window function $W[m]=\Pi_{[-\beta/2,\beta/2-1]}(m)$;
- For each central frequency $n=0,1,3,\cdots,3\beta/2\cdots$, apply an β-point inverse DFT to $W[m]H[m+n]$ in order to calculate the DST coefficients $s[k,n]$, where: $k=0,L/\beta,2L/\beta,\cdots,(\beta-1)L/\beta$;

Inverse DOST:

- For each central frequency $n=0,1,3,\cdots,3\beta/2\cdots$, apply an β-point DFT to $s[k,n]$ with respect to time index k to obtain the windowed Fourier trum $W[m]H[m+n]$. Note that $W[m]=1$, for $m\in\left[-\frac{\beta}{2},\cdots,\frac{\beta}{2}-1\right]$ that returns $H[n]$, $n=\beta,\beta+1,\cdots,2\beta-1$, β Fourier coefficients of the signal;
- Apply an N-point inverse DFT to $H[n]$to recover the original signal $h[l]$;

The corresponding key parameters for the DOST are given as follows:

- FSR: β appropriated to the central frequency $n=3\beta/2$
- TSR: L/β
- Window bandwidth: β.

3.3 Generalization of Discrete STs

As we can observe, the computational framework of both the DST and the DOST is essentially the same, that is, they rely on the DFT operations to the Fourier spectrum of the signal within different frequency sub-bands. The differences between the DST and the DOST are the selections of the key parameters as listed above. This motivates us to unify and extend the current DST and DOST to a more general setting by varying the frequency/time sampling resolution, the window function and the window bandwidth.

One of the main issues is the partition of the frequency domain into the union of non-overlapping sub frequency bands. Let F denotes the entire frequency domain and $\{F_\nu\}_{\nu=0}^{L-1}$ as a partition of F, such that:

$$F=\cup_{\nu=0}^{L-1}F_\nu \quad \text{and} \quad F_{\nu_1}\cap F_{\nu_2}=\emptyset \ \text{if} \ \nu_1\neq\nu_2 \qquad (13)$$

We denote the length of interval F_ν by α_ν, i.e., defining the frequency sampling resolution of the time-frequency representation.

For each sub-band F_ν $(\nu=0,1,\cdots,L-1)$, we define a frequency sub-band D_ν such that

$$F_\nu\subseteq D_\nu \qquad (14)$$

Let β_ν denote the length of interval D_ν, where N/β_ν determines the time sampling resolution corresponding to the frequency sub-band F_ν. We also denote the center frequency of F_ν by n_ν.

For each pair (F_ν,D_ν), we can define a frequency window function $W_\nu[m]$ satisfying:

$$\begin{cases} W_\nu[m]\neq 0 & \text{when } m\in F_\nu \\ W_\nu[m]=0 & \text{when } m\notin D_\nu \end{cases} \qquad (15)$$

The frequency window bandwidth can be defined as

$$\sum_{m\in F_\nu}\|W_\nu[m]\|^2\,(m-n_\nu) \qquad (16)$$

Now, following the notation of F_ν, D_ν, α_ν, β_ν, $n_\nu$and W_ν, and generalizing Eq (6) and (11), we define the generalized discrete Stockwell transform (GDST) as the following:

$$s[\tau,\nu]=s[k,n]=\sum_{l=0}^{N-1}\frac{1}{\sqrt{N\beta_\nu}}W_\nu[l]e^{\frac{i2\pi(l-n_\nu)k}{N}}H[l]$$

where $\nu\in[0,\cdots,L-1]$ is the index of frequency sub band, time index $k=\tau N/\beta_\nu$, $\tau\in[0,\cdots,\beta_\nu-1]$, is left end sampling point of the τ-th time interval corresponding to sub-band ν, and the central frequency of sub-band ν is n_ν.

By a variable change $l-n_\nu+\beta_\nu/2=m$, we get

$$s[\tau,\nu]=s[k,n] \\ =\sqrt{\frac{\beta_\nu}{N}}e^{-i\pi\tau}\frac{1}{\beta_\nu}\sum_{m=0}^{\beta_\nu-1}W_\nu[m+n_\nu-\beta_\nu/2]e^{\frac{i2\pi m\tau}{\beta_\nu}}H[m+n_\nu-\beta_\nu/2] \qquad (17)$$

Eq (17) leads to a fast implementation of the GDST by utilizing the FFT.

Similar to the DST and the DOST, the inverse GDST can be defined straightforwardly as the following:

$$h[k]=\frac{1}{N}\sum_{l=0}^{N-1}\left[\sum_{\nu=0}^{L-1}\frac{\Pi_{D_\nu}(l)}{W_\nu[l]}\sum_{\tau=0}^{\beta_\nu-1}e^{i\pi\tau}\sqrt{\frac{N}{\beta_\nu}}s[\tau,\nu]e^{\frac{-i2\pi(l-n_\nu+\beta_\nu/2)\tau}{\beta_\nu}}\right]e^{\frac{-i2\pi lk}{N}} \qquad (18)$$

Therefore, Eqs (17) and (18) define the forward and inverse GDST, respectively.

To summarize, the GDST can be implemented in a similar numerical procedure as the DOST:

Forward GDST:

- Apply an N-point DFT to calculate the Fourier spectrum of the signal $H[m]$;
- Multiple $H[m+n_\nu-\beta_\nu/2]$ with the rectangular window function $W_\nu[m+n_\nu-\beta_\nu/2]$;
- For each central frequency n_ν, apply an β_ν-point inverse DFT in order to calculate the GDST coefficients $s[k,n]$;

Inverse GDST:

- For each central frequency n_ν, apply a β_ν-point DFT to $s[k,n]$with respect to time index k to obtain the windowed Fourier spectrum $W_\nu H_\nu[m+n_\nu-\beta_\nu/2]$. Note that $W_\nu[m]\neq 0$, for $m\in D_\nu$ that returns $H[n]$, $n\in F_\nu$, Fourier coefficients of the signal;

- Apply an N-point inverse DFT to $H[n]$ to recover the original signal $h[l]$;

For each sub frequency band ν, the corresponding key parameters for the GDST:

- FSR: α_ν
- TSR: N/β_ν
- Windows bandwidth given by (16)

Within the framework defined by the GDST, the following set of parameters defines the DST:

$$F_\nu = \{\nu\}, D_\nu = [0, \cdots, N-1]$$
$$\alpha_\nu = 1, \beta_\nu = N, n_\nu = \nu,$$
$$W_\nu[m] = e^{-\frac{2\pi^2(m-\nu)^2}{\nu^2}}$$

where $\nu = [0, \cdots, N-1]$.

And the DOST can be given by the parameters:

$$F_\nu = D_\nu = [\beta_\nu, 2\beta_\nu - 1]$$
$$\alpha_\nu = \beta_\nu = \{0,1,2,4, \cdots, 2^{p-1}\}, n_\nu = 3\beta_\nu/2,$$
$$W_\nu[m] = \Pi_{[\beta_\nu,\ 2\beta_\nu - 1]}(m)$$

where $\nu = [0, \cdots, L-1]$, $p = log_2(N) - 1$.

As we can see, the GDST offers a flexible way to define a class of the DSTs by varying the partition of the frequency domain and the values of the other parameters. The blockness appearance of the resulting time-frequency representation of the GDST is determined by the time and frequency sampling resolutions, while the frequency window bandwidth affects the time-frequency resolution.
Depending on the selection of parameters, the various forms of the DSTs produce time-frequency representations with different amount of information redundancy. Using the notation of the GDST, for each frequency sub-band ν, we can define a redundancy ratio measuring the relative amount of information redundancy of a time-frequency representation along time:

$$RR_\nu = (\beta_\nu - \alpha_\nu)/(L - \alpha_\nu) \quad (19)$$

where $0 \leq RR_\nu \leq 1$. For instance, the redundancy ratio for the DOST is zero consistent with the orthonormality of the DOST, while the redundancy ratio for the DST is one giving the maximum amount of information redundancy. More interestingly, as long as $\beta_\nu = \alpha_\nu$, the redundancy ratio is zero which implies that the associated GDST is orthogonal.

4. OTHER EXAMPLES OF THE GDST

Section 3.3 provides a flexible and constructive way to design any new GDST tailored to specific needs. These include orthogonal or non-orthogonal DSTs. Here we give two new DSTs with different partitions in the frequency domain.

4.1 The GDST with Uniform Partition

We partition the frequency domain uniformly into sub-bands of the same length denoted as N/L. The parameters can be set as:

$$F_\nu = \left[\frac{\nu * N}{L}, \frac{(\nu+1) * N}{L}\right], F_\nu \subseteq D_\nu$$
$$\alpha_\nu = \beta_\nu = \{N/L\}, n_\nu = \frac{(\nu + 1/2) * N}{L},$$
$$W_\nu[m] = \Pi_{F_\nu}(m)$$

where $\nu = [0, \cdots, L-1]$.
In this case, the redundancy ratio is zero and therefore the resulting time-frequency representation is non-redundant. This defines an orthogonal DST, yielding N time-frequency coefficients, the same length as the original signal.
We can relax the restriction condition of D_ν and W_ν. For example, increasing the length of D_ν will improve the time sampling resolution in the time-frequency representation, but will introduce higher level of information redundancy and increase the number of coefficients. The window function can be replaced by Gaussian or other admissible window functions as well.

4.2 The GDST with Adaptive Spectral Energy Partition

It is common to modulate and band-pass filter signals arose from applications of signal processing. In practice, energy of a signal often concentrates in certain frequency bands where better frequency sampling resolution should be applied in order to reveal more details. On the other hand, for the frequency bands where less spectral energy exists, we can have large frequency sampling resolution to save computational resources.
Therefore, we propose a partition whose frequency sampling resolution is adaptive to spectral energy: better frequency sampling resolution is assigned to higher spectral energy areas and coarser frequency sampling resolution for lower spectral energy areas. By further choosing of other parameters D_ν , β_ν and W_ν, we can design a spectral energy GDST tailored to a specific signal. One of the practical ways to implement such a GDST is to employ the binary set partition to automatically partition the frequency domain into a set of non-overlapping sub-bands, so that all frequency sub-bands have roughly the same amount of spectral energy.

5. EXPERIMENT AND SIMULATION

Fig.1 demonstrates performance of various forms of the GDSTs for a synthetic signal:

$$h[n] = cos\left(2\pi n\left(0.2 + \frac{6}{1+n}\cos(6\pi n/1024)\right)\right) \quad (20)$$

whose frequency has a cosinoidal fluctuates around 0.2 sampling frequency over time as shown in Fig. 1 a). Figures 1 b)-f) are the amplitude of the time-frequency spectrum given by the DOST, the orthogonal GDST with uniform partition, the orthogonal GDST with energy adaptive partition, the GDST with uniform partition and its redundancy ratio being one, the GDST with adaptive spectral energy partition and its redundancy ratio being one, and the original DST, respectively. For the DOST and the various GDSTs in

Fig 1. b)-f), we partition the frequency domain into 12 non-overlapping frequency sub-bands.

As we can see, the original DST provides a smooth and detailed description on the temporal variation of the frequency variable using N^2 coefficients. The DOST and the orthogonal GDST with uniform or adaptive energy partition all generate a time-frequency representation with N coefficients.

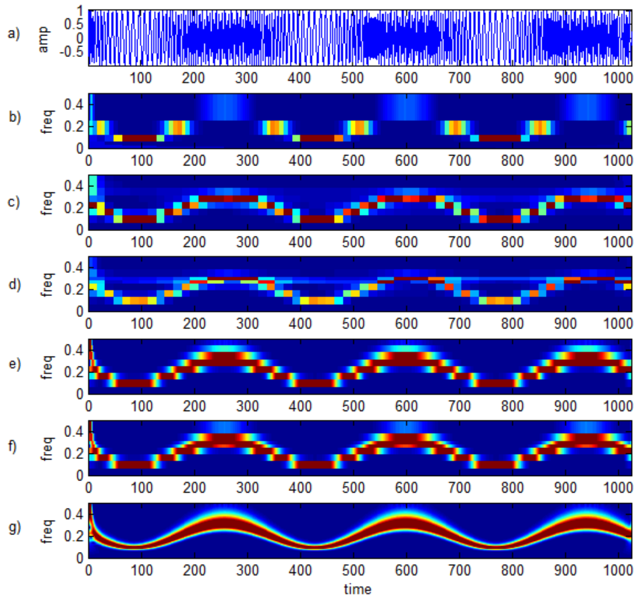

Figure 1 – Generalizations of discrete Stockwell transforms of a) a discrete signal given by Eq. (20) , b) the DOST, c) the orthogonal DST with uniform partition, d) the orthogonal DST with adaptive spectral energy partition, e) the DST with uniform partition, f) the DST with adaptive spectral energy partition and g) the DST, respectively.

All of the three orthogonal DSTs give a rough description on how frequency changes over time. But the DOST with the octave partition as shown in Fig.1 b) assigns more coefficients to the low frequency range that contains little information. The GDST with uniform partition spread the coefficients evenly in the time-frequency domain, sufficiently revealing the cosinoidal fluctuation of the frequency. The GDST with adaptive spectral energy partition however assigns more coefficients to energy concentrated frequency regions at which better frequency resolution is given.

Compare Fig. 1 e)-g), the time-frequency spectra given by the non-orthogonal GDSTs and the DST. The DST certainly produces the most smooth spectrum using N^2 coefficients. On the other hand, the non-orthogonal GDSTs use only 2.5% of the N^2 coefficients clearly revealing the cosinoidal pattern in the frequency, as shown in Fig. 1 e) and f). Similar to Fig. 1 c)-d), the difference between Fig.1. e) and f) is that the adaptive energy partition results better frequency resolution at high energy frequency regions and poor frequency resolution at low energy frequency region and the uniform partition treats all the frequencies the same.

6. SUMMARY

In this paper, we propose a class of generalized discrete Stockwell transforms in a constructive framework. The key features of a GDST are driven by the four parameters: the frequency/time sampling intervals, the ratio of redundancy and the window functions. By adjusting these parameters, we can not only yield the existing DST and the DOST, but also flexibly design a GDST tailored to a specific signal or application. For the spectral partition in Eqs (13) and (14), if $F_v = D_v$, the resulting GDSTs are orthogonal and provide coarse non-redundant time-frequency representations; if $F_v \subset D_v$, the GDSTs have nonzero redundancy ratios providing smoother redundant representations. The window functions will determine the time-frequency resolutions in the representation, the ability to distinguish different frequency components and their time occurrence. Due to the page limit, the detailed investigation of the impact of these parameters to the generalized discrete Stockwell transforms will be discussed in a separate paper. In summary, the proposed GDST framework will provide us a highly flexible way to analyze and process a signal. More practical examples of the GDST are worth of exploring for specific applications.

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