MINIMAX ROBUST POWER ALLOCATION OVER PARALLEL COMMUNICATION CHANNELS

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ABSTRACT

Adapting the transmission strategy under partial or uncertain channel state information (CSI) at the transmitter is of importance in practical systems with imperfect channel knowledge. In this work, we consider the problem of power allocation in a system of parallel communication sub-channels with uncertain CSI. We adopt the robust (minimax) optimization methodology and attempt to arrive at a novel precoding strategy that guarantees a certain worst-case performance, but without being too conservative in terms of average performance. To this end, we precisely model the underlying sources of uncertainty, and characterize the typical norm-constrained uncertainty set by an additional unbiasedness constraint. This helps to reduce the severity of the worst-case scenario, and in turn leads to a less conservative, robust minimax power allocation strategy with superior average performance.

1. INTRODUCTION

Adapting the transmission strategy with respect to the channel conditions leads to an improved transmission performance. In case, perfect CSI is available at the transmitter, the transmission strategy can be adapted via a precoder designed to optimize a certain objective function such as receiver *mean square error (MSE)*, *signal-to-noise ratio (SNR)* or the transmission rate. On the other hand, if no CSI is available, only a trivial fixed precoder can be used, and the potential precoding gains are lost. Between these two extremes of precoder design, based on perfect CSI and no CSI, the case of partial CSI at the transmitter is of interest in practical systems, where the CSI is obtained at the transmitter either via estimation on the reverse link or via the quantized feedback from the receiver, both leading to a CSI that is not perfect and possibly erroneous.

Broadly speaking, the precoder design approaches with partial CSI knowledge can be classified into two frameworks. The *stochastic framework* assumes a certain distribution for the channel coefficients, and designs the precoder based on the knowledge of the mean or the covariance of this distribution [1–3]. The criterion here is to optimize a certain objective function w.r.t the given CSI distribution. The precoder, thus obtained, is optimal in terms of average performance, but since the knowledge of CSI statistics is needed, it is inherently prone to CSI modeling errors.

The *deterministic framework*, on the other hand, models the channel coefficients as being lying in an *uncertainty region* around the nominal channel and does not require any statistical knowledge about the CSI. The only assumptions are on the size of this uncertainty region. The *robust optimization* [4] strategy is then pursued, leading to a minimax/maximin problem formulation, and as such the precoder promises a certain guaranteed level of performance [5, 6]. The deterministic framework was pursued independently in [5, 6] to design precoders under uncertain CSI with the objective of maximizing the *worst-case (WC)* received SNR for spherical and more general uncertainty classes respectively. In [7], the problem of robust power allocation is addressed for the maximization of channel capacity in a simple parallel sub-channel setting. However, owing to the non-convex nature of the objective function, the consequent duality gap leads to a possibly too conservative solution. Payaro *et al.* applied the deterministic framework in [8] to solve the complementary problem, i.e. designing a precoder that minimizes the transmit power while satisfying some *Quality of Service (QoS)* constraints. Recently, [9] presents a unified view on precoder design for SNR maximization or transmit power minimization for ellipsoidal uncertainty class and different power constraints.

In this work, we pursue the deterministic framework to design a precoder for the maximization of SNR in a system of parallel sub-channels, given an uncertain knowledge of the sub-channel gains at the transmitter. In this context, the precoder design problem reduces to the one of finding the optimal power assignment to each sub-channel. Our work differs from the previous approaches, in the way we characterize the uncertainty class. Traditionally, the WC precoder designs (e.g. [5–7]) are based on the norm based uncertainty class for the minimax problem formulation. In this work, motivated by the unbiasedness of the estimator (or the quantizer for feedback) employed to obtain the CSI at the transmitter, we impose an additional constraint that the average value of the transmitter uncertainty (i.e. the estimation/quantization error) is zero. This additional and natural constraint effectively reduces the size of the uncertainty class employed in the minimax problem formulation, and consequently reduces the conservativeness of the maximin robust precoder.

The rest of the paper is organized as follows. Section 2 presents the system model and the preliminaries needed to pose the robust optimization problem. In Section 3, we detail the proposed and conventional approaches to optimize power allocation under channel uncertainty. Section 4 makes a simulation based comparison of the proposed design with conventional ones, while Section 5 presents the conclusion.

2. PRELIMINARIES

2.1 System Model & Objective Function

We consider a system of N parallel interference-free sub-channels. This originates, for instance, in the context of an orthogonal multi-carrier system, where the transmission on each sub-carrier can be modeled as a flat fading AWGN channel having a gain determined by the channel frequency response at that sub-carrier. Thus, the system can be represented via set of scalar equations,

$$\tilde{y}_i = \tilde{h}_i x_i + \tilde{\eta}_i \tag{1}$$

for $i = 1, 2, \ldots, N$, with $\tilde{h}_i \in \mathbb{C}$ denoting the complex fading coefficients, and $\tilde{\eta}_i \sim \mathcal{N}(0, \sigma_\eta^2)$ denoting the zero mean, circularly symmetric complex Gaussian noise. Here x_i denote the independent (unit power) transmit symbols, so we re-write the system model as

$$\tilde{y}_i = \tilde{h}_i p_i^{1/2} x_i + \tilde{\eta}_i, \tag{2}$$

where $p_i \in \mathbb{R}_+$ denotes the power allocated to the $i-$th sub-channel, and serves as *a degree of freedom* to optimize the transmission performance. Throughout this work, we assume a perfect receiver CSI[1], thus the receiver can pursue the matched filtering (and scaling) to obtain $y_i = \frac{1}{|\tilde{h}_i|}\tilde{h}_i^* \tilde{y}_i$, such that

$$y_i = h_i p_i^{1/2} x_i + \eta_i, \tag{3}$$

[1]This scenario serves at least as an approximation for the scenario where the receiver CSI is of sufficiently reliable quality.

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where $h_i = |\tilde{h}_i| \in \mathbb{R}_+$, and $\eta_i = \frac{1}{|\tilde{h}_i|}\tilde{h}_i^* \tilde{\eta}_i$ still follows the Gaussian distribution $\mathcal{N}(0, \sigma_\eta^2)$. We build this work on the system model given by (3). While the receiver CSI is perfect, we assume that the transmitter CSI is possibly erroneous.[2] Let the estimated channel gains at the transmitter be denoted as $h_{\mathrm{t}i} \in \mathbb{R}_+$, then we have the actual channel gains given as

$$h_i = h_{\mathrm{t}i} + h_{\mathrm{u}i}, \tag{4}$$

where $h_{\mathrm{u}i} \in \mathbb{R}$ denote the associated estimation (quantization) errors i.e. the uncertainties at the transmitter. Without limiting the scope of our contribution, we assume that the sub-channels are sorted and indexed in the order of decreasing channel gains w.r.t. the transmitter CSI $h_{\mathrm{t}i}$.

Note that in the matrix-vector notation, the system model in (3) can be expressed as $\boldsymbol{y} = \boldsymbol{H}\boldsymbol{P}^{1/2}\boldsymbol{x} + \eta$ with $\boldsymbol{H}$ being the diagonal channel matrix and $\boldsymbol{P}$ being a diagonal matrix determining the power assignments. We aim to optimize the receiver SNR, $\frac{\mathrm{tr}(\boldsymbol{HPH}^{\mathrm{H}})}{\mathrm{tr}(\boldsymbol{R}_\eta)}$, via precoder optimization. Thus, in the scalar notations, we write the objective function explicitly in terms of the power assignments $\{p_i\}$ and the channel uncertainties $\{h_{\mathrm{u}i}\}$ as

$$\varphi(\{p_i\}, \{h_{\mathrm{u}i}\}) = \sum_{i=1}^{N} p_i \left(h_{\mathrm{t}i} + h_{\mathrm{u}i}\right)^2, \tag{5}$$

where we dropped the constant term involving σ_η^2, i.e. $\mathrm{SNR} = \frac{\varphi(\{p_i\},\{h_{\mathrm{u}i}\})}{N\sigma_\eta^2}$. It may be pointed out here that besides measuring the received SNR, this objective function also has a relationship [10, Sec. 3.4] with the *Pairwise Error Probability (PEP)* — the probability that a codeword has a better ML metric than the actually transmitted codeword.

2.2 Uncertainty Model

The transmitter is assumed to obtain its CSI via one of the two mechanisms:

- a possibly erroneous, out-dated, and quantized feedback from the receiver, or
- estimation on the reverse link by exploiting channel reciprocity.

In both cases, the obtained CSI is imperfect, and $h_{\mathrm{u}i} = h_i - h_{\mathrm{t}i}$ denotes the associated uncertainty about the actual channel magnitude at the transmitter. It is clear that this uncertainty is real valued, i.e. $h_{\mathrm{u}i} \in \mathbb{R}$, and that the uncertainty has no bias, otherwise it can be removed as a preprocessing step. We assume ergodicity of this unbiasedness, so that the uncertainty is also unbiased along the channel index. This assumption is justified especially in scenarios where CSI is obtained by pilot symbols at intermittent sub-channels, followed by an unbiased interpolation / estimation at the intermediate sub-channels.

The uncertainty region, which is a crucial *design parameter* for minimax problem formulation in worst-case designs, is therefore selected via incorporation of two constraints. First, that the channel uncertainty at the transmitter has a bounded norm, and second that, owing to the unbiasedness of the estimator/quantizer employed, it has a zero sum. Mathematically, we define the uncertainty class as the following set of channel uncertainties,

$$\mathcal{U}_{\mathrm{h}} = \left\{ \{h_{\mathrm{u}i}\} : \frac{1}{N}\sum_{i=1}^{N} {h_{\mathrm{u}i}}^2 \leq \delta, \quad \frac{1}{N}\sum_{i=1}^{N} h_{\mathrm{u}i} = 0 \right\}. \tag{6}$$

Thus, while pursuing the minimax WC design, we consider only unbiased uncertainties of magnitudes not exceeding a threshold. The threshold δ can be interpreted as a design parameter in practical implementations to trade-off robustness with conservativeness, but *if some statistics about the uncertainty are available*, it can indeed be selected in a non-ad-hoc manner, as outlined in earlier works, for instance [6].

[2]Note that the transmitter needs a knowledge of only the sub-channel gains $\{h_i\}$ instead of the actual fading coefficients $\{\tilde{h}_i\}$ to optimize its power allocation.

An important consequence of the zero sum constraint in (6), in comparison to earlier works, is that it effectively reduces the manifold of potential minimax robust solutions, thereby introducing a regularization effect which turns out to be favorable. The reason being that the minimax optimization for the considered scenario in absence of such a constraint leads to optimal uncertainties $h_{\mathrm{u}i}^\star \leq 0$ for all $i = 1, 2, \ldots, N$. Besides estimator biasedness, this inherently means that the estimation error is correlated — a property which any reasonable (especially the MMSE) estimator should exploit to improve the estimate. Thus, given an unbiased estimator for CSI at the transmitter, it is natural to pursue the minimax optimization with this unbiasedness constraint in place, and it helps ultimately to arrive at a minimax robust power assignment scheme that is less conservative. We remark, that in Section 4, we also evaluate the performance of various schemes, including the proposed one, for Gaussian uncertainties without an explicit zero sum constraint.

3. MAXIMIN ROBUST POWER ALLOCATION (MRPA)

The maximin robust power allocation design problem in presence of an uncertainty $\{h_{\mathrm{u}i}\}$ at the transmitter can be posed as,

$$\max_{\{p_i\}\in\mathcal{P}} \quad \min_{\{h_{\mathrm{u}i}\}\in\mathcal{U}_{\mathrm{h}}} \quad \varphi(\{p_i\}, \{h_{\mathrm{u}i}\}), \tag{7}$$

where $\mathcal{P}$ denotes the set of valid power assignments, defined below, and $\mathcal{U}_{\mathrm{h}}$ denotes the channel uncertainty class, as defined for instance in (6). Put in words, the inner minimization identifies the worst-case scenario, while the outer maximization optimizes the worst-case performance. We note that,

- $\mathcal{P} = \{\{p_i\} : \sum_{i=1}^{N} p_i \leq P_{\mathrm{T}}, \ \ p_i \geq 0\}$ forms a convex set.
- $\mathcal{U}_{\mathrm{h}}$ in (6) is an intersection of two convex sets, and is therefore a convex set as well.
- $\varphi(\{p_i\}, \{h_{\mathrm{u}i}\})$ is concave (linear) in p_i and is convex in $h_{\mathrm{u}i}$.

Hence, the objective function is concave-convex in the two optimization variables, and the regions $\mathcal{P}$ and $\mathcal{U}_{\mathrm{h}}$ are convex and compact, so that the minimax theorem [11, Sec. 2.6] holds, i.e. the optimal power assignment and the WC uncertainty form a saddle point. In the sequel, we solve this maximin problem for the proposed and the conventional uncertainty class.

3.1 Proposed MRPA – Optimization with Unbiasedness Constraint

We solve here the maximin robust power allocation problem (7) with the uncertainty class defined in (6). To this end, we note that since the inner minimization is convex, we may employ the Lagrangian duality (for inner minimization) to equivalently write the problem as

$$\max_{\{p_i\}\in\mathcal{P}} \ \max_{\bar{\lambda}\geq 0,\bar{\mu}} \ \min_{\{h_{\mathrm{u}i}\}} \ \sum_{i=1}^{N} \left[p_i (h_{\mathrm{t}i} + h_{\mathrm{u}i})^2 + \frac{\bar{\lambda}}{N} {h_{\mathrm{u}i}}^2 + \frac{\bar{\mu}}{N} h_{\mathrm{u}i} \right] - \bar{\lambda}\delta \tag{8}$$

where $\lambda \geq 0$ and μ are the Lagrangian multipliers dualizing respectively the inequality and equality constraints in (6). Defining,

$$\lambda = \frac{\bar{\lambda}}{N}, \qquad \mu = \frac{\bar{\mu}}{2N}, \tag{9}$$

we may rewrite the Lagrangian function in (8) as,

$$\phi\left(\{p_i\},\{h_{\mathrm{u}i}\},\lambda,\mu\right) = \sum_{i=1}^{N} \left[p_i (h_{\mathrm{t}i} + h_{\mathrm{u}i})^2 + \lambda h_{\mathrm{u}i}^2 + 2\mu h_{\mathrm{u}i} \right] - \lambda N \delta. \tag{10}$$

The KKT condition for the minimization problem yield the following optimal (WC) uncertainties

$$h_{\mathrm{u}i}^\star = -\left(\frac{p_i h_{\mathrm{t}i} + \mu}{p_i + \lambda} \right), \tag{11}$$

for $i = 1, 2, \ldots, N$. We observe that back substitution of $h_{\mathrm{u}i}^{\star}$ in (10), taking into account

$$h_{\mathrm{t}i} + h_{\mathrm{u}i}^{\star} = \frac{\lambda h_{\mathrm{t}i} - \mu}{p_i + \lambda}, \tag{12}$$

leads to the dual function

$$\theta\left(\{p_i\}, \lambda, \mu\right) = \sum_{i=1}^{N} \left[\frac{p_i h_{\mathrm{t}i}(\lambda h_{\mathrm{t}i} - 2\mu) - \mu^2}{p_i + \lambda} \right] - \lambda N \delta, \tag{13}$$

such that the maximin problem (8) reduces to the following maximization problem.

$$\max_{\lambda \geq 0, \mu} \quad \max_{\{p_i\} \in \mathcal{P}} \quad \theta\left(\{p_i\}, \lambda, \mu\right) \tag{14}$$

Since the inner maximization is convex, we may again use the Lagrangian duality to write the problem as

$$\max_{\lambda \geq 0, \mu} \quad \min_{\nu \geq 0} \quad \max_{\{p_i\}: p_i \geq 0} \quad \tilde{\phi}\left(\{p_i\}, \lambda, \mu, \nu\right), \tag{15}$$

with $\tilde{\phi}\left(\{p_i\}, \lambda, \mu, \nu\right) = \theta\left(\{p_i\}, \lambda, \mu\right) + \nu \left(P_{\mathrm{T}} - \sum_{i=1}^{N} p_i\right)$, i.e. we used $\nu \geq 0$ as the Lagrangian multiplier to dualize the sum power constraint in $\mathcal{P}$. The KKT conditions now lead to the following optimal (robust) power assignments

$$p_i^{\star} = \left(\frac{\lambda h_{\mathrm{t}i} - \mu}{\sqrt{\nu}} - \lambda \right)^{+}, \tag{16}$$

where $(\bullet)^{+} = \max(\bullet, 0)$ ensures that the power assignments are non-negative.[3] Let N_{a} denote the number of active modes with $p_i > 0$ and N_{b} denote the number of inactive modes, i.e. $N_{\mathrm{a}} + N_{\mathrm{b}} = N$, then we may express the resulting dual function, by substituting $p_i^{\star}$ into $\tilde{\phi}\left(\{p_i\}, \lambda, \mu, \nu\right)$, to get

$$\begin{aligned}\tilde{\theta}\left(\lambda, \mu, \nu\right) = &\sum_{i=1}^{N_{\mathrm{a}}} \left[\frac{h_{\mathrm{t}i}(\lambda^2 h_{\mathrm{t}i}^2 - 3\mu\lambda h_{\mathrm{t}i} + 2\mu^2) - 2\sqrt{\nu}(\lambda h_{\mathrm{t}i} - \mu)^2}{(\lambda h_{\mathrm{t}i} - \mu)} \right] \\ &+ N_{\mathrm{a}} \nu \lambda - N_{\mathrm{b}} \frac{\mu^2}{\lambda} + \nu P_{\mathrm{T}} - \lambda N \delta.\end{aligned}$$

Since $\tilde{\theta}\left(\lambda, \mu, \nu\right)$ is by definition convex in ν and concave in μ, the optimal Lagrangian multipliers can be obtained via the first order optimality conditions and some algebraic manipulations, such that

$$\mu^{\star} = \frac{-\lambda \alpha P_{\mathrm{T}}}{N\lambda + P_{\mathrm{T}}(N_{\mathrm{b}}/N_{\mathrm{a}})}, \tag{17}$$

$$\nu^{\star} = \left(\frac{N \lambda \alpha}{N\lambda + P_{\mathrm{T}}(N_{\mathrm{b}}/N_{\mathrm{a}})} \right)^2, \tag{18}$$

with the scalar $\alpha = \frac{1}{N_{\mathrm{a}}} \sum_{i=1}^{N_{\mathrm{a}}} h_{\mathrm{t}i}$. Back substitution leads to

$$\begin{aligned}\tilde{\theta}\left(\lambda, \mu^{\star}, \nu^{\star}\right) = &\left[\sum_{i=1}^{N_{\mathrm{a}}} \frac{h_{\mathrm{t}i}(\lambda^2 h_{\mathrm{t}i}^2 - 3\mu^{\star}\lambda h_{\mathrm{t}i} + 2\mu^{\star 2})}{(\lambda h_{\mathrm{t}i} - \mu^{\star})} \right] - N_{\mathrm{b}} \frac{\mu^{\star 2}}{\lambda} \\ &+ \frac{3N^2 N_{\mathrm{a}} \lambda^2 \alpha^2 (\lambda + P_{\mathrm{T}}/N_{\mathrm{a}})}{(N\lambda + P_{\mathrm{T}}(N_{\mathrm{b}}/N_{\mathrm{a}}))^2} - \lambda N \delta\end{aligned} \tag{19}$$

We note that $\tilde{\theta}\left(\lambda, \mu^{\star}, \nu^{\star}\right)$ is by definition concave in $\lambda \geq 0$, so a simple single variable, numerical convex optimization may now be pursued to obtain $\lambda^{\star}$, which can be employed to obtain the robust power allocation $\{p_i^{\star}\}$ and its corresponding WC uncertainty $\{h_{\mathrm{u}i}^{\star}\}$. An algorithmic summary of how to arrive at the proposed minimax robust power allocation is given below.

[3] It is worth mentioning here that the quantity $\lambda h_{\mathrm{t}i} - \mu$ is non-negative for all i. This holds because $\lambda h_{\mathrm{t}i} \geq 0$ for all i, and the Lagrangian multiplier $\mu \leq 0$. Note that the non-negativity of $\lambda h_{\mathrm{t}i} - \mu$ also confirms that $h_{\mathrm{t}i} + h_{\mathrm{u}i}^{\star}$ in (12) is non-negative for all i.

Algorithm 1 Algorithm for the proposed MRPA.

Inputs: $\{h_{\mathrm{t}i}\}_{i=1}^{N}, P_{\mathrm{T}}, \delta$
Outputs: $\{h_{\mathrm{u}i}^{\star}\}_{i=1}^{N}, \{p_i^{\star}\}_{i=1}^{N}$
Choose an initial guess λ.
while Desired optimality not attained **do**
  for $N_{\mathrm{a}} = 1$ to N **do**
    Compute α, and then $\mu^{\star}$ and $\nu^{\star}$ via (17) and (18).
    Compute $\{p_i\}$ via (16).
    if $\sum_{i=1}^{N} p_i == P_{\mathrm{T}}$ **then**
      Break
    end if
  end for{Correct value of N_{a} determined.}
  Evaluate $\tilde{\theta}\left(\lambda, \mu^{\star}, \nu^{\star}\right)$ at given N_{a}, α , $\mu^{\star}$ and $\nu^{\star}$ via (19).
  Make an appropriate correction in λ. We employ the `fmincon` routine from the MATLAB optimization toolbox.
end while
$\lambda^{\star} \leftarrow \lambda$
Compute $\{p_i^{\star}\}_{i=1}^{N}$ from (16), and $\{h_{\mathrm{u}i}^{\star}\}_{i=1}^{N}$ from (11).

3.2 Conventional MRPA — Optimization without Unbiasedness Constraint

The maximin robust power assignment design problem without the uncertainty unbiasedness constraint can be posed similarly as in (7), but with $\mathcal{U}_{\mathrm{h}}$ replaced by $\acute{\mathcal{U}}_{\mathrm{h}}$ which denotes the conventional channel uncertainty class, constrained only via the uncertainty norm, i.e.

$$\acute{\mathcal{U}}_{\mathrm{h}} = \left\{ \{h_{\mathrm{u}i}\} : \frac{1}{N} \sum_{i=1}^{N} h_{\mathrm{u}i}^2 \leq \delta \right\}. \tag{20}$$

To solve (7) with $\acute{\mathcal{U}}_{\mathrm{h}}$ instead of $\mathcal{U}_{\mathrm{h}}$, we make use of the Lagrangian dualities in a way similar to that in Sec. 3.1. The WC uncertainty without the unbiasedness constraint solves to

$$h_{\mathrm{u}i}^{\star} = -\left(\frac{p_i h_{\mathrm{t}i}}{p_i + \acute{\lambda}} \right), \tag{21}$$

while the optimal power assignments can be given as

$$p_i^{\star} = \left(\frac{\acute{\lambda} h_{\mathrm{t}i}}{\sqrt{\acute{\nu}}} - \acute{\lambda} \right)^{+}. \tag{22}$$

The associated optimal Lagrangian multipliers, obtained via the first order optimality conditions, read as

$$\acute{\nu}^{\star} = \left(\frac{\acute{\lambda} \alpha}{\acute{\lambda} + P_{\mathrm{T}}/N_{\mathrm{a}}} \right)^2, \tag{23}$$

$$\acute{\lambda}^{\star} = \frac{P_{\mathrm{T}}}{N_{\mathrm{a}}} \left(\sqrt{\frac{\alpha^2}{\alpha^2 - \beta + \delta N/N_{\mathrm{a}}}} - 1 \right), \tag{24}$$

with $\beta = \frac{1}{N_{\mathrm{a}}} \sum_{i=1}^{N_{\mathrm{a}}} h_{\mathrm{t}i}^2$. The optimal Lagrangian multipliers $\acute{\lambda}^{\star}$ and $\acute{\nu}^{\star}$ can now be plugged in (22) to obtain the robust power allocation $\{p_i^{\star}\}$, cf. [9] for instance.

We reiterate, that unlike the proposed robust power allocation in Sec. 3.1, which incorporates the uncertainty unbiasedness as an additional constraint in its maximin design formulation, the conventional approach does not take this into consideration and as such has a larger uncertainty class. This inherently means that the conventional WC optimization pursued in this sub-section leads to a possibly too conservative power allocation strategy.

3.3 Worst-Case Unbiased Uncertainty for a Given Power Allocation

A related problem, that we need to solve for a comparison of the proposed robust power assignment with some conventional power allocation schemes (such as the one in Sec. 3.2) is to determine the worst-case *unbiased* uncertainty for a given power allocation. To this end, we need to solve the following optimization problem

$$\min_{\{h_{\mathrm{u}i}\} \in \mathcal{U}_{\mathrm{h}}} \quad \varphi(\{p_i\}, \{h_{\mathrm{u}i}\}) \tag{25}$$

where $\{p_i\}$ is some arbitrary fixed power allocation. This problem can be immediately recognized as being identical to the inner minimization problem in (7), and solves to

$$h_{\mathrm{u}i}^{\star} = -\left(\frac{p_i h_{\mathrm{t}i} + \tilde{\mu}}{p_i + \tilde{\lambda}}\right), \quad (26)$$

which leads to the dual problem

$$\max_{\tilde{\lambda}\geq 0, \tilde{\mu}} \quad \sum_{i=1}^{N}\left[\frac{p_i h_{\mathrm{t}i}(\tilde{\lambda} h_{\mathrm{t}i} - 2\tilde{\mu}) - \tilde{\mu}^2}{p_i + \tilde{\lambda}}\right] - \tilde{\lambda} N \delta. \quad (27)$$

By the first order optimality condition for $\tilde{\mu}$, we get

$$\tilde{\mu}^{\star} = -\frac{\sum_{i=1}^{N} \frac{p_i h_{\mathrm{t}i}}{p_i + \tilde{\lambda}}}{\sum_{i=1}^{N} \frac{1}{p_i + \tilde{\lambda}}}, \quad (28)$$

leading to the function $\phi(\{p_i\}, \{h_{\mathrm{u}i}^{\star}\}, \tilde{\lambda}, \tilde{\mu}^{\star})$ to be numerically maximized over $\tilde{\lambda}$ to get $\tilde{\lambda}^{\star}$ that can be plugged in (28) to obtain $\tilde{\mu}^{\star}$. Finally, via (26), we obtain $\{h_{\mathrm{u}i}^{\star}\}$ — the WC unbiased uncertainty associated with the given power assignment.

4. SIMULATION RESULTS

In order to analyze the effectiveness of the proposed power allocation strategy, we carry out a simulation based comparison for a system of N parallel sub-channels. We assume that the transmitter has an imperfect knowledge of sub-channel gains, and that it has a limited power budget P_{T} that can be distributed among the parallel sub-channels, via one of the following four schemes.

- Uniform Power Allocation (UPA), i.e. $p_i = P_{\mathrm{T}}/N$ for all $i = 1, 2, \ldots, N$.
- Best Sub-channel Power Allocation (BSPA), with $p_i = P_{\mathrm{T}}$ only for the strongest channel mode (w.r.t. the uncertain transmitter CSI) and $p_i = 0$ for all other i.
- Maximin Robust Power Allocation (MRPA) *without unbiasedness* constraint, i.e. $p_i^{\star}$ as in Sec. 3.2.
- Maximin Robust Power Allocation (MRPA) *with unbiasedness* constraint, i.e. $p_i^{\star}$ as in Sec. 3.1.

Let us first take a look at the intuitive results. We note that with SNR as the objective function, the optimal power assignment – in case of perfect CSI – is to assign all the available power P_{T} to the strongest sub-channel. For the scenario of uncertain CSI at the transmitter, the solution would stay the same unless and until the uncertainty is large enough to cast doubt on the identity of the strongest sub-channel. In the extreme case, if the uncertainty is very high, we may end up in an equal power assignment to all sub-channels. The UPA and the BSPA are therefore the two extreme cases to which the maximin robust power allocation converges at very high and very low uncertainties respectively.

4.1 Worst-case SNR Performance

In this section, we present the comparison of the four schemes in terms of their WC received SNRs, i.e., the received SNR achieved by them for their respective WC uncertainties as derived in Section 3.3. To this end, we generate random nominal channel realizations $\{h_{\mathrm{t}i}\}$, determine the power allocation and corresponding WC uncertainties for each scheme, and then measure their received SNRs. The SNRs are averaged over several (1000) nominal channel realizations, and are plotted in Fig. 1 as a function of the degree of uncertainty in transmitter CSI. For better intuition, we label the x-axis in terms of the normalized uncertainty measure $\tilde{\delta}$, such that

$$\frac{\sum_{i=1}^{N} h_{\mathrm{u}i}^2}{\sum_{i=1}^{N} h_{\mathrm{t}i}^2} \leq \tilde{\delta}. \quad (29)$$

Hence, values of $\tilde{\delta} = \{0, 1\}$ correspond to the extreme cases of zero and maximum uncertainty in the transmitter CSI. In order to focus on the effect of channel uncertainty only, we normalize the achieved SNR by the effective channel power $\frac{1}{N}\sum_{i=1}^{N}(h_{\mathrm{t}i} + h_{\mathrm{u}i})^2$, so that the performance of UPA stays constant, regardless of the uncertainty level.

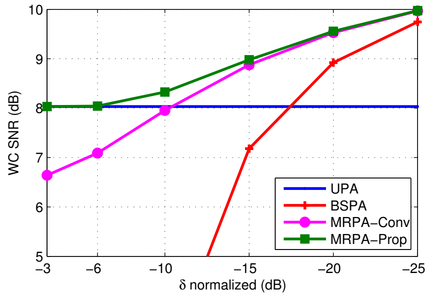


Figure 1: Worst-Case SNR as a function of $\tilde{\delta}$ at $N = 8$.

The comparison in Fig. 1 corresponds to a system with $N = 8$ sub-channels with unity receive noise power at each sub-channel, and a transmit power budget P_{T} such that $P_{\mathrm{T}}/(N\sigma_\eta^2) = 8$ dB. We note that at high uncertainty, $\tilde{\delta}$ greater than -10 dB or so, the performance of BSPA suffers badly because of the uncertainty being high enough to cast doubt on the identity of the best sub-channel. The UPA, on the other hand, offers a reasonable performance here, because it distributes the power uniformly over all the sub-channels — the optimal strategy in absence of CSI. As the uncertainty decreases, the performance of BSPA improves however, and eventually becomes superior to UPA. As expected, the proposed robust power allocation with unbiasedness constraint (MRPA-Proposed) always performs the best among all four schemes, at all uncertainty levels.

4.2 Coded BER Performance

A superior performance of the proposed scheme in terms of WC SNR, as observed in the last sub-section, is a somewhat expected result. In this section, we present the comparison in terms of the coded BER, which is of prime concern in many practical systems. To this end, we employ a rate $1/3$ turbo code (as specified in the LTE standard [12]). We consider the transmission of an identical fixed length sequence of coded bits over the same channel with different power allocation schemes. Since different schemes employ different number of active modes, but same transmit constellation (QPSK), to enable a fair comparison, we normalize the power budget per transmission slot such that the total transmit power needed for the transmission of this fixed length sequence is the same for all schemes.

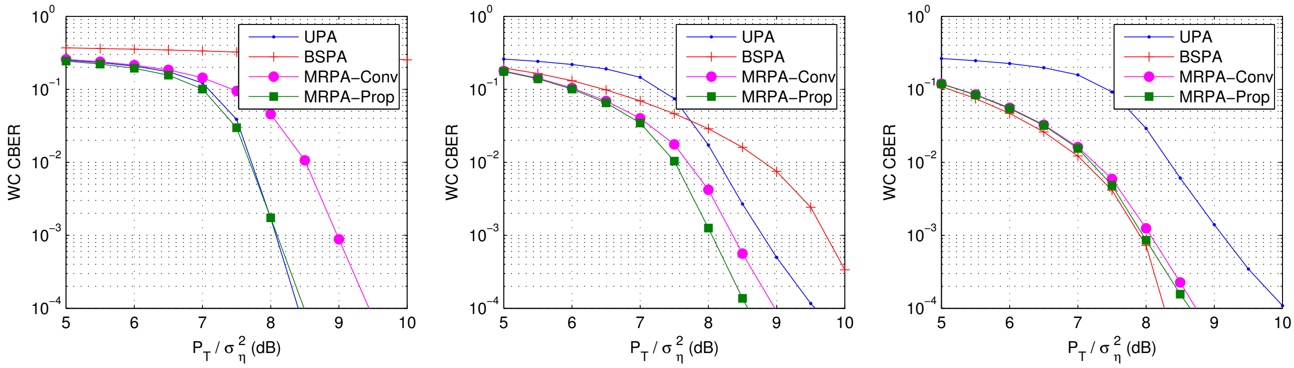
Fig. 2 shows the CBER comparison of the four schemes at their respective *worst-case channels* as derived in Sec. 3.3, for three different uncertainty levels, namely $\tilde{\delta} = -10$ dB, -20 dB and -30 dB. We observe that at high uncertainty (left sub-plot), UPA and the proposed MRPA perform the best. At moderate uncertainty (center sub-plot), proposed MRPA beats all other power assignments, while at low uncertainty (right sub-plot), the proposed MRPA performs as good as the BSPA. [4]

Finally, in Fig. 3, we present the comparison of the power allocation schemes in terms of their *average* coded BER, obtained by generating various (100) random uncertainties (instead of the WC uncertainty), and then averaging over various (1000) transmitter CSI realizations. It is worth mentioning that this comparison is made by generating zero mean Gaussian uncertainties of bounded norm, but without the unbiasedness constraint explicitly imposed.

[4] It may be mentioned here that the WC performance in Fig. 2 and in Table 1 is presented in terms of WC coded BER, so the performance of proposed scheme is not necessarily best.

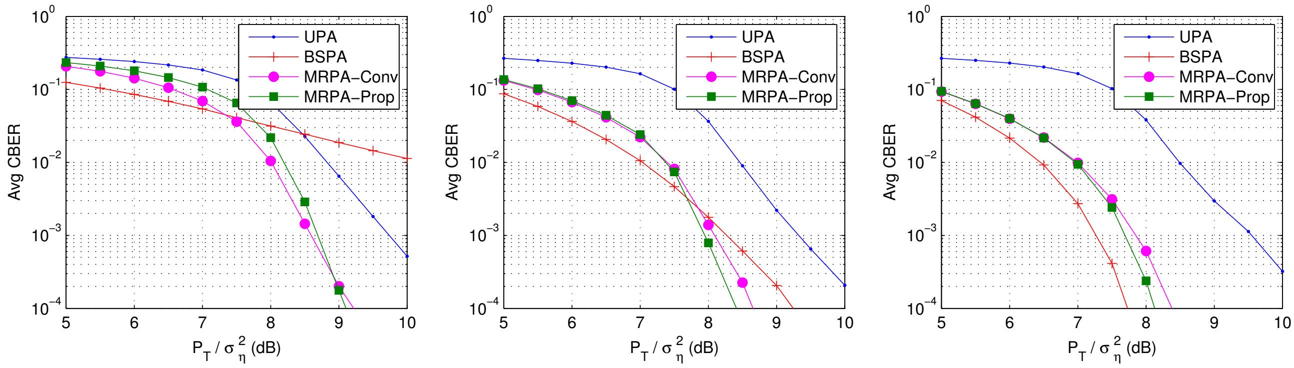
Table 1: Relative gains of proposed MRPA over other schemes at CBER of 10^{-3}

Scheme	Worst-case performance, at $\tilde{\delta}$			Average performance, at $\tilde{\delta}$		
	-10 dB	-20 dB	-30 dB	-10 dB	-20 dB	-30 dB
UPA	0.0 dB	0.74 dB	0.39 dB	1.05 dB	1.38 dB	0.85 dB
BSPA	∞ dB	1.67 dB	-0.05 dB	≈ 5 dB	0.32 dB	-0.42 dB
$\mathrm{MRPA_{CV}}$	0.87 dB	0.31 dB	0.12 dB	-0.10 dB	0.15 dB	0.16 dB



(a) $\tilde{\delta} = -10$ dB (b) $\tilde{\delta} = -20$ dB (c) $\tilde{\delta} = -30$ dB

Figure 2: Coded BER at worst-case uncertainties as a function of transmit power budget at different values of $\tilde{\delta}$.



(a) $\tilde{\delta} = -10$ dB (b) $\tilde{\delta} = -20$ dB (c) $\tilde{\delta} = -30$ dB

Figure 3: Average Coded BER as a function of transmit power budget at different values of $\tilde{\delta}$.

It is encouraging to see that besides offering the best WC performance, the proposed robust power allocation scheme also offers a reasonably good average coded BER performance. The relative gains of the proposed scheme in terms of *WC CBER* as well as *average CBER* at the reference coded BER of 10^{-3} are summarized in Table 1. It reveals that even in terms of *average CBER*, the performance of the proposed MRPA is superior especially at moderate uncertainty levels.

5. CONCLUSION

We conclude that the unbiasedness of estimator / quantizer via which the transmitter CSI is obtained, provides us with a natural constraint that restricts the size of the uncertainty class for the design of maximin robust power allocation scheme. This restriction of the uncertainty class helps us to arrive at a robust power allocation with an average performance that is much superior than preceding robust designs. Obviously, the robustness is slightly reduced, but given the unbiasedness of practical estimators, this constrained uncertainty class provides a nice and practically relevant trade-off between robustness and conservativeness.

REFERENCES

[1] A. Narula, M. J. Lopez, M. D. Trott, and G. W. Wornell, "Efficient Use of Side Information in Multiple-Antenna Data Transmission over Fading Channels," *IEEE Journal on Selected Areas in Communications*, vol. 16, pp. 1423–1436, October 1998.

[2] E. Visotsky and U. Madhow, "Space-Time Transmit Precoding with Imperfect Feedback," *IEEE Transactions on Signal Processing*, vol. 47, pp. 2632–2639, September 2001.

[3] S. A. Jafar, S. Vishwanath, and A. Goldsmith, "Channel Capacity and Beamforming for Multiple Transmit and Receive Antennas with Covariance Feedback," in *IEEE International Conference on Communications*, June 2001, pp. 2266–2270.

[4] A. Ben-Tal, L. El Ghaoui, and A. Nemirovski, *Robust Optimization*, Princeton University Press, 2009.

[5] A. A. Samad, T. N. Davidson, and A. B. Gershman, "Robust Transmit Eigen Beamforming based on Imperfect Channel State Information," *IEEE Transactions on Signal Processing*, vol. 54, pp. 1596–1609, May 2006.

[6] A. Pascual-Iserte, D. P. Palomar, A. I. Perez-Neira, and M. A. Lagunas, "A Robust Maximin Approach for MIMO Communications with Imperfect Channel State Information based on Convex Optimization," *IEEE Transactions on Signal Processing*, vol. 54, pp. 346–360, May 2006.

[7] A. Wiesel, Y. C. Eldar, and S. Shamai, "Robust Power Allocation for Maximizing the Compound Capacity," in *NEWCOM-ACoRN Joint Workshop*, September 2006.

[8] M. Payaro, A. Pascual-Iserte, and M. A. Lagunas, "Robust Power Allocation Designs for Multiuser and Multiantenna Downlink Communication Systems through Convex Optimization," *IEEE Journal on Selected Areas in Communications*, vol. 25, pp. 1390–1401, September 2007.

[9] J. Wang and D. P. Palomar, "Worst-Case Robust MIMO Transmission with Imperfect Channel Knowledge," *IEEE Transactions on Signal Processing*, vol. 57, pp. 3086–3100, August 2009.

[10] E. Biglieri, R. Calderbank, A. Constantinides, A. Goldsmith, A. Paulraj, and H. V. Poor, *MIMO Wireless Communications*, Cambridge University Press, 2007.

[11] D. P. Bertsekas, A. Nedic, and A. E. Ozdaglar, *Convex Analysis and Optimization*, Athena Scientific, 2003.

[12] Standardization Committee 3GPP, "Evolved Universal Terrestrial Radio Access (E-UTRA); Physical Channels and Modulation, TS 36.211," *Online, http://www.3gpp.org*, 2010.